# Symmetry Characterization Theorems
# for
# Homogeneous Convex Cones and Siegel Domains

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**April 1 – 4, 2009**

## Homogeneous Open Convex Cones

$V$: a real vector space with inner product
$V \supset \Omega$: a regular open convex cone

- $G(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}$: linear automorphism group of $\Omega$
  a Lie group as a closed subgroup of $GL(V)$
- $\Omega$ is **homogeneous** $\overset{\text{def}}{\Longleftrightarrow}$ $G(\Omega) \curvearrowright \Omega$ is transitive

**Example:** $V = \mathrm{Sym}(r, \mathbb{R}) \supset \Omega := \mathrm{Sym}(r, \mathbb{R})^{++}$:
$GL(r, \mathbb{R}) \curvearrowright \Omega$ by $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx{}^t g \in \Omega$

This is a selfdual homogeneous open convex cone (**symmetric cone**).
$\Omega$ is **selfdual** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \langle \cdot \,|\, \cdot \rangle$ s.t. $\Omega = \big\{ y \in V\ ;\ \langle\, x \,|\, y \,\rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \big\}$
(the RHS is the dual cone taken relative to $\langle \cdot \,|\, \cdot \rangle$)

Symmetric Cones $\rightleftarrows$ Euclidean Jordan Algebras

$\Omega \rightleftarrows V$: algebraic str. in the ambient VS ($\equiv$ tangent space at a ref. pt.)

**List of Irreducible Symmetric Cones**:

$\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$, $\quad A \circ B := \frac{1}{2}(AB + BA)$

$\Omega = \mathrm{Herm}(r, \mathbb{C})^{++} \subset V = \mathrm{Herm}(r, \mathbb{C})$

$\Omega = \mathrm{Herm}(r, \mathbb{H})^{++} \subset V = \mathrm{Herm}(r, \mathbb{H})$

$\Omega = \mathrm{Herm}(3, \mathbb{O})^{++} \subset V = \mathrm{Herm}(3, \mathbb{O})$

$\Omega = \Lambda_n \subset V = \mathbb{R}^n$ ($n$-dimensional Lorentz cone)

**Non-Selfdual Homogeneous Open Convex Cones**: **Vinberg cone** (1960)

$$V = \left\{ x = \begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & 0 \\ x_4 & 0 & x_5 \end{pmatrix} \; ; \; x_1, \ldots, x_5 \in \mathbb{R} \right\} \supset \Omega := \left\{ x \in V \; ; \; \begin{matrix} x_1 > 0 \\ x_1 x_3 - x_2^2 > 0 \\ x_1 x_5 - x_4^2 > 0 \end{matrix} \right\}$$

the lowest-dimensional homogeneous non-selfdual open convex cone

## Classification of Irreducible Homogeneous Convex Cones (dim $\leq 10$)

(Kaneyuki–Tsuji, 1974)

There are $\mathbf{135}$ (up to linear isom.) in which $\mathbf{12}$ are selfdual.

$\mathbb{R}_{>0}$, $\Lambda_n \subset \mathbb{R}^n$ (Lorentz cones with $\dim = 3, 4, \ldots, 10$),
$\mathrm{Sym}(3, \mathbb{R})^{++}$ (6-dim), $\mathrm{Herm}(3, \mathbb{C})^{++}$ (9-dim), $\mathrm{Sym}(4, \mathbb{R})^{++}$ (10-dim)

By Vinberg's theory (1963)

Homogeneous Open Convex Cones $\rightleftarrows$ Clans with unit element

$\Omega \rightleftarrows V$: algebraic str. in the ambient VS ($\equiv$ tangent space at a ref. pt.)

- The Case of Symmetric Cones: $G(\Omega)$ is reductive.

  JA str. of $V$: $V \equiv T_e(\Omega) \equiv \mathfrak{p}$ of the Cartan decomposition $\mathfrak{g}(\Omega) = \mathfrak{k} + \mathfrak{p}$
  (The product is commutative.)

- The Case of General Homogeneous Convex Cones:
  simply transitive action of Iwasawa subgroup of $G(\Omega)$

  Clan str. of $V$: $V \equiv T_e(\Omega) \equiv$ Iwasawa subalgebra $\mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}(\Omega)$
  (The product is non-commutative, in general.)

$\Omega$: homogeneous open convex cone, $G(\Omega)$: linear automorphism group of $\Omega$,
$H$: Iwasawa subgroup of $G(\Omega)$.
$H$ is a split solvable Lie group, acting simply transitively on $\Omega$.

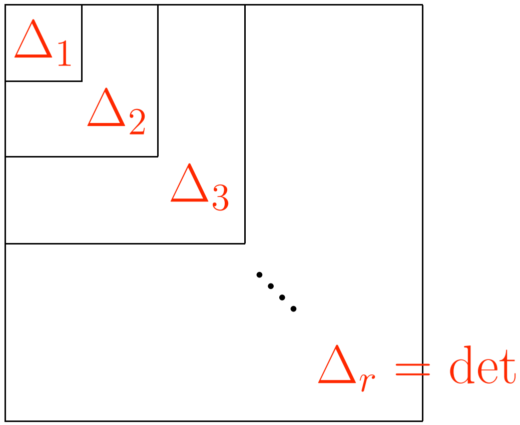
a function $f$ on $\Omega$, is **relatively invariant** (w.r.t. $H$)
$\overset{\text{def}}{\iff}$ $\exists\chi$: 1-dim. rep. of $S$ s.t. $f(gx) = \chi(g)f(x)$ (for all $g \in H$, $x \in \Omega$).

**Theorem** [Ishi 2001].
$\exists\Delta_1, \ldots, \Delta_r$ $(r := \mathrm{rank}(\Omega))$: relat. inv. irred. polynomial functions on $V$ s.t
any relat. inv. polynomial function $P(x)$ on $V$ is written as

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \mathrm{const.},\ (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

**Theorem** [Ishi–N., 2008].
$W$: the complexification of the Clan $V$,
$R(w)$: the right multiplication operator by $w$ in $W$
$\Longrightarrow$ irreducible factors of $\det R(w)$ are just $\Delta_1(w), \ldots, \Delta_r(w)$.

**Example:** $\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$




• Product in $V$ as a clan: $x \Delta y = \underline{x}\, y + y\, {}^t(\underline{x})$, where for $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$,

we put $$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & & & & 0 \\ & \frac{1}{2}x_{22} & & & \\ & & \ddots & & \\ & & x_{ji} & \ddots & \\ & & & & \frac{1}{2}x_{nn} \end{pmatrix} \quad (i < j).$$ Thus $x = \underline{x} + {}^t(\underline{x})$.

In this case we have $\det R(y) = \Delta_1(y) \cdots \Delta_r(y)$.

## The case of general irreducible symmetric cone $\Omega \subset V$

- $\Delta_k(y)$ is the $k$-th Jordan algebra principal minor.
- $\mathfrak{h} \ni X \mapsto Xe \in V$ is a linear isomorphism ($e$ is the unit element of $V$).
- The inverse map is denoted as $V \ni v \mapsto X_v \in \mathfrak{h}$. Thus $X_v e = v$ for any $v \in V$. (This is related to the linear isomorphism $\mathfrak{g}_{(\alpha_j-\alpha_i)/2} \cong V_{ij}$ $(i > j)$).

- Euclidean Jordan algebra $V$ is now a clan by the product $x \Delta y := X_x y = R(y)x$.

- $\boxed{\det R(y) = \Delta_1(y)^d \cdots \Delta_{r-1}(y)^d \Delta_r(y)}$, where $d =$ common dim. of $V_{ij}$ $(i < j)$: $d = 1$ for $\mathrm{Sym}(r, \mathbb{R})$, $d = \dim_{\mathbb{R}} \mathbb{K}$ for $\mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$, $r = 2$, $d = n - 2$ for $\Omega = \Lambda_n$ $(n \geqq 3)$.

The boxed formula is nice in view of $\dim V = r + \frac{d}{2} \cdot r(r-1)$, because

$$\deg\bigl(\Delta_1(y)^d \cdots \Delta_{r-1}(y)^d \Delta_r(y)\bigr) = d\bigl(1 + \cdots + (r-1)\bigr) + r = r + \frac{d}{2} \cdot r(r-1).$$

## Why do I become interested in these things? ......

**Proposition.** $w \in \mathrm{Sym}(r, \mathbb{C})$.

$$\mathrm{Re}\, w \gg 0 \implies \mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r),$$

where we understand $\Delta_0(w) \equiv 1$.

This follows from the following two lemmas.

**Lemma 1.** Suppose $w \in \mathrm{Sym}(r, \mathbb{C})$ and $\Delta_k(w) \neq 0$ for $k = 1, \ldots, r$.

Then we have $w = na\,{}^t n$ with $n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix}.$

Each $a_k$ $(k = 1, \ldots, r)$ is given by $a_k = \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)}$.

**Lemma 2.** Let $n, a$ be as above. Then

$$\mathrm{Re}(na\,{}^t n) \gg 0 \implies \mathrm{Re}\, a_k > 0 \quad (k = 1, \ldots, r).$$

## Generalization to Irreducible Symmetric Cones

$\Omega$: an irreducible symmetric cone $(\mathrm{rk}(\Omega) = r) \subset V$: a Euclidean JA
$\Delta_1, \ldots, \Delta_r$: JA principal minors (basic relative invariants).

**Theorem** [Ishi–N. 2008]. Suppose $w \in W := V_{\mathbb{C}}$. Then

$$w \in \Omega + iV \implies \mathrm{Re}\,\frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r).$$

# Further Generalization to Homogeneous Open Convex Cones?

$\Omega$: Irreducible homogeneous open convex cone $(\mathrm{rank}(\Omega) = r) \subset V$: a clan
$\Delta_1, \ldots, \Delta_r$: basic relative invariants

**Problem.** $w \in \Omega + iV \implies \mathrm{Re}\, \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r)$ ?

**Immediately seen that**

if one observes for $w \in \Omega \cap \{\text{diagonal type}\}$, we must have the gap between $\deg \Delta_k(w)$ and $\deg \Delta_{k-1}(w)$ equal to $1$.

**diagonal type elements**:

$\exists E_1, \ldots, E_r \in \overline{\Omega}$: complete orthogonal system of primitive idempotents
(the sum is equal to the unit element)

diagonal type elements $= c_1 E_1 + \cdots + c_r E_r$ ($c_k > 0$ for $\forall k = 1, \ldots, r$)

**Problem.** $w \in \Omega + iV \implies \mathrm{Re}\,\dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \dots, r)$ ?

**Answer to the problem**:

(1) For non-selfdual irred. homog. open convex cones, the answer is in general No.

(2) If the answer is always No, then one is happy ( $\implies$ obtains a symmetry characterization). But this is not the case.

(3) In dim $\leq 10$, there are **123** non-selfdual irred. homog. open convex cones up to linear isomorphisms). But the **only one** such gives the answer Yes. It is of $8$-dimension.

(4) In the following, we give such cone (non-selfdual irred. homog. open convex cone that gives an affirmative answer to the problem) with any rank ($\geq 3$).

$I_m$: $m \times m$ unit matrix
$\mathbb{R}^{rm}$: column vectors of size $r \times m$.

$$V := \left\{ x = \left( \begin{array}{c|c} x_0 \otimes I_m & \boldsymbol{y} \\ \hline {}^t\boldsymbol{y} & z \end{array} \right) \; ; \; x_0 \in \mathrm{Sym}(r, \mathbb{R}),\ \boldsymbol{y} \in \mathbb{R}^{rm},\ z \in \mathbb{R} \right\}.$$

Note $V \subset \mathrm{Sym}(rm+1, \mathbb{R})$.
When $m = r = 2$, $x$ is the following $5 \times 5$ matrix:

$$x = \begin{pmatrix} x_{11} & 0 & x_{21} & 0 & y_{11} \\ 0 & x_{11} & 0 & x_{21} & y_{12} \\ x_{21} & 0 & x_{22} & 0 & y_{21} \\ 0 & x_{21} & 0 & x_{22} & y_{22} \\ y_{11} & y_{12} & y_{21} & y_{22} & z \end{pmatrix} \qquad \text{(in this case } \dim V = 8\text{)}.$$

- For open convex cone $\Omega$, we take the set of positive definite ones in $V$:

$$\Omega := \{ x \in V \; ; \; x \gg 0 \} \qquad (\mathrm{rank}(\Omega) = r + 1).$$

**Assumption:** $m \geqq 2$, $r \geqq 2$ :
($m = 1 \implies \Omega = \mathrm{Sym}^{++}(r+1, \mathbb{R})$, $\quad r = 1 \implies \Omega = \Lambda_{m+2}$)

## Homogeneity of $\Omega$

$$A := \left\{ a = \left( \begin{array}{c|c} a_0 \otimes I_m & 0 \\ \hline 0 & a_{r+1} \end{array} \right) \; ; \; \begin{array}{l} a_0 := \mathrm{diag}[a_1, \dots, a_r] \text{ with} \\ a_1 > 0, \dots, a_r > 0 \text{ and } a_{r+1} > 0 \end{array} \right\},$$

$$N := \left\{ n = \left( \begin{array}{c|c} n_0 \otimes I_m & 0 \\ \hline {}^t\boldsymbol{\xi} & 1 \end{array} \right) \; ; \; \begin{array}{l} n_0 \text{ is strictly lower triangular in } GL(r, \mathbb{R}), \\ \boldsymbol{\xi} \in \mathbb{R}^{rm} \end{array} \right\}.$$

We have $H := N \ltimes A \curvearrowright \Omega$ by $H \times \Omega \ni (h, x) \mapsto h\, x\, {}^t h \in \Omega$

**This action is simply transitive**. In fact, given $x \in \Omega$, the equation $x = na\,{}^t n = na^{1/2} I_{rm+1} a^{1/2} ({}^t n)$ $(a \in A,\ n \in N)$ has unique solution: For $a_k$ we have

$$a_k = \frac{\Delta_k(x)}{\Delta_{k-1}(x)} \ (k = 1, 2, \dots, r+1), \text{ with } \Delta_0(x) \equiv 1, \text{ where}$$

$$\begin{cases} \Delta_k(x) := \Delta_k^0(x_0) & (k = 1, \dots, r), \\ \Delta_{r+1}(x) := z \det(x_0) - {}^t\boldsymbol{y} ({}^{\mathrm{co}}x_0 \otimes I_m) \boldsymbol{y}. \end{cases}$$

${}^{\mathrm{co}}T$: the cofactor matrix of $T$. Thus $T({}^{\mathrm{co}}T) = ({}^{\mathrm{co}}T)T = (\det T)I$.

- We note $\deg \Delta_k(x) = k$ $(k = 1, 2, \dots, r+1)$.

## To understand $\Delta_{r+1}(x)$

For each $x = \left(\begin{array}{c|c} x_0 \otimes I_m & \boldsymbol{y} \\ \hline {}^t\boldsymbol{y} & z \end{array}\right) \in V$, we set

$$ {}^{\mathrm{d}}x := \left(\begin{array}{ccc|c} x_{11} & \cdots & x_{r1} & \boldsymbol{y}_1 \\ \vdots & & \vdots & \vdots \\ x_{r1} & \cdots & x_{rr} & \boldsymbol{y}_r \\ \hline {}^t\boldsymbol{y}_1 & \cdots & {}^t\boldsymbol{y}_r & z \end{array}\right), \quad x_0 = (x_{ij}), \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_r \end{pmatrix}, \quad \boldsymbol{y}_j \in \mathbb{R}^m $$

Since $\Delta_{r+1}(x) = z \det(x_0) - \sum_{i=1}^{r} \sum_{j=1}^{r} ({}^{\mathrm{co}}x_0)_{ij}\, \boldsymbol{y}_i \cdot \boldsymbol{y}_j$, we have $\Delta_{r+1}(x) = \det {}^{\mathrm{d}}x$.

Here we compute $\det {}^{\mathrm{d}}x$ as if it were an ordinary determinant, and the product of ${}^t\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ should be interpreted as the inner product $\boldsymbol{y}_i \cdot \boldsymbol{y}_j$.

- $\Delta_1(x), \ldots, \Delta_r(x), \Delta_{r+1}(x)$ are basic relative invariants.

$\delta_k(x)$ $(k = 1, \ldots, rm+1)$: the $k$-th principal minor of $x = \left(\begin{array}{c|c} x_0 \otimes I_m & \boldsymbol{y} \\ \hline {}^t\boldsymbol{y} & z \end{array}\right) \in V$.

Then we have

$$\begin{cases} \delta_{km+j}(x) = \Delta_k(x)^{m-j}\Delta_{k+1}(x)^j & (0 \leqq k \leqq r-1,\ 1 \leqq j \leqq m), \\ \delta_{rm+1}(x) = \Delta_r(x)^{m-1}\Delta_{r+1}(x). \end{cases}$$

Hence

$$x \in \Omega \iff \Delta_j(x) > 0 \text{ for any } j = 1, \ldots, r+1.$$

- This description of $\Omega$ in terms of basic relative invarinats is true in general. (Ishi 2001)

Regarding $\Delta_k$ $(k = 1, \ldots, r+1)$ as polynomial functions on $W := V_{\mathbb{C}}$ naturally, we get

**Proposition.** $w \in \Omega + iV \implies \mathrm{Re}\,\dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r+1).$

## Non-selfdual Irreducible Homogeneous Convex Cones linearly isomorphic to the dual cones

- $\Omega$ is selfdual $\iff$ $\exists T$: positive definite selfadjoint operator s.t. $T(\Omega) = \Omega^*$
  $\big(\Omega^* := \big\{ y \in V \,;\, \langle\, x \,|\, y \,\rangle > 0 \quad \text{for } \forall x \in \overline{\Omega} \setminus \{0\} \big\}\big)$

- Even though there is no positive definite selfadjoint operator $T$ s.t. $T(\Omega) = \Omega^*$, we might find such $T$ if we do not require the positive definiteness.

- If one accepts reducible ones, then $\Omega_0 \oplus \Omega_0^*$ just gives an example. Thus the irreducibility counts for much here.

- The list in [Kaneyuki–Tsuji] of irreducible homogeneous open convex cones $(\dim \leq 10)$ is described up to linear isomoprhisms. There is one non-selfdual irreducible homogeneous convex cone ($7$-dimensional) identified with its dual cone.

- In an exercise of Faraut–Korányi's book, a hint is given to prove that the Vinberg cone is never linerly isomorphic to its dual cone. Then the non-selfduality follows from this immediately.

$$\boldsymbol{e} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m+1}, \qquad I_{m+1}\text{: } (m+1)\times(m+1) \text{ unit matrix}$$

$$V := \left\{ x := \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \;;\; \begin{array}{l} x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, \quad X \in \mathrm{Sym}(m,\mathbb{R}) \\ \boldsymbol{\xi} \in \mathbb{R}^{m+1},\ \boldsymbol{x}' \in \mathbb{R}^m,\ \boldsymbol{x}'' \in \mathbb{R}^m \end{array} \right\}$$

We note $V \subset \mathrm{Sym}(2m+2,\mathbb{R})$, and take $\Omega := \{x \in V \;;\; x \gg 0\}$.

When $m = 1$, we have $x = \begin{pmatrix} x_1 & 0 & x' & \xi_1 \\ 0 & x_1 & 0 & \xi_2 \\ x' & 0 & X & x'' \\ \xi_1 & \xi_2 & x'' & x_2 \end{pmatrix}$.

## Homogeneity of $\Omega$

$$\mathbf{H} := \left\{ h := \left(\begin{array}{c|cc} h_1 I_{m+1} & 0 & 0 \\ \hline \boldsymbol{h}'{}^t\boldsymbol{e} & H & 0 \\ {}^t\boldsymbol{\zeta} & {}^t\boldsymbol{h}'' & h_2 \end{array}\right) \; ; \; \begin{array}{c} h_j > 0 \; (j=1,2),\, \boldsymbol{h}' \in \mathbb{R}^m,\, \boldsymbol{h}'' \in \mathbb{R}^m \\ \boldsymbol{\zeta} \in \mathbb{R}^{m+1}, H \in GL(m,\mathbb{R}) \end{array} \right\}$$

$\mathbf{H}$ acts on $\Omega$ by $\mathbf{H} \times \Omega \ni (h, x) \mapsto hx{}^th$. The action is in fact simply transitive.
We have $\mathbf{H} = \mathbf{N} \rtimes \mathbf{A}$ with

$$\mathbf{A} := \left\{ a := \left(\begin{array}{c|cc} a_1 I_{m+1} & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & a_2 \end{array}\right) \; ; \; \begin{array}{c} a_j > 0 \quad (j=1,2), \\ A \in GL(m,\mathbb{R}) \text{ is a diagonal} \\ \text{matrix with positive diagonals} \end{array} \right\},$$

$$\mathbf{N} := \left\{ n := \left(\begin{array}{c|cc} I_{m+1} & 0 & 0 \\ \hline \boldsymbol{n}'{}^t\boldsymbol{e} & N & 0 \\ {}^t\boldsymbol{\nu} & {}^t\boldsymbol{n}'' & 1 \end{array}\right) \; ; \; \begin{array}{c} \boldsymbol{n}', \boldsymbol{n}'' \in \mathbb{R}^m,\, \boldsymbol{\nu} \in \mathbb{R}^{m+1} \\ N \in GL(m,\mathbb{R}) \text{ is strictly} \\ \text{lower triangular} \end{array} \right\}.$$

In solving $x = na{}^tn$ ($x \in \Omega$: given) we obtain basic relative invariants as before.

**Basic relative invariants**: For $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \in V$,

$$\Delta_1(x) := x_1,$$

$$\Delta_j(x) := \det\left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}'_{j-1} \\ \hline \boldsymbol{x}'_{j-1} & X_{j-1} \end{array}\right) \qquad (j = 2, \dots, m+1)$$

$$\left( X_k := \begin{pmatrix} x_{11} & \cdots & x_{k1} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix}, \quad \boldsymbol{x}'_k := \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} \in \mathbb{R}^k \right),$$

$$\Delta_{m+2}(x) := x_1 \det \begin{pmatrix} x_1 & {}^t\boldsymbol{x}' & \xi_1 \\ \boldsymbol{x}' & X & \boldsymbol{x}'' \\ \xi_1 & {}^t\boldsymbol{x}'' & x_2 \end{pmatrix} - \left(\|\boldsymbol{\xi}\|^2 - \xi_1^2\right) \det \left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}' \\ \hline \boldsymbol{x}' & X \end{array}\right)$$

$$\left({}^t\boldsymbol{\xi} = (\xi_1, \dots, \xi_{m+1})\right).$$

Note $\deg \Delta_{m+2} = m + 3$.

**Inner product in $V$**: For $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right)$, $y = \left(\begin{array}{c|cc} y_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{y}' & \boldsymbol{\eta} \\ \hline \boldsymbol{y}'\,{}^t\boldsymbol{e} & Y & \boldsymbol{y}'' \\ {}^t\boldsymbol{\eta} & {}^t\boldsymbol{y}'' & y_2 \end{array}\right)$

$$\langle\, x \,|\, y \,\rangle := x_1 y_1 + \mathrm{tr}(XY) + x_2 y_2 + 2(\boldsymbol{x}' \cdot \boldsymbol{y}' + \boldsymbol{x}'' \cdot \boldsymbol{y}'' + \boldsymbol{\xi} \cdot \boldsymbol{\eta})$$

$$\Omega^* := \left\{ y \in V \ ;\ \langle\, x \,|\, y \,\rangle > 0 \quad \text{for } \forall x \in \overline{\Omega} \setminus \{0\} \right\}$$

Define a linear operator $T_0$ on $V$ by $T_0 x = \left(\begin{array}{c|cc} x_2 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}'' J & \boldsymbol{\xi} \\ \hline J\boldsymbol{x}''\,{}^t\boldsymbol{e} & JXJ & J\boldsymbol{x}' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}' J & x_1 \end{array}\right)$ $(x \in V)$,

where $J \in \mathrm{Sym}(m, \mathbb{R})$ is given by $J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$.

| |
|---|
| **Theorem** [Ishi-N. 2009]. $\Omega^* = T_0(\Omega)$. |

**Conjecture**. $\Omega$ with $\operatorname{rank}\Omega = r$ is selfdual $\iff$ the degrees of basic relative invariants associated to $\Omega$ and $\Omega^*$ are both $1, 2, \ldots, r$.