Symmetry Characterization Theorems

for

Homogeneous Convex Cones and Siegel Domains

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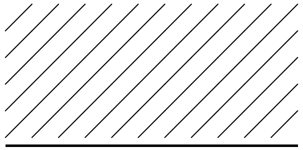
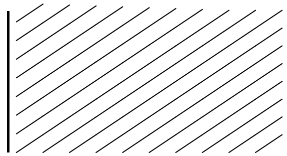
Siegel domains (Piatetski–Shapiro, 1957)

- generalization of  or  to higher dimensions
- holomorphically equivalent to bounded domains

Examples.

(1) $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Sym}^{++}(r, \mathbb{R})$, $\Omega + i\,V$: (Siegel right half-space)

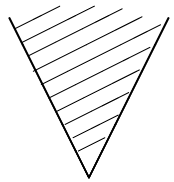
(2) $\{(u, w) \in \mathbb{C}^m \times \mathbb{C}\ ;\ w + \overline{w} - \frac{1}{2}\|u\|^2 > 0\}$

(holomorphically equivalent to the unit ball in $\mathbb{C}^{m+1}$)

In general:

$$D = \{(u, w) \in U \times V_{\mathbb{C}} \; ; \; w + \overline{w} - Q(u, u) \in \Omega\}.$$

U: (finite-dimensional) complex vector space,
V: (finite-dimensional) real vector space,
$\Omega \subset V$: open convex cone containing no entire line



$Q(u, v)$: $V_{\mathbb{C}}$-valued Hermitian form which is **Ω-positive**

($\overset{\text{def}}{\Longleftrightarrow}$ $Q(u, u) \in \overline{\Omega} \setminus \{0\}$ if $u \neq 0$)

- $U = \{0\}$ is allowed $\rightsquigarrow D = \Omega + i\,V$

Piatetski–Shapiro's motivation (1957)

- Application to automorphic functions
  — just needed a half-plane type realization of Hermitian symmetric space
- $\exists$ Hermitian symmetric spaces that cannot be realized as $\Omega + iV$

Earlier study

E. Cartan (1935): Any homogeneous bounded domain in $\mathbb{C}^2$ or $\mathbb{C}^3$ is symmetric.

- D is symmetric $\overset{\text{def}}{\Longleftrightarrow}$

  $\forall z \in D$, $\exists \sigma_z \in \mathrm{Hol}(D)$ with $\sigma_z^2 = \mathrm{Id}$ s.t. z is an isolated fixed point of σ_z.

Cartan left a question: **What happens in $\mathbb{C}^n$ for $n \geqq 4$?**

The most unexpected application

Discovery of many non-symmetric homogeneous bounded domains:

1959: P.-S.'s examples of non-symmetric homogeneous Siegel (hence bounded) domains in $\mathbb{C}^4$, $\mathbb{C}^5$.

Later: In $\mathbb{C}^n$ $(n \geqq 7)$, $\exists$mutually inequivalent non-symmetric Siegel domains with continuous parameter

For non-symmetric $\Omega + i\,V$, one needed non-selfdual Ω.

Vinberg (1963): Theory of homogeneous open convex cones
non-selfdual Ω with minimum dimension $=$ **5** (1960)

Natural Question. How do we characterize symmetric Siegel domains (among homogeneous Siegel domains)?

Symmetry characterization theorems

- **Before P.-S.'s example**

  A. Borel (1954), L. Koszul (1955):
  A bounded domain is symmetric if it is a homogeneous space
  of a semisimple Lie group: weakened to "*unimodular*" by J. Hano (1957)
  (∃ left and right invariant Haar measure)

- **In terms of defining data of Siegel domains**

  I. Satake (book, 1980), **J. Dorfmeister** (Habilitationsschrift, 1978)

- **Geometric conditions** (curvature etc. . .)

  J. D'Atri and I. Dotti (1983), **K. Azukawa** (1985)

Siegel domains. — Definition —

V : a real vector space ($\dim V < \infty$)
$\cup$
Ω : a regular open convex cone ($\overset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)
$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)
U : another complex vector space ($\dim U < \infty$)
$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

$$\textit{i.e.,}\quad \begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \quad (0 \neq \forall u \in U) \end{cases}$$

Siegel domain (of type II)

$$D := \{(u, w) \in U \times W \; ; \; w + w^* - Q(u, u) \in \Omega\}$$

- $U = \{0\}$ is allowed. In this case $D = \Omega + iV$.

Assume that D is homogeneous, *i.e.*, $\mathrm{Hol}(D) \curvearrowright D$ transitively.

Then Ω is also homogeneous: $G(\Omega) := \{g \in GL(V) \; ; \; g\Omega = \Omega\} \curvearrowright \Omega$ transitively

D : a homogeneous Siegel domain, $\mathbf{G} := \mathrm{Hol}(D)^\circ$: identity component
Fix e $\in D$, $\mathbf{K} := \mathrm{Stab}_{\mathsf{e}}\mathbf{G}$.
Then $\mathbf{K} \curvearrowright T_{\mathsf{e}}(D)$ linearly (isotropy representation)

D'Atri–Dorfmeister–Zhao's work (1985)

The following (1)~(4) are equivalent:
(1) D is symmetric.
(2) Almost $\mathbb{C}$ structure on $T_{\mathsf{e}}(D)$ is represented by an operator of the infinitesimal isotropy representation.
(3) $\nexists$ non-trivial $\mathbf{G}$-invariant vector field.
(4) The algebra $\mathbf{D}(D)^{\mathbf{G}}$ of $\mathbf{G}$-invariant differential operators on D is commutative.

(2) is well-known for Hermitian symmetric spaces.
(4) is well-known for Riemannian symmetric spaces.
More is known: $\mathbf{D}(D)^{\mathbf{G}} \cong \mathbb{C}[t_1, \ldots, t_r]$ $(r := \mathrm{rank}(D))$.
For Hermitian symmetric spaces, generators are of even degrees $\rightsquigarrow$ (3).

What is interesting here is . . .

Well-known properties for symmetric spaces are already characteristic of symmetric domains among homogeneous Siegel domains.

$\mathcal{L}$: Laplace–Beltrami operator (w.r.t. a standard Kähler metric)

Theorem 1 [N, 2001]. $\mathcal{L}$ commutes with the Berezin transform
$\iff$ D is symmetric and the metric considered is the Bergman
(up to positive multiple).

Theorem 2 ([N, 2003]). The Poisson–Hua kernel is annihilated by $\mathcal{L}$
$\iff$ D is symmetric and the metric considered is the Bergman
(up to positive multiple).

In Theorem 2, if the metric is assumed to be Bergman from the beginning, then the theorem is due to

Hua–Look (1959), Korányi (1965) for $\Leftarrow$
Xu (1979) for $\Rightarrow$

homogeneous Siegel domains $\leftrightarrow$ normal j-algebras (Piatetski-Shapiro algebras)

D: homogeneous Siegel domain

Then $\exists G \subset \mathrm{Hol}_{\mathrm{Aff}}(D)$: split solvable $\curvearrowright D$ simply transitively.
$\mathfrak{g} := \mathrm{Lie}(G)$ has a structure of Piatetski-Shapiro algebra (normal j-algebra):

$$\begin{cases} \exists J\text{: integrable almost complex structure on } \mathfrak{g}, \\ \exists \omega\text{: admissible linear form on } \mathfrak{g}, \textit{ i.e., } \langle\, x \,|\, y \,\rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines} \\ \qquad \text{a } J\text{-invariant positive definite inner product on } \mathfrak{g}. \end{cases}$$

<u>Example</u> (Koszul '55) **Koszul form**

$$\langle x, \beta \rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

This β is admissible.

- In fact, $\langle\, x \,|\, y \,\rangle_\beta$ is the real part of the Hermitian inner product on $\mathfrak{g} \equiv T_{\mathsf{e}}(D)$ defined by the Bergman metric on $D \approx G$ (up to a positive scalar multiple).

Structure of $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$ ($\mathfrak{a}$: abelian, $\mathfrak{n}$: sum of $\mathfrak{a}$-root spaces (positive roots only))

- Always contains a product of $ax+b$ algebra:

$\exists H_1, \ldots, H_r$: a basis of $\mathfrak{a}$ $(r := \operatorname{rank} \mathfrak{g})$ s.t. $[H_j, E_k] = \delta_{jk} E_k$,
where $E_k := -JH_k \in \mathfrak{n}$,

Possible forms of roots: ($\alpha_1, \ldots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$)

$\frac{1}{2}(\alpha_k \pm \alpha_j)\ (j < k),\ \ \alpha_1, \ldots, \alpha_r,\ \ \frac{1}{2}\alpha_1, \ldots, \frac{1}{2}\alpha_r$

- $\mathfrak{g}_{\alpha_k} = \mathbb{R}E_k\ (k = 1, \ldots, r)$.
- $\mathfrak{g}_\alpha$ are mutually orthogonal w.r.t. $\langle \cdot \mid \cdot \rangle_\omega$ ($\forall\omega$: admissible)
- tube domains $\iff$ none of $\frac{1}{2}\alpha_k$ is present.

Let us define $E_k^* \in \mathfrak{g}^*$ by $\langle E_k, E_k^* \rangle = 1$ and $= 0$ on $\mathfrak{a}$ and $\mathfrak{g}_\alpha$ $(\alpha \neq \alpha_k)$.

- Admissible linear forms are $\mathfrak{a}^* \oplus \{0\} \oplus \sum_{k=1}^{r} \mathbb{R}_{>0} E_k^*$.

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we put $E^*_{\mathbf{s}} := \sum_{k=1}^{r} s_k E^*_k \in \mathfrak{g}^*$.

- If $s_1 > 0, \ldots, s_r > 0$ (we will write $\mathbf{s} > 0$), then $\langle\, x \,|\, y \,\rangle_{\mathbf{s}} := \langle [Jx, y], E^*_{\mathbf{s}} \rangle$ is a J-invariant inner product on $\mathfrak{g}$

$\rightsquigarrow$ left invariant Riemannian metric on G

$\rightsquigarrow$ $\mathcal{L}_{\mathbf{s}}$: the corresponding Laplace–Beltrami operator on G.

Berezin transforms

- Berezin transform is an important operator for Berezin quantization.
- If D is symmetric, then Helgason's (spherical) Fourier transformation theory gives an explicit spectral decomposition of the Berezin transform.
  (Berezin (1978), Unterberger–Upmeier (1994), Arazy–Zhang (1995), etc. . .)
- For general D, Arazy–Upmeier (2004) made analysis by using *non-unimodular Plancherel theory* for simply transitive split solvable Lie group. However, its relation to the *ordinary* spectral decomposition is not so clear (in particular for symmetric cases. . .).

κ: the Bergman kernel of D (reproducing kernel of $L^2(D) \cap \mathcal{O}(D)$)

the Berezin kernels

$$A_\lambda(z_1, z_2) := \left(\frac{|\kappa(z_1, z_2)|^2}{\kappa(z_1, z_1)\kappa(z_2, z_2)} \right)^\lambda \quad (z_j \in D;\ \lambda \in \mathbb{R})$$

- A_λ is G-invariant: $A_\lambda(g \cdot z_1,\ g \cdot z_2) = A_\lambda(z_1, z_2)$.

Since $D \approx G$, we work on G:

$$a_\lambda(g) := A_\lambda(g \cdot \mathsf{e},\ \mathsf{e}) \quad (g \in G,\ \mathsf{e} \in D : \text{fixed reference point})$$

- $a_\lambda \in L^1(G)$ if $\lambda > \lambda_0$ ($0 < \lambda_0 < 1$: explicitly calculated).
  (non-vanishing condition for Hilbert spaces of holomorphic functions on D, in which κ^λ is the reproducing kernel.)

Berezin transform on G

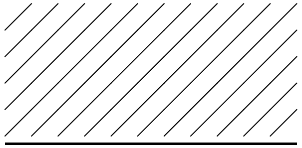
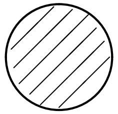
$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x)$$

$B_\lambda \in \mathbf{B}(L^2(G))$: selfadjoint, positive.

Recall the Koszul form $\beta \in \mathfrak{g}$. We have $\beta|_{\mathfrak{n}} = E^*_{\mathbf{c}}|_{\mathfrak{n}}$ for some $\mathbf{c} > 0$.

Theorem 1. $\lambda > \lambda_0$: fixed.
B_λ commutes with $\mathcal{L}_{\mathbf{s}} \iff D$ is symmetric and $\mathbf{s} = \gamma \mathbf{c}$ for some $\gamma > 0$.

Poisson–Hua kernel

In  or  the Szegö kernel S and the Poisson kernel P are related by

$$(*) \qquad P(z,\zeta) = \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in \text{domain}, \ \zeta \in \text{boundary}).$$

In a general Siegel domain D, we still have the Szegö kernel. Then Hua defined a Poisson kernel by $(*)$, where boundary $=$ Shilov boudary Σ:

$$\Sigma = \{(u,w)\ ;\ w + w^* - Q(u,u) = 0\}.$$

Poisson–Hua kernel

$S(z_1, z_2)$: the Szegö kernel of D ($=$ the reproduding kernel of the Hardy space)

- **Hardy space**

  Hilbert space of holomorphic functions F on D s.t.

$$\sup_{t\in\Omega}\int_U dm(u)\int_V \left|F\left(u, t+\tfrac{1}{2}Q(u,u)+ix\right)\right|^2 dx < \infty$$

- We know $S(z_1, z_2) = \eta(w_1 + w_2 - Q(u_1, u_2))$ $(z_j = (u_j, w_j) \in D)$ for some holomorphic function η on $\Omega + iV$.
- $S(z, \zeta)$ for $z \in D$ and $\zeta \in \Sigma$ still has a meaning. Then define a Poisson kernel by the formula

$$P(z,\zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in D,\ \zeta \in \Sigma).$$

We transfer it to G: $P^G_\zeta(g) := P(g\cdot \mathsf{e},\ \zeta)$ $(g \in G)$.

Theorem 2. $\mathcal{L}_{\mathbf{s}} P_{\zeta}^{G} = 0$ for $\forall \zeta \in \Sigma$
$\iff$ D is symmetric and $\mathbf{s} = \gamma \mathbf{c}$ for some $\gamma > 0$.