Analysis and Geometry Related to Homogeneous Siegel Domains and Homogeneous Convex Cones

Part II

Basic Relative Invariants Associated to Homogeneous Convex Cones

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September 28, 2010

Homogeneous Open Convex Cones

V **: a real vector space (with inner product, for simplpicity)**
$V \supset \Omega$ **: a regular open convex cone**

- $G(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}$**: linear automorphism group of** Ω
  linear Lie group as a closed subgroup of $GL(V)$
- Ω **is homogeneous** $\overset{\text{def}}{\iff}$ $G(\Omega) \curvearrowright \Omega$ **is transitive**

Example: $V = \mathrm{Sym}(r, \mathbb{R}) \supset \Omega := \mathrm{Sym}(r, \mathbb{R})^{++}$**:**
$GL(r, \mathbb{R}) \curvearrowright \Omega$ **by** $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx\,{}^t g \in \Omega$

This is a selfdual homogeneous open convex cone (symmetric cone).
Ω **is selfdual** $\overset{\text{def}}{\iff}$ $\exists \langle \cdot | \cdot \rangle$ **s.t.** $\Omega = \left\{ y \in V\ ;\ \langle\, x \,|\, y \,\rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \right\}$
(the RHS is the dual cone taken relative to $\langle \cdot | \cdot \rangle$**)**

Symmetric Cones $\rightleftarrows$ Euclidean Jordan Algebras

$\Omega \rightleftarrows V$: **alg. str. in the ambient VS ($\equiv$ tangent space at a ref. pt.)**

- V **with a bilinear product** xy **is called a Jordan algebra if**
  (1) $xy = yx$,
  (2) $x^2(xy) = x(x^2y)$.
- **A real Jordan algebra is said to be Euclidean if** $\exists \langle \cdot \,|\, \cdot \rangle$ **s.t.**

$$\langle\, xy \,|\, z \,\rangle = \langle\, x \,|\, yz \,\rangle \qquad (\forall x, y)$$

List of Irreducible Symmetric Cones:

$\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$, $\quad A \circ B := \frac{1}{2}(AB + BA)$

$\Omega = \mathrm{Herm}(r, \mathbb{C})^{++} \subset V = \mathrm{Herm}(r, \mathbb{C})$

$\Omega = \mathrm{Herm}(r, \mathbb{H})^{++} \subset V = \mathrm{Herm}(r, \mathbb{H})$

$\Omega = \mathrm{Herm}(3, \mathbb{O})^{++} \subset V = \mathrm{Herm}(3, \mathbb{O})$

$\Omega = \Lambda_n \subset V = \mathbb{R}^n$ **(n-dimensional Lorentz cone)**

Non-Selfdual Homogeneous Open Convex Cones:

Vinberg cone (1960)

$$V = \left\{ x = \begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & 0 \\ x_4 & 0 & x_5 \end{pmatrix} \ ; \ x_1, \ldots, x_5 \in \mathbb{R} \right\} \supset \Omega := \left\{ x \in V \ ; \ \begin{matrix} x_1 > 0 \\ x_1 x_3 - x_2^2 > 0 \\ x_1 x_5 - x_4^2 > 0 \end{matrix} \right\}$$

the lowest-dimensional homogeneous non-selfdual open convex cone

Classification of Irred. Homogeneous Convex Cones (dim ≤ 10)

(Kaneyuki–Tsuji, 1974)

There are 135 (up to linear isom.) in which 12 are selfdual.

$\mathbb{R}_{>0}$, $\Lambda_n \subset \mathbb{R}^n$ **(Lorentz cones with** $\dim = 3, 4, \ldots, 10$**),**
$\mathrm{Sym}(3, \mathbb{R})^{++}$ **(**6-dim**),** $\mathrm{Herm}(3, \mathbb{C})^{++}$ **(**9-dim**),** $\mathrm{Sym}(4, \mathbb{R})^{++}$ **(**10-dim**)**

By Vinberg's theory (1963)

Homogeneous Open Convex Cones $\rightleftarrows$ **Clans with unit element**

$\Omega \rightleftarrows V$**: alg. str. in the ambient VS (**$\equiv$ **tangent space at a ref. pt.)**

- V **with a bilinear product** $x\Delta y = L(x)y = R(y)x$ **is called a Clan if**

(1) $[L(x), L(y)] = L(x\Delta y - y\Delta x)$**,**

(2) $\exists s \in V^*$ **s.t.** $\langle x\Delta y, s\rangle$ **defines an inner product,**

(3) Each $L(x)$ **has only real eigenvalues.**

(1) The Case of Symmetric Cones:

- $G(\Omega)$ **is reductive.**
- **Jordan algebra structure of** V**:**

  $V \equiv T_e(\Omega) \equiv$ "$\mathfrak{p}$ **of the Cartan decomposition** $\mathfrak{g}(\Omega) = \mathfrak{k} + \mathfrak{p}$"

  Indeed $\mathfrak{p} = \{M(x)\ ;\ x \in V\}$**, the space of Jordan multiplication operators.**
- **The Jordan product is commutative.**

(2) The Case of General Homogeneous Convex Cones:

- **simply transitive action of Iwasawa subgroup of** $G(\Omega)$
- **Clan structure of** V**:**

  $V \equiv T_e(\Omega) \equiv$ "**Iwasawa subalgebra** $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$ **of** $\mathfrak{g}(\Omega)$"

  Indeed $\mathfrak{s} = \{L(x)\ ;\ x \in V\}$**, the space of left clan multiplication operators.**
- **The clan product is non-commutative, in general.**
- **Of course one can consider the clan structure in the case of symmetric cones.**

Ω: homogeneous open convex cone,
$G(\Omega)$: linear automorphism group of Ω,
S: Iwasawa subgroup of $G(\Omega)$.
S is a split solvable Lie group, acting simply transitively on Ω.

a function f on Ω is relatively invariant (w.r.t. S)
$\overset{\text{def}}{\Longleftrightarrow}$ **$\exists\chi$: 1-dimensional representation of S s.t.**

$$f(gx) = \chi(g)f(x) \quad \text{(for all } g \in S,\ x \in \Omega\text{)}.$$

Theorem [Ishi 2001]. One can find irreducible relatively invariant polynomial functions

$$\Delta_1, \ldots, \Delta_r \qquad (r := \mathrm{rank}(\Omega))$$

on V s.t. any relatively invariant polynomial function $P(x)$ on V is written as

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \text{const.},\ (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

More intrinsic description of $\Delta_1, \ldots, \Delta_r$

Theorem [Ishi–N., 2008].
$W = V_{\mathbb{C}}$ **: the complexification of the Clan** V**,**
$R(w)$ **: the right multiplication operator by** w **in** W**:**

$$R(w)z = z \triangle w.$$

$\Longrightarrow$ **irreducible factors of** $\det R(w)$ **are just** $\Delta_1(w), \ldots, \Delta_r(w)$**.**

- $\Delta_1, \ldots, \Delta_r$**: Basic relative invariants associated to** Ω**.**
- **Just avoid the zeros of** $\Delta_1, \ldots, \Delta_r$

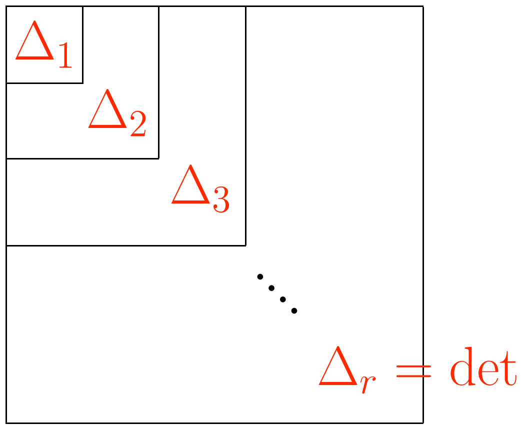
  $\Longrightarrow$ **we are safe for holomorphy for the pseudo-inverse map** $\mathcal{I}_{\mathbf{s}}(w) = E^*_{\mathbf{s}} \circ R(w)^{-1}$**.**

Example: $\Omega = \mathrm{Sym}(r,\mathbb{R})^{++} \subset V = \mathrm{Sym}(r,\mathbb{R})$.

$GL(r,\mathbb{R})$ **acts on** Ω **by** $\rho(g)x := gx\,{}^tg \quad (g \in GL(r,\mathbb{R}),\ x \in \Omega)$.

Let $GL(r,\mathbb{R}) \supset S := \left\{ \begin{pmatrix} a_1 & & & \mathbf{0} \\ & a_2 & & \\ & & \ddots & \\ * & & & a_r \end{pmatrix} ;\ a_1 > 0, \ldots, a_r > 0 \right\}$.

The basic relative S**-invariants are**



$$\begin{pmatrix} A & O \\ B & C \end{pmatrix} \begin{pmatrix} X & Y \\ {}^tY & Z \end{pmatrix} \begin{pmatrix} {}^tA & {}^tB \\ O & {}^tC \end{pmatrix} = \begin{pmatrix} AX\,{}^tA & * \\ * & * \end{pmatrix}$$

Product in $V = \mathrm{Sym}(r, \mathbb{R})$ as a clan: $x \Delta y = \underline{x}\, y + y\, {}^t(\underline{x})$,

where for $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$, we put

$$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & & & \\ x_{21} & \frac{1}{2}x_{22} & & 0 \\ \vdots & & \ddots & \\ x_{r1} & x_{r2} & & \frac{1}{2}x_{rr} \end{pmatrix} \in \mathfrak{s} := \mathrm{Lie}(S).$$

Thus $x = \underline{x} + {}^t(\underline{x})$.

In this case we have $\det R(y) = \Delta_1(y) \cdots \Delta_r(y)$.

Remark: This can be seen without computation, once one accepts the previous theorem.

- **In fact,** $\deg \det R(y) = V = \frac{1}{2} \cdot r(r+1)$.
- **On the other hand,** $\deg\big(\Delta_1(y) \cdots \Delta_r(y)\big) = 1 + \cdots + r = \frac{1}{2} \cdot r(r+1)$.

The case of general irreducible symmetric cone $\Omega \subset V$

- $\Delta_k(y)$ is the k-th Jordan algebra principal minor of $y \in V$.
- Iwasawa subgroup $S \subset G(\Omega)$ acts on Ω simply transitively
  $\rightsquigarrow$ The orbit map $S \ni g \mapsto ge \in \Omega$ is a diffeomorphism
  ($e \in \Omega$ is the unit element of V)
  The differential $\mathfrak{s} \ni X \mapsto Xe \in V$ is a linear isomorphism.
- The inverse map is denoted as $V \ni v \mapsto X_v \in \mathfrak{s}$.
  $\rightsquigarrow$ Then $X_v e = v$ for any $v \in V$.

- Euclidean Jordan algebra V now has a structure of clan by

$$x \triangle y := X_x y = R(y)x.$$

Proposition [N, Preprint]: $\det R(y) = \Delta_1(y)^d \cdots \Delta_{r-1}(y)^d \Delta_r(y)$,
where $d = 1$ for $\mathrm{Sym}(r, \mathbb{R})$, $d = \dim_{\mathbb{R}} \mathbb{K}$ for $\mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$,
$r = 2$, $d = n - 2$ for $\Omega = \Lambda_n$ $(n \geqq 3)$.

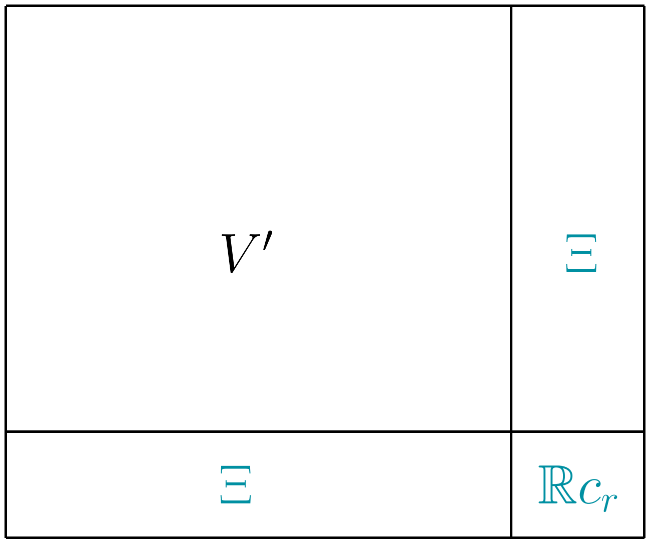
- The formula is nice in view of $\dim V = r + \frac{d}{2} \cdot r(r-1)$, because

$$\deg\bigl(\Delta_1(y)^d \cdots \Delta_{r-1}(y)^d \Delta_r(y)\bigr) = d\bigl(1 + \cdots + (r-1)\bigr) + r = r + \frac{d}{2} \cdot r(r-1).$$

Inductive structure of right multiplications

- **Just computing** $\det R(y)$ **is not so difficult.**
  One can see the corresponding one-dimensional representation of S **without much difficulty.**

$V =$



$$\mathrm{rank}(V) = r, \ \mathrm{rank}(V') = r - 1$$

$$V = V' \oplus \Xi \oplus \mathbb{R}c_r$$

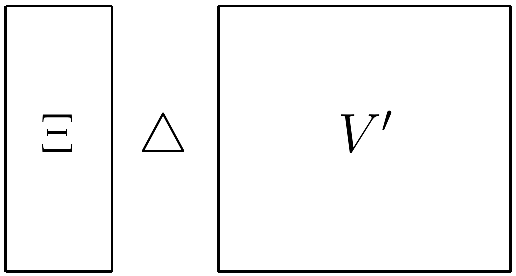
$$W := \Xi \oplus \mathbb{R}c_r$$

- W **is a 2-sided ideal in the clan structure of** V**.**

According to the decomposition $V = V' \oplus \underbrace{\Xi \oplus \mathbb{R}c_r}_{W}$, we have

$$R_v = \left(\begin{array}{c|c|c} R'(v') & \multicolumn{2}{c}{O} \\ \hline \ast & \phi(v') & \langle \cdot \,|\, c_r \rangle_W \, \xi \\ \cline{2-3} & \langle \cdot \,|\, \xi \rangle_W \, c_r & v_r I_{V_{rr}} \end{array}\right) \qquad (v = v' + \xi + v_r c_r)$$

$\langle \cdot \,|\, \cdot \rangle$ **: renormalized inner product in W,**
$\phi(v')\xi_0 := R(v')\xi_0$ **(by definition)**



Proposition: $R(V')\Xi \subset \Xi$, **and** $v' \mapsto \phi(v') := R(v')\big|_{\Xi} \in \mathrm{End}(\Xi)$ **is a Jordan algebra representation of** V'**:**

$$\phi(v_1' v_2') = \frac{1}{2}\left(\phi(v_1')\phi(v_2') + \phi(v_2')\phi(v_1')\right)$$

Question: Ω**: irred. homogeneous open convex cone of rank** r**.**
Then Ω **is selfdual**
$\iff$ **the degrees of basic relative invariants are** $1, 2, \ldots, r$**.**

[$\Rightarrow$] **Jordan algebra principal minors are of degree** $1, 2, \ldots, r$**.**
[$\Leftarrow$] **False in any rank** ≥ 3**:**
$\exists\Omega$**: non-selfdual homogeneous open convex cone**
s.t. the associated basic relative invariants are of
degree $1, 2, \ldots, r$**.**

Example

I_m: $m \times m$ **unit matrix,** $\mathbb{R}^{rm}$: **column vectors of size** $r \times m$**.**

$$V := \left\{ x = \begin{pmatrix} x_0 \otimes I_m & \boldsymbol{y} \\ {}^t\boldsymbol{y} & z \end{pmatrix} \;;\; x_0 \in \mathrm{Sym}(r, \mathbb{R}),\ \boldsymbol{y} \in \mathbb{R}^{rm},\ z \in \mathbb{R} \right\}.$$

Note $V \subset \mathrm{Sym}(rm+1, \mathbb{R})$**.**

When $m = r = 2$**,** x **is the following** 5×5 **matrix:**

$$x = \begin{pmatrix} x_{11} & 0 & x_{21} & 0 & y_{11} \\ 0 & x_{11} & 0 & x_{21} & y_{12} \\ x_{21} & 0 & x_{22} & 0 & y_{21} \\ 0 & x_{21} & 0 & x_{22} & y_{22} \\ y_{11} & y_{12} & y_{21} & y_{22} & z \end{pmatrix} \qquad (\textbf{in this case } \dim V = 8).$$

- **For** Ω**, we take the set of positive definite ones in** V**:**

$$\Omega := \{x \in V \;;\; x \gg 0\} \qquad (\mathrm{rank}(\Omega) = r + 1).$$

Assumption: $m \geqq 2$**,** $r \geqq 2$ **:**

($m = 1 \Rightarrow \Omega = \mathrm{Sym}(r+1, \mathbb{R})^{++}$**,** $\quad r = 1 \Rightarrow \Omega = \Lambda_{m+2}$**)**

Homogeneity of Ω:

$$A := \left\{ a = \left(\begin{array}{c|c} a_0 \otimes I_m & 0 \\ \hline 0 & a_{r+1} \end{array} \right) \; ; \; \begin{array}{l} a_0 := \mathrm{diag}[a_1, \dots, a_r] \text{ with} \\ a_1 > 0, \dots, a_r > 0 \text{ and } a_{r+1} > 0 \end{array} \right\},$$

$$N := \left\{ n = \left(\begin{array}{c|c} n_0 \otimes I_m & 0 \\ \hline {}^t\boldsymbol{\xi} & 1 \end{array} \right) \; ; \; \begin{array}{l} n_0 \text{ is strictly lower triangular in } GL(r, \mathbb{R}), \\ \boldsymbol{\xi} \in \mathbb{R}^{rm} \end{array} \right\}.$$

We have $H := N \ltimes A \curvearrowright \Omega$ **by** $H \times \Omega \ni (h, x) \mapsto h\, x\, {}^t h \in \Omega$.

This action is simply transitive. In fact, given $x \in \Omega$**, the equation** $x = na{}^tn = na^{1/2} I_{rm+1} a^{1/2} ({}^tn)$ $(a \in A,\, n \in N)$ **has unique solution:**
for a_k **we have**

$$a_k = \frac{\Delta_k(x)}{\Delta_{k-1}(x)} \; (k = 1, 2, \dots, r+1), \text{ with } \Delta_0(x) \equiv 1, \text{ where}$$

$$\begin{cases} \Delta_k(x) := \Delta_k^0(x_0) & (k = 1, \dots, r), \\ \Delta_{r+1}(x) := z \det(x_0) - {}^t\boldsymbol{y}({}^{\mathrm{co}}x_0 \otimes I_m)\boldsymbol{y}. \end{cases}$$

${}^{\mathrm{co}}T$**: the cofactor matrix of** T **— thus** $T({}^{\mathrm{co}}T) = ({}^{\mathrm{co}}T)T = (\det T)I$.

- $\Delta_1(x), \dots, \Delta_r(x), \Delta_{r+1}(x)$ **are basic relative invariants.**
  We note $\deg \Delta_k(x) = k$ $(k = 1, 2, \dots, r+1)$**. But** Ω **is not selfdual.**

- **understanding** $\Delta_{r+1}(x) = z \det(x_0) - {}^t\boldsymbol{y}({}^{\mathrm{co}}x_0 \otimes I_m)\boldsymbol{y}$

For each $x = \left(\begin{array}{c|c} x_0 \otimes I_m & \boldsymbol{y} \\ \hline {}^t\boldsymbol{y} & z \end{array}\right) \in V$**, we set**

$$
{}^{\mathrm{d}}x := \left(\begin{array}{ccc|c} x_{11} & \cdots & x_{r1} & \boldsymbol{y}_1 \\ \vdots & & \vdots & \vdots \\ x_{r1} & \cdots & x_{rr} & \boldsymbol{y}_r \\ \hline {}^t\boldsymbol{y}_1 & \cdots & {}^t\boldsymbol{y}_r & z \end{array}\right), \quad x_0 = (x_{ij}), \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_r \end{pmatrix}, \quad \boldsymbol{y}_j \in \mathbb{R}^m
$$

We have $\Delta_{r+1}(x) = \det {}^{\mathrm{d}}x$**. Here we compute** $\det {}^{\mathrm{d}}x$ **as if it were an ordinary determinant, and the product of** ${}^t\boldsymbol{y}_i$ **and** $\boldsymbol{y}_j$ **should be interpreted as the inner product** $\boldsymbol{y}_i \cdot \boldsymbol{y}_j$**.**

Conjecture:
Ω: **Irreducible homogeneous open convex cone of rank** r,
Ω^*: **the dual cone of** Ω.
If the degrees of basic relative invariants associated to Ω **and** Ω^* **are both** $1, 2, \ldots, r$, **then** Ω **is selfdual.**

- **The degrees of the basic relative invariants associated to** Ω^* **for the previous** Ω **with** $r = 3$ **are** $1, 2, 4$.
- **The conjecture is proved in a weaker form by Y. Watanabe (Master thesis, Kyoto University, 2006).**

Non-selfdual Irreducible Homogeneous Convex Cones linearly isomorphic to the dual cones

- $\Omega \subset V$ **with** $\langle \cdot \,|\, \cdot \rangle$ **is selfdual**
  $\iff \exists T$**: positive definite selfadjoint operator s.t.** $T(\Omega) = \Omega^*$**.**
  $$\bigl(\Omega^* := \bigl\{ y \in V \; ; \; \langle \, x \,|\, y \, \rangle > 0 \quad \textbf{for } \forall x \in \overline{\Omega} \setminus \{0\} \bigr\} \bigr)$$
- **Even though there is no positive definite selfadjoint operator** T **s.t.** $T(\Omega) = \Omega^*$**, we might be able to find such** T **if we do not require the positive definiteness.**
- **If one accepts reducible ones, then** $\Omega_0 \oplus \Omega_0^*$ **just gives an example. Thus the irreducibility counts for much here.**
- **In an exercise of Faraut–Korányi's book, a hint is given to prove that the Vinberg cone is never linearly isomorphic to its dual cone. Then the non-selfduality follows from this immediately.**

$$\boldsymbol{e} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m+1}, \qquad I_{m+1}\text{: } (m+1)\times(m+1) \text{ unit matrix}$$

$$V := \left\{ x := \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \; ; \; \begin{array}{lll} x_1 \in \mathbb{R}, & x_2 \in \mathbb{R}, & X \in \mathrm{Sym}(m, \mathbb{R}) \\ \boldsymbol{\xi} \in \mathbb{R}^{m+1}, & \boldsymbol{x}' \in \mathbb{R}^m, & \boldsymbol{x}'' \in \mathbb{R}^m \end{array} \right\}$$

We note $V \subset \mathrm{Sym}(2m+2, \mathbb{R})$**, and take** $\Omega := \{x \in V \; ; \; x \gg 0\}$**.**

When $m = 1$**, we have** $x = \begin{pmatrix} x_1 & 0 & x' & \xi_1 \\ 0 & x_1 & 0 & \xi_2 \\ x' & 0 & X & x'' \\ \xi_1 & \xi_2 & x'' & x_2 \end{pmatrix}.$

Homogeneity of Ω

$$\mathbf{H} := \left\{ h := \left(\begin{array}{c|cc} h_1 I_{m+1} & 0 & 0 \\ \hline \boldsymbol{h}'\,{}^t\boldsymbol{e} & H & 0 \\ {}^t\boldsymbol{\zeta} & {}^t\boldsymbol{h}'' & h_2 \end{array} \right) ; \begin{array}{c} h_1 > 0,\ h_2 > 0 \\ \boldsymbol{\zeta} \in \mathbb{R}^{m+1},\ \boldsymbol{h}' \in \mathbb{R}^m,\ \boldsymbol{h}'' \in \mathbb{R}^m, \\ H \in GL(m,\mathbb{R}) \text{ is lower triangular} \\ \text{with positive diagonals} \end{array} \right\}.$$

H acts on Ω by $\mathbf{H} \times \Omega \ni (h, x) \mapsto hx\,{}^t h$.

This action is simply transitive. We have $\mathbf{H} = \mathbf{N} \rtimes \mathbf{A}$ with

$$\mathbf{A} := \left\{ a := \left(\begin{array}{c|cc} a_1 I_{m+1} & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & a_2 \end{array} \right) ; \begin{array}{c} a_j > 0 \quad (j = 1, 2), \\ A \in GL(m,\mathbb{R}) \text{ is a diagonal} \\ \text{matrix with positive diagonals} \end{array} \right\},$$

$$\mathbf{N} := \left\{ n := \left(\begin{array}{c|cc} I_{m+1} & 0 & 0 \\ \hline \boldsymbol{n}'\,{}^t\boldsymbol{e} & N & 0 \\ {}^t\boldsymbol{\nu} & {}^t\boldsymbol{n}'' & 1 \end{array} \right) ; \begin{array}{c} \boldsymbol{n}', \boldsymbol{n}'' \in \mathbb{R}^m,\ \boldsymbol{\nu} \in \mathbb{R}^{m+1} \\ N \in GL(m,\mathbb{R}) \text{ is strictly} \\ \text{lower triangular} \end{array} \right\}.$$

In solving $x = na\,{}^t n$ ($x \in \Omega$: given) we obtain basic relative invariants as before.

Basic relative invariants: **For** $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \in V$**,**

$$\Delta_1(x) := x_1,$$

$$\Delta_j(x) := \det\left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}'_{j-1} \\ \hline \boldsymbol{x}'_{j-1} & X_{j-1} \end{array}\right) \qquad (j = 2, \ldots, m+1)$$

$$\left(X_k := \begin{pmatrix} x_{11} & \cdots & x_{k1} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix}, \quad \boldsymbol{x}'_k := \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} \in \mathbb{R}^k \right),$$

$$\Delta_{m+2}(x) := x_1 \det\begin{pmatrix} x_1 & {}^t\boldsymbol{x}' & \xi_1 \\ \boldsymbol{x}' & X & \boldsymbol{x}'' \\ \xi_1 & {}^t\boldsymbol{x}'' & x_2 \end{pmatrix} - \left(\|\boldsymbol{\xi}\|^2 - \xi_1^2\right) \det\left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}' \\ \hline \boldsymbol{x}' & X \end{array}\right)$$

$$\left({}^t\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{m+1})\right).$$

- $\deg \Delta_{m+2} = m + 3$**.**

For $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right)$, $y = \left(\begin{array}{c|cc} y_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{y}' & \boldsymbol{\eta} \\ \hline \boldsymbol{y}'\,{}^t\boldsymbol{e} & Y & \boldsymbol{y}'' \\ {}^t\boldsymbol{\eta} & {}^t\boldsymbol{y}'' & y_2 \end{array}\right)$ **we set**

$$\langle\, x \,|\, y \,\rangle := x_1 y_1 + \mathrm{tr}(XY) + x_2 y_2 + 2(\boldsymbol{x}' \cdot \boldsymbol{y}' + \boldsymbol{x}'' \cdot \boldsymbol{y}'' + \boldsymbol{\xi} \cdot \boldsymbol{\eta}).$$

Let $\Omega^* := \big\{ y \in V \;;\; \langle\, x \,|\, y \,\rangle > 0 \quad$ **for** $\forall x \in \overline{\Omega} \setminus \{0\} \big\}$.

Define $T_0 \in \mathrm{End}(V)$ **by** $T_0 x = \left(\begin{array}{c|cc} x_2 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}'' J & \boldsymbol{\xi} \\ \hline J\boldsymbol{x}''\,{}^t\boldsymbol{e} & JXJ & J\boldsymbol{x}' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}' J & x_1 \end{array}\right)$ $(x \in V)$, **where**

$J \in \mathrm{Sym}(m, \mathbb{R})$ **is given by** $J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$.

Theorem: $\Omega^* = T_0(\Omega)$.

- **Current works with students**

(1) V **: Euclidean Jordan algebra**
$\rightsquigarrow$ **selfadjoint Jordan algebra representation of** V
$\rightsquigarrow$ **define a clan without unit element**
$\rightsquigarrow$ **What are the irreducible factors of** $\det R(y)$**?**

(2) To get concrete ordinary matrix realizations of homogeneous convex cones, in particular those cones for $\dim \leq 10$**.**
The description should be free from the general theory of T**-algebra developed by Vinberg.**

(3) Rewrite Vinberg's correspondence clan $\leftrightarrow$ homogeneous cone in a more straightforward way.

(4) Write up homogeneous open convex cones which are root-multiplicity-free.

- **Long-term project**

Develop a harmonic analysis, group-invariant decomposition of L^2**-space over homogeneous convex cones and homogeneous Siegel domains**