Symmetry Characterization Theorems for Homogeneous Convex Cones and Siegel Domains

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Interpretation via Cayley transform

- Canonical bounded model for a symmetric Siegel domain
  $\cdots\cdots$ Harish-Chandra model of a non-compact Hermitian symmetric space
  (Open unit ball of a positive Hermitian JTS w.r.t the spectral norm)

- Canonical bounded model for a quasisymmetric Siegel domain
  $\cdots\cdots$ by Dorfmeister (1980)
  The image of the Siegel domain under the Cayley transform naturally defined in terms of the relevant Jordan algebra structure.
  (but non-convex unless it is symmetric. By C. Kai (2006))

- For general homogeneous Siegel domains

We can consider

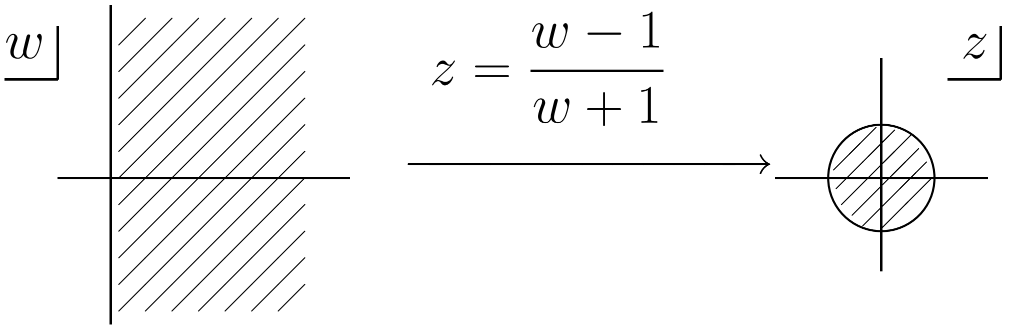
- Cayley transform associated to the characteristic function of the cone (due to **R. Penney**, 1996)
- Cayley transform associated to the Bergman kernel (**N**, 2001)
- Cayley transform associated to the Szegö kernel

etc. . .

More generally, we can define Cayley transforms associated to the admissible linear forms $E_{\mathbf{s}}^*$ $(\mathbf{s} > 0)$ (**N**, 2003).

Cayley transforms:

(1)



$$z = \frac{w-1}{w+1}$$

(2) $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Sym}^{++}(r, \mathbb{R})$, $\Omega + iV$

$$z = (w - E)(w + E)^{-1} \quad (E : \text{unit matrix})$$

Siegel right half space $\ni w \mapsto z \in$ Siegel disk

(3) General symmetric tube domain

$\Omega + iV$ (Ω: a selfdual open convex cone in V)

V (hence $V_{\mathbb{C}}$) can be equipped with Jordan algebra structure.

$$z = (w - e)(w + e)^{-1} \quad (e\text{: the unit element in } V)$$

- For $\mathrm{Sym}(r, \mathbb{C})$, Jordan algebra product is:

$$\begin{cases} A \circ B = \frac{1}{2}(AB + BA), \\ \text{Jordan algebra inverse} = \text{inverse matrix.} \end{cases}$$

Thus $(w - e) \circ (w + e)^{-1} = (w - e)(w + e)^{-1}$.

Symmetric tube domain $\longrightarrow$ Open unit ball in $V_{\mathbb{C}}$ (w.r.t some norm)

(4) $D := \{(u,w) \in \mathbb{C}^m \times \mathbb{C}\ ;\ w + \overline{w} - \frac{1}{2}\|u\|^2 > 0\}$.
(rank 1 (symmetric) Siegel domain)

$$\mathcal{C}(u,w) := \left(\frac{u}{w+1}, \frac{w-1}{w+1}\right)$$

$$\boxed{\mathcal{C} : D \longrightarrow \text{open unit ball in } \mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}}$$

(5) For general Siegel domain, one first needs something like $(w+e)^{-1}$. Then

$$(w-e)(w+e)^{-1} = e - 2(w+e)^{-1}.$$

Recall the formula

$$-\frac{d}{dt}\log\det(x+tv)^{-1}\Big|_{t=0} = \operatorname{tr}(x^{-1}v) \quad (x \in \mathrm{Sym}^{++}(r,\mathbb{R}),\ v \in \mathrm{Sym}(r,\mathbb{R}))$$

- $\operatorname{tr}(xy)$ is the inner product to identify $\mathrm{Sym}(r,\mathbb{R})$ with the dual vector space $\mathrm{Sym}(r,\mathbb{R})^*$ so that the cone $\mathrm{Sym}^{++}(r,\mathbb{R})$ coincides with its dual cone.

Compound power functions (after Gindikin)

$\exists H \subset G$ s.t. $H \curvearrowright \Omega$ linearly and simply transitively

$E \in \Omega$ (canonically fixed base point)

Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

- We have $G = N \rtimes A,\ H = N_0 \rtimes A$ with $A := \exp \mathfrak{a}$.

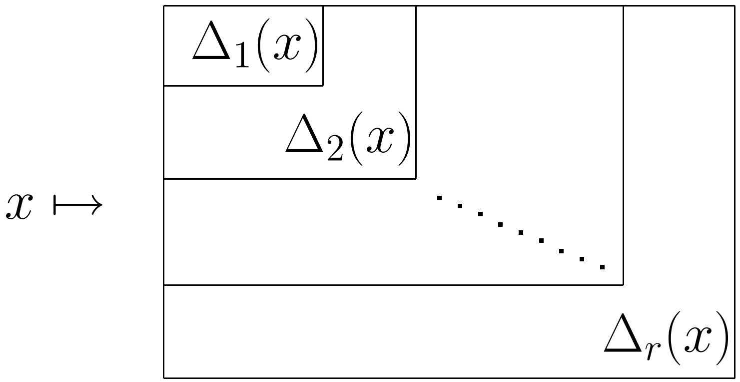
For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we put $\alpha_{\mathbf{s}} := \sum_{j=1}^{r} s_j \alpha_j \in \mathfrak{a}^*$

($\alpha_1, \ldots, \alpha_r$: the basis of $\mathfrak{a}^*$ taken to describe roots of $\mathfrak{g}$)

$\chi_{\mathbf{s}}(\exp x) := \exp\langle x, \alpha_{\mathbf{s}}\rangle$ $(x \in \mathfrak{a})$: 1-dim. representation of A, hence of H

$\rightsquigarrow$ function $\Delta_{\mathbf{s}}$ on Ω by $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$.

Example: If $\Omega = \mathrm{Sym}^{++}(r, \mathbb{R})$, then $\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$, where $\Delta_1(x), \ldots, \Delta_r(x)$: principal minors of x:



$$\Delta_1(x) := x_{11}$$

$$\Delta_2(x) := x_{11}x_{22} - x_{21}^2$$

$$\vdots$$

$$\Delta_r(x) := \det x$$

[end of Example]

Fact (**Gindikin, Ishi (2000)**): $\Delta_{\mathbf{s}}$ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^*, where

$$\Omega^* := \{\xi \in V^* \,;\, \langle x, \xi \rangle > 0 \ \ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

Pseudoinverse map associated to $\boldsymbol{E}_{\mathbf{s}}^*$

For each $x \in \Omega$, define $\mathcal{I}_{\mathbf{s}}(x) \in V^*$ by $\left(D_v f(x) := \frac{d}{dt} f(x+tv)\big|_{t=0}\right)$

$$\langle v, \mathcal{I}_{\mathbf{s}}(x) \rangle := -D_v \log \Delta_{-\mathbf{s}}(x) \quad (v \in V).$$

- $\mathcal{I}_{\mathbf{s}}(\lambda x) = \lambda^{-1} \mathcal{I}_{\mathbf{s}}(x) \quad (\lambda > 0)$.

Proposition. Suppose $E_{\mathbf{s}}^*$ is admissible (i.e., $\mathbf{s} > 0$).

(1) $\mathcal{I}_{\mathbf{s}}(x) \in \Omega^*$ and $\mathcal{I}_{\mathbf{s}} : \Omega \twoheadrightarrow \Omega^*$ is bijective.

(2) $\mathcal{I}_{\mathbf{s}}$ extends analytically to a rational map $W \to W^*$.

(3) One also has an explicit formula for $\mathcal{I}_{\mathbf{s}}^{-1} : \Omega^* \twoheadrightarrow \Omega$, which continues analytically to a rational map $W^* \to W$. Thus $\mathcal{I}_{\mathbf{s}}$ is birational.

(4) $\mathcal{I}_{\mathbf{s}} : \Omega + iV \twoheadrightarrow \mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is biholomorphic.

Remark. Bergman kernel and Szegö kernel are of the form (up to positive const.) $\eta(z_1, z_2) = \Delta_{-\mathbf{s}}\big(w_1 + w_2^* - Q(u_1, u_2)\big) \quad (z_j = (u_j, w_j))$, and the characteristic function of Ω is $\Delta_{-\mathbf{s}}$ for some $\mathbf{s} > 0$ (up to positive const.).

Theorem [Kai-N, 2005]. $\mathcal{I}_{\mathbf{s}}(\Omega + iV) = \Omega^* + iV^*$
$\iff$ $s_1 = \cdots = s_r$ and Ω is selfdual.

Given a homogeneous convex cone Ω in V, we define a partial order $x \succeq_\Omega y$ in V by the condition $x - y \in \overline{\Omega}$.

Theorem [Kai, 2008].
$\mathcal{I}_{\mathbf{s}}$ is order-reversing (i.e., $x \succeq_\Omega y$ implies $\mathcal{I}_{\mathbf{s}}(x) \preceq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(y)$)
$\iff$ Ω is selfdual and $s_1 = s_2 = \cdots = s_r$.

Cayley transform

One has $E_{\mathbf{s}}^* = \mathcal{I}_{\mathbf{s}}(E) \in \Omega^*$.

$C_{\mathbf{s}}(w) := E_{\mathbf{s}}^* - 2\,\mathcal{I}_{\mathbf{s}}(w+E)$ for tube domains $\left(1 - \frac{2}{w+1} = \frac{w-1}{w+1}\right)$

$$\mathcal{C}_{\mathbf{s}}(u,w) := \underbrace{2\,\langle Q(u,\cdot),\, \mathcal{I}_{\mathbf{s}}(w+E)\rangle}_{\in U^\dagger} \oplus \underbrace{C_{\mathbf{s}}(w)}_{\in W^*},$$

where $U^\dagger$ stands for the space of antilinear forms on U.

Proposition.
(1) $\mathcal{C}_{\mathbf{s}} : D \twoheadrightarrow \mathcal{C}_{\mathbf{s}}(D)$ is birational and biholomorphic.
(2) $\mathcal{C}_{\mathbf{s}}^{-1}$ can be written explicitly.

Theorem [N, 2003]. $\mathcal{C}_{\mathbf{s}}(D)$ is bounded (in $U^{\dagger} \oplus W^{*}$).

Remark. For general $\mathbf{s} > 0$, the image $\mathcal{C}_{\mathbf{s}}(D)$ even for symmetric D is *not* the standard Harish-Chandra model of a non-compact Hermitian symmetric space.
In fact

Theorem [Kai, 2007].
$\mathcal{C}_{\mathbf{s}}(D)$ is convex $\iff$ D is symmetric and $s_1 = s_2 = \cdots = s_r$.

Berezin transform on $L^2(G)$

written as a right convolution operator by $a_\lambda \in L^1(G)$, where $a_\lambda(g) := A_\lambda(g \cdot \mathsf{e}, \mathsf{e})$ and

$$B_\lambda f(x) = \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a(x) \quad (dy\text{: left Haar measure}).$$

- Laplace-Beltrami operator $\mathcal{L}_{\mathbf{s}}$ on G is left invariant differential operator ($\rightsquigarrow$ expressed by an element of $U(\mathfrak{g})$, $\mathfrak{g} := \mathrm{Lie}(G)$).

- $\langle x \,|\, y \rangle_{\mathbf{s}} := \langle [Jx, y], E^*_{\mathbf{s}} \rangle$: J-invariant inner product on $\mathfrak{g}$ defined by $E^*_{\mathbf{s}}$
  $\rightsquigarrow$ Upon $G \equiv D$ by $g \mapsto g \cdot \mathsf{e}$, we have Hermitian inner prod. on $T_{\mathsf{e}}(D) \equiv U \oplus W$
  $\rightsquigarrow$ Hermitian inner product $(\cdot \,|\, \cdot)_{\mathbf{s}}$ and norm $\| \cdot \|_{\mathbf{s}}$ on the *dual* vector space $U^\dagger \oplus W^*$.

- Take $\Psi_{\mathbf{s}} \in \mathfrak{g}$ so that $\mathrm{tr}\, \mathrm{ad}(x) = \langle x \,|\, \Psi_{\mathbf{s}} \rangle_{\mathbf{s}}$ $(\forall x \in \mathfrak{g})$. We know $\Psi_{\mathbf{s}} \in \mathfrak{a}$.

- Recall that $\beta|_{\mathfrak{n}} = E^*_{\mathbf{c}}|_{\mathfrak{n}}$ for some $\mathbf{c} > 0$, so that $\Delta_{-\mathbf{c}}(w_1 + w_2^* - Q(u_1, u_2))$ is the Bergman kernel of D (up to positive const.).

Key Formula. For any $g \in G$

$$\mathcal{L}_{\mathbf{s}} a_\lambda(g) = \lambda a_\lambda(g)\big(-\lambda\|\mathcal{C}_{\mathbf{c}}(g \cdot \mathsf{e})\|_{\mathbf{s}}^2 + \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{c}}\rangle\big).$$

Observations. (1) $a_\lambda(g) = a_\lambda(g^{-1})$ for $\forall g \in G$.

(2) B_λ commutes with $\mathcal{L}_{\mathbf{s}} \iff \mathcal{L}_{\mathbf{s}} a_\lambda(g) = \mathcal{L}_{\mathbf{s}} a_\lambda(g^{-1})$ for $\forall g \in G$.

Therefore B_λ commutes with $\mathcal{L}_{\mathbf{s}} \iff \|\mathcal{C}_{\mathbf{c}}(g \cdot \mathsf{e})\|_{\mathbf{s}} = \|\mathcal{C}_{\mathbf{c}}(g^{-1} \cdot \mathsf{e})\|_{\mathbf{s}} \quad (\forall g \in G)$.

Theorem [N, 2001]. $\|\mathcal{C}_{\mathbf{c}}(g \cdot \mathsf{e})\|_{\mathbf{s}} = \|\mathcal{C}_{\mathbf{c}}(g^{-1} \cdot \mathsf{e})\|_{\mathbf{s}}$ for $\forall g \in G$
$\iff D$ is symmetric and $\mathbf{s} = \gamma \mathbf{c}$ with $\gamma > 0$.

Since $\mathcal{C}_{\mathbf{c}}(\mathsf{e}) = 0$, this can be rephrased as:

Theorem. $\|h \cdot 0\|_{\mathbf{s}} = \|h^{-1} \cdot 0\|_{\mathbf{s}}$ for $\forall h \in \mathcal{C}_{\mathbf{c}} \circ G \circ \mathcal{C}_{\mathbf{c}}^{-1}$
$\iff \mathcal{D} := \mathcal{C}_{\mathbf{c}}(D)$ is symmetric and $\mathbf{s} = \gamma \mathbf{c}$ with $\gamma > 0$.

If D is symmetric, $\mathcal{D}$ is essentially the Harish-Chandra model of a non-compact Hermitian symmetric space.

$\mathsf{G} := \mathrm{Hol}(\mathcal{D})^\circ$: semisimple Lie group,
$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal compact subgroup of G.

Using $\mathsf{G} = \mathsf{KAK}$ with $\mathsf{A} := \mathcal{C}_{\mathbf{c}} \circ A \circ \mathcal{C}_{\mathbf{c}}^{-1}$, one can prove easily that $\|h \cdot 0\|_{\mathbf{c}} = \|h^{-1} \cdot 0\|_{\mathbf{c}}$ for any $h \in \mathsf{G}$.

- For the unit disk in $\mathbb{C}$, we have the following picture (next sheet)

The case of unit disk $\mathbb{D} \subset \mathbb{C}$

$SU(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \; ; \; |\alpha|^2 - |\beta|^2 = 1 \right\}$ acts on $\mathbb{D}$ by

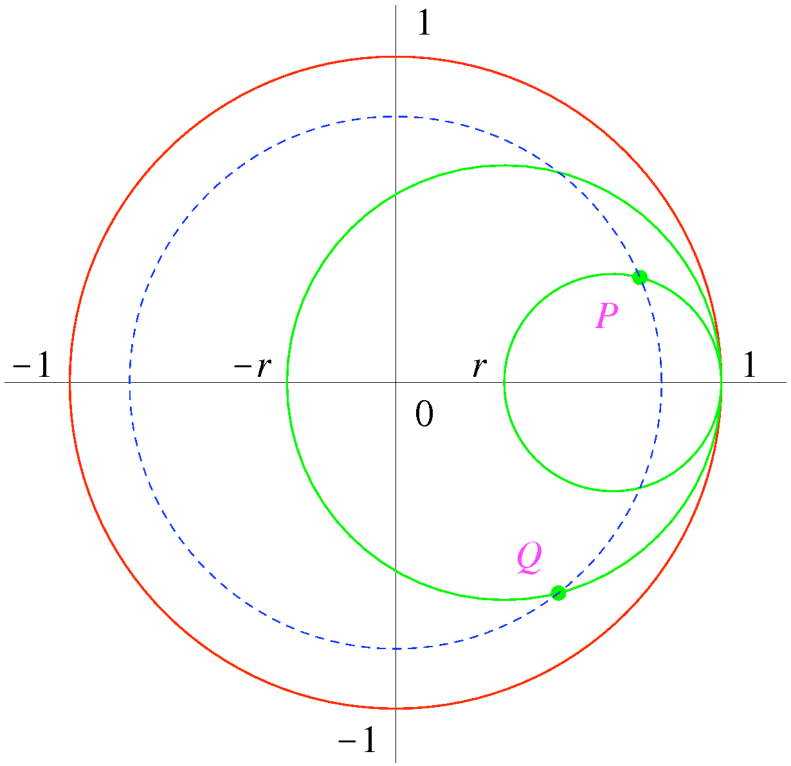
$$g \cdot z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad (z \in \mathbb{D}).$$

$$\begin{cases} g \cdot 0 = \dfrac{\beta}{\overline{\alpha}} \\ g^{-1} \cdot 0 = -\dfrac{\beta}{\alpha} \end{cases} \Longrightarrow |g \cdot 0| = |g^{-1} \cdot 0|.$$

However, if one stays within the Iwasawa subgroup, we have an interesting picture.

$$\mathsf{A} := \left\{ a_t := \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \; ; \; t \in \mathbb{R} \right\}, \mathsf{N} := \left\{ n_\xi := \begin{pmatrix} 1 - \frac{i}{2}\xi & \frac{i}{2}\xi \\ -\frac{i}{2}\xi & 1 + \frac{i}{2}\xi \end{pmatrix} \; ; \; \xi \in \mathbb{R} \right\}.$$

Then $\mathcal{C} \circ G \circ \mathcal{C}^{-1} = \mathsf{NA}$.



$r := a_t \cdot 0 = \tanh(t/2)$

P : $n_\xi a_t \cdot 0 = n_\xi \cdot r$
$\in \mathsf{N} \cdot r$: horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at r.

Q : $(n_\xi a_t)^{-1} \cdot 0 = n_{-e^{-t}\xi} a_{-t} \cdot 0 = n_{-e^{-t}\xi} \cdot (-r)$
$\in \mathsf{N} \cdot (-r)$: horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at $-r$.

Poisson kernel

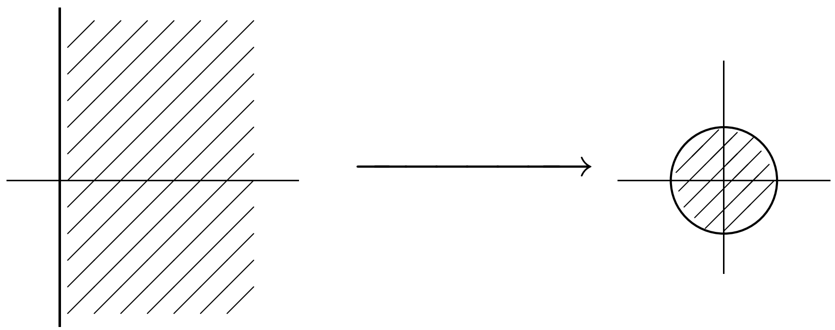
• Take $\mathbf{b} > 0$ so that $\Delta_{-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2))$ is the Szegö kernel of D (up to positive const.).

Key Formula. $\mathcal{L}_{\mathbf{s}} P_\zeta^G = (-\|\mathcal{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 + \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle) P_\zeta^G.$

Therefore, $\mathcal{L}_{\mathbf{s}} P_\zeta^G = 0 \ (\forall \zeta \in \Sigma) \iff \|\mathcal{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle \ (\forall \zeta \in \Sigma).$

Theorem [N, 2003]. $\|\mathcal{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle$ for $\forall \zeta \in \Sigma$
$\iff$ D is symmetric and $\mathbf{s} = \gamma \mathbf{b}$ with $\gamma > 0$.
In this case we also have $\mathbf{s} = \gamma' \mathbf{c}$ with $\gamma' > 0$.

Recall $\mathbf{c} > 0$ is taken so that $\beta|_{\mathfrak{n}} = E_{\mathbf{c}}^*|_{\mathfrak{n}}$, where β is the Koszul form.



D : symmetric $\Longrightarrow$

$\mathcal{D} := \mathcal{C}_{\mathbf{c}}(D)$ is the Harish-Chandra model of a Hermitian symmetric space

In particular, $\mathcal{D}$ is circular (Note $\mathcal{C}_{\mathbf{c}}(\mathsf{e}) = 0$).

$\mathsf{G} := \mathrm{Hol}(\mathcal{D})^{\circ}$: semisimple Lie group (with trivial center)

$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal compact subgroup of G

- Circularity of $\mathcal{D}$ ($\Longrightarrow$ K is linear) + K-invariance of the Bergman metric

$\Longrightarrow$ K $\subset$ Unitary group

$$\begin{cases} \mathcal{C}_{\mathbf{c}} : \Sigma \ni 0 \mapsto -E_{\mathbf{c}}^{*}, \\ \text{Shilov boundary } \Sigma_{\mathcal{D}} \text{ of } \mathcal{D} = \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}). \end{cases}$$

Since $\Sigma_{\mathcal{D}}$ is also a G-orbit $\Sigma_{\mathcal{D}} = \mathsf{G} \cdot (-E_{\mathbf{c}}^{*})$, and since Σ is an orbit of a nilpotent subgroup of $G \subset \mathrm{Hol}(D)^{\circ}$, we get

$$\begin{aligned} \mathcal{C}_{\mathbf{c}}(\Sigma) \subset \mathsf{G} \cdot (-E_{\mathbf{c}}^{*}) &= \Sigma_{\mathcal{D}} = \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}) \\ &\subset \{z \,;\, \|z\|_{\mathbf{c}} = \|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}\}. \end{aligned}$$

We see easily that $\|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}^{2} = \langle \Psi_{\mathbf{c}}, \alpha_{\mathbf{b}} \rangle$ in this case (because $\mathbf{b}$ is a multiple of $\mathbf{c}$).