Analysis and Geometry Related to Homogeneous Siegel Domains and Homogeneous Convex Cones

Part I

Cayley Transforms of Homogeneous Siegel Domains

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Siegel Domains

- **Generalization of the upper half plane to higher dimensions**
- **Introduced by Piatetski-Shapiro in 1957**
- **Biholomorphically equivalent to bounded domains**
- **Question posed by Cartan (1935)**
  If $n \geqq 4$, ∃? non-symmetric homogeneous bounded domains $\mathbb{C}^n$
  was solved in the affirmative (1959)

- **Dorfmeister–Nakajima (Kazufumi Nakajima) solved (1988)**

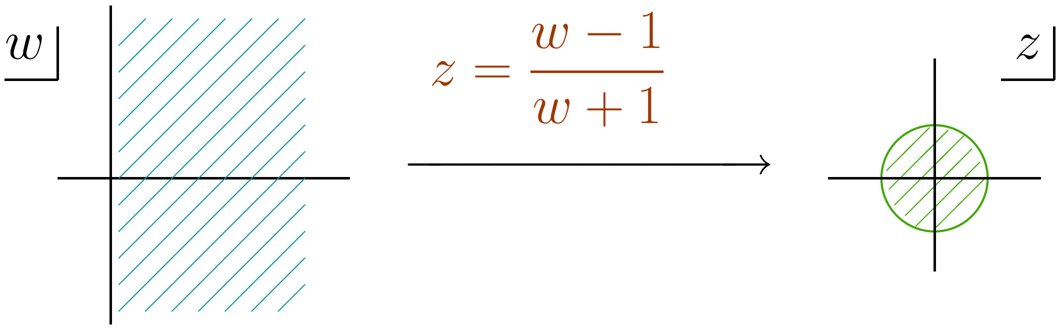
the fundamental conjecture on homogeneous Kähler manifold

(by Vinberg–Gindikin in 1965)

Any Homogeneous Kähler Mfd **= Holomorphic Fiber Bundle over a Homog. Bdd Dom.** **Fiber = Flat Homog. Kähler Mfd** **× Cpt Simply Connected Homog. Kähler Mfd**

Examples of Cayley transform

(I)



$\operatorname{Re} w > 0$ $\qquad\qquad$ $|z| < 1$

(II) $\quad V = \mathrm{Sym}(r, \mathbb{R})$, $\quad \Omega = \mathrm{Sym}(r, \mathbb{R})^{++}$, $\quad \Omega + iV$

$$z = (w - E)(w + E)^{-1} \quad (E\text{: unit matrix})$$

Siegel right half space $\ni w \mapsto z \in$ **Siegel disk**

$\{z \in \mathrm{Sym}(r, \mathbb{C})\ ;\ E - z^* z \gg 0\}.$

(III) General symmetric tube domain

$\Omega + iV$ (**Ω: a selfdual open convex cone in V**)
V **(hence $V_{\mathbb{C}}$) can be equipped with Jordan algebra structure.**

$$z = (w-e)(w+e)^{-1} \quad (e\text{: the unit element in } V)$$

Symmetric tube domain $\longrightarrow$ Open unit ball (w.r.t some norm)

e.g., Siegel disk $= \{z \in \mathrm{Sym}(r, \mathbb{C})\ ;\ \|z\|_{\mathrm{op}} < 1\}$.

For $\mathrm{Sym}(r, \mathbb{C})$**, Jordan algebra product is:**

$$A \circ B = \frac{1}{2}(AB + BA)$$

Jordan algebra inverse $=$ **inverse matrix.**
Thus $(w-e) \circ (w+e)^{-1} = (w-e)(w+e)^{-1}$.

- **Domain** D **is symmetric**

$$\overset{\text{def}}{\Longleftrightarrow} \forall z \in D,\ \exists \sigma_z \in \mathrm{Hol}(D) \text{ s.t. } \begin{cases} (1)\ \sigma_z^2 = \mathrm{Id} \\ (2)\ z \text{ is an isolated fixed point of } \sigma_z \end{cases}$$

(IV) Siegel domain of rank one

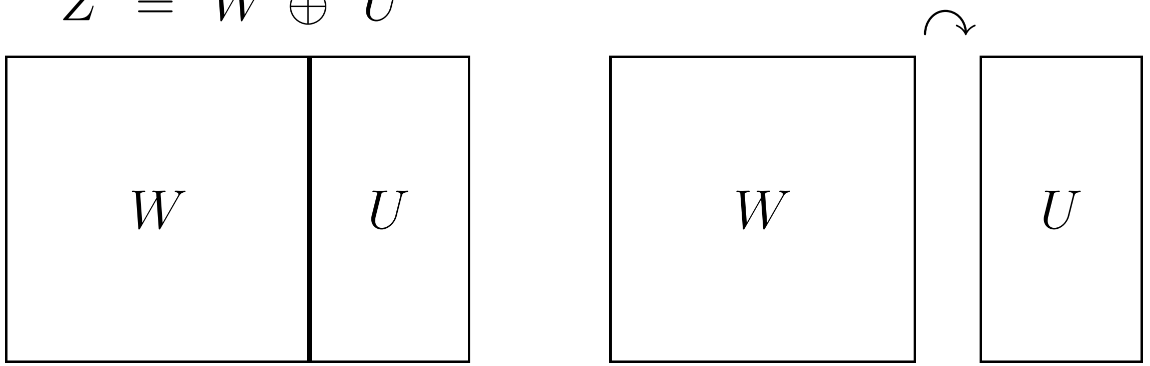
$$D := \{(u,w) \in \mathbb{C}^m \times \mathbb{C}\ ;\ \operatorname{Re} w - \|u\|^2 > 0\}.$$

$$\mathcal{C}(u,w) := \left(\frac{2u}{w+1},\ \frac{w-1}{w+1}\right)$$

$$\boxed{\mathcal{C} : D \longrightarrow \textbf{open unit ball in } \mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}}$$

(V) General symmetric Siegel domain $D \subset Z \cong \mathbb{C}^N$

$Z = W \oplus U$ **natural action**



complex semisimple Jordan algebra $W = V_{\mathbb{C}}$ **with** V **Euclidean JA**

Jordan algebra $*$-repre. φ **of** W

$$\mathcal{C}(u, w) := (2\varphi(w+e)^{-1}u,\ (w-e)(w+e)^{-1})$$

$D \longrightarrow$ **Open unit ball (w.r.t spectral norm)**

↖ Harish-Chandra realization of a non-cpt Hermitian symmetric space

Siegel domains. — Definition —

V : a real vector space $(\dim V < \infty)$

$\cup$

Ω : a regular open convex cone ($\overset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ $(w \mapsto w^*$: conjugation w.r.t. $V)$

U : another complex vector space $(\dim U < \infty)$

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $Q(u', u) = Q(u, u')^*$, $Q(u, u) \in \overline{\Omega} \setminus \{0\}$ $(0 \neq \forall u \in U)$

Siegel domain (of type II)

$$D := \{(u, w) \in U \times W \; ; \; w + w^* - Q(u, u) \in \Omega\}$$

- $U = \{0\}$ is allowed.

In this case $D = \Omega + iV$ (tube domain or type I domain)

Assume that D is homogeneous, ***i.e.***, $\mathrm{Hol}(D) \curvearrowright D$ transitively.

Then Ω is also homogeneous.

Piatetski-Shapiro algebras — normal j-algebras —

$\exists G \subset \mathrm{Hol}_{\mathrm{Aff}}(D)$: **split solvable** $\curvearrowright D$ **simply transitively**

$\mathfrak{g} := \mathrm{Lie}(G)$ **has a structure of Piatetski-Shapiro algebra.**

(**normal j-algebra**)

$$\begin{cases} \exists J \ : \text{ integrable almost complex structure on } \mathfrak{g}, \\ \exists \omega \ : \text{ admissible linear form on } \mathfrak{g}, \textit{ i.e.}, \ \langle\, x \,|\, y \,\rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines} \\ \qquad \text{a } J\text{-invariant (positive definite) inner product on } \mathfrak{g}. \end{cases}$$

Example of admissible linear form (Koszul '55).

$$\langle x, \beta \rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

- **In fact,** $\langle\, x \,|\, y \,\rangle_\beta$ **is the real part of the Hermitian inner product on** $\mathfrak{g} \equiv T_e(D)$ **defined by the Bergman metric on** $D \approx G$ **(up to a positive scalar multiple).**

Structure of $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$ $\begin{cases} \mathfrak{a} : \text{abelian}, \\ \mathfrak{n} : \text{sum of } \mathfrak{a}\text{-root spaces (positive roots only)} \end{cases}$

$\exists H_1, \ldots, H_r$ **: a basis of** $\mathfrak{a}$ $(r := \operatorname{rank} \mathfrak{g})$ **s.t.**
if one puts $E_j := -JH_j \in \mathfrak{n}$**, then** $[H_j, E_k] = \delta_{jk} E_k$**.**

Possible forms of roots:

$$\tfrac{1}{2}(\alpha_k \pm \alpha_j) \ (j < k), \ \ \alpha_1, \ldots, \alpha_r, \ \ \tfrac{1}{2}\alpha_1, \ldots, \tfrac{1}{2}\alpha_r,$$

where $\alpha_1, \ldots, \alpha_r$ **is the basis of** $\mathfrak{a}^*$ **dual to** $H_1, \ldots, H_r$**.**

- $\mathfrak{g}_{\alpha_k} = \mathbb{R}E_k \ (k = 1, \ldots, r)$**.**
- $\mathfrak{g}_\alpha$ **are mutually orthogonal w.r.t.** $\langle \cdot | \cdot \rangle_\omega$ **(**$\forall \omega \in \mathfrak{g}^*$**: admissible).**
- **Putting** $\mathfrak{g}(0) := \mathfrak{a} + \sum_{j<k} \mathfrak{g}_{\frac{1}{2}(\alpha_k - \alpha_j)}, \ \mathfrak{g}(1/2) := \sum_{j=1}^{r} \mathfrak{g}_{\frac{1}{2}\alpha_j}, \ \mathfrak{g}(1) := \sum_{j \leqq k} \mathfrak{g}_{\frac{1}{2}(\alpha_j + \alpha_k)}$**,**

  we have $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$**.**

Define $E_k^* \in \mathfrak{g}^*$ **by** $\langle E_k, E_k^* \rangle = 1$, **and** $= 0$ **on** $\mathfrak{a}$ **and** $\mathfrak{g}_\alpha$ $(\alpha \neq \alpha_k)$**.**

Fact. Admissible linear forms are $\mathfrak{a}^* \oplus \{0\} \oplus \sum_{k=1}^{r} \mathbb{R}_{>0} E_k^*$**.**

For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$**, we put** $E_{\mathbf{s}}^* := \sum_{k=1}^{r} s_k E_k^* \in \mathfrak{g}^*$**.**

If $s_1 > 0, \dots, s_r > 0$ **(we will write** $\mathbf{s} > 0$**), then** $\langle x \,|\, y \rangle_{\mathbf{s}} := \langle [Jx, y], E_{\mathbf{s}}^* \rangle$ **is a** J**-invariant inner product on** $\mathfrak{g}$

Fact (Dorfmeister, 1985) **The inner products** $\langle x \,|\, y \rangle_{\mathbf{s}}$ $(\mathbf{s} > 0)$ **exhaust homogeneous Kähler metrics on** D **(up to conjugations by** $\mathrm{Hol}(D)$**).**

- **A Kähler metric** f **is homogeneous if** $\mathrm{Aut}(D, f)$ **of biholomorphic isometries acts transitively on** D**.**

Compound power functions (After Gindikin)

$\mathfrak{h} := \mathfrak{g}(0)$, $H := \exp \mathfrak{h}$. $E := E_1 + \cdots + E_r \in \mathfrak{g}(1) =: V$.
Then $\Omega = HE$ **(Adjoint action) and** $H \approx \Omega$ **(diffeo) by** $h \mapsto hE$.

- **Note**

$$\mathfrak{g} = \mathfrak{a} + \overbrace{\underbrace{\sum_{j<k} \mathfrak{g}_{(\alpha_k - \alpha_j)/2}}_{=:\mathfrak{n}_0} + \mathfrak{g}(1/2) + \mathfrak{g}(1)}^{\mathfrak{n}=[\mathfrak{g},\mathfrak{g}].}$$

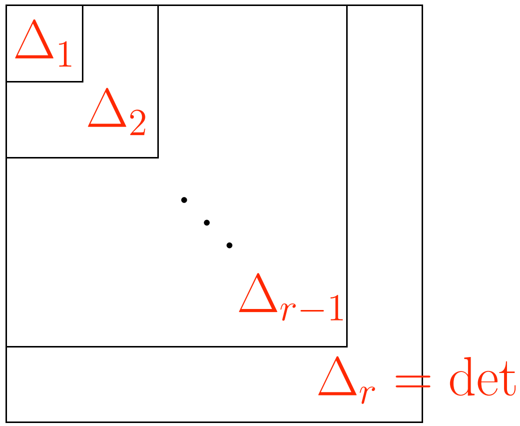
Then, $G = N \rtimes A$, $H = N_0 \rtimes A$ **with** $A := \exp \mathfrak{a}$, $N_0 \subset N := \exp \mathfrak{n}$.

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, **put** $\alpha_{\mathbf{s}} := \sum_{j=1}^{r} s_j \alpha_j \in \mathfrak{a}^*$
($\alpha_1, \ldots, \alpha_r$: **basis of** $\mathfrak{a}^*$ **dual to** $H_1, \ldots, H_r$).

$\chi_{\mathbf{s}}(\exp x) := \exp\langle x, \alpha_{\mathbf{s}} \rangle$ $(x \in \mathfrak{a})$ **defines a 1-dim. representation of** A, **hence of** H

$\rightsquigarrow$ **function** $\Delta_{\mathbf{s}}$ **on** Ω **by** $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$.

Example : If $\Omega = \mathrm{Sym}(r, \mathbb{R})^{++}$, then $\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2}\Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$, where $\Delta_1(x), \ldots, \Delta_r(x)$ are the principal minors of x:



take minors from the left-upper corner

General case

Fact (Gindikin, Ishi (2000)): $\Delta_{\mathbf{s}}$ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^*, where

$$\Omega^* := \{\xi \in V^* \,;\, \langle x, \xi \rangle > 0 \;\; \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

Pseudoinverse map associated to E^*_{s}

For $\forall x \in \Omega$**, define** $\mathcal{I}_{\mathrm{s}}(x) \in V^*$ **by** $\langle v, \mathcal{I}_{\mathrm{s}}(x)\rangle := -D_v \log \Delta_{-\mathrm{s}}(x) \ \ (v \in V)$.

$$\left(D_v f(x) := \tfrac{d}{dt} f(x+tv)\big|_{t=0}\right)$$

- $\mathcal{I}_{\mathrm{s}}(\lambda x) = \lambda^{-1}\mathcal{I}_{\mathrm{s}}(x) \qquad (\lambda > 0)$

Proposition. **Suppose** E^*_{s} **is admissible (i.e.,** $\mathrm{s} > 0$**)**

(1) $\mathcal{I}_{\mathrm{s}}(x) \in \Omega^*$ **and** $\mathcal{I}_{\mathrm{s}} : \Omega \twoheadrightarrow \Omega^*$ **is bijective.**
(2) $\mathcal{I}_{\mathrm{s}}$ **extends analytically to a rational map** $W \to W^*$**.**
(3) One also has an explicit formula for $\mathcal{I}_{\mathrm{s}}^{-1} : \Omega^* \twoheadrightarrow \Omega$**, which continues analytically to a rational map** $W^* \to W$**.**
Thus $\mathcal{I}_{\mathrm{s}}$ **is birational.**
(4) $\mathcal{I}_{\mathrm{s}} : \Omega + iV \twoheadrightarrow \mathcal{I}_{\mathrm{s}}(\Omega + iV)$ **is biholomorphic.**

Remarks. (1) Recall

$$-D_v \log \det(x)^{-1} = \mathrm{tr}(x^{-1}v) \qquad (x \in \mathrm{Sym}(r,\mathbb{R})^{++},\ v \in \mathrm{Sym}(r,\mathbb{R})).$$

(2) Rationality of $\mathcal{I}_{\mathrm{s}}$ can be seen as follows:
Consider $x \triangle y := [Jx, y]$ $(x, y \in V := \mathfrak{g}(1))$.
Then $(V, \triangle)$ **is a clan in the sense of Vinberg (⇝ Part II).**

Fact: $\mathcal{I}_{\mathbf{s}}(y) = E_{\mathbf{s}}^* \circ R(y)^{-1}$ $(x \in \Omega)$, **where** $R(y)x := x \triangle y$.

The analytic continuation of $\mathcal{I}_{\mathrm{s}}$ **can be obtained by just considering the complexification of the non-associative algebra** $(V, \triangle)$.

(3) The Bergman kernel and the Szegö kernel are of the form

$$\eta(z_1, z_2) = \Delta_{-\mathbf{s}}\big(w_1 + w_2^* - Q(u_1, u_2)\big) \qquad (z_j = (u_j, w_j) \in D),$$

(up to positive const.) and the characteristic function ϕ **of** Ω **equals** $\Delta_{-\mathrm{s}_0}$ **for some** $\mathrm{s}_0 > 0$ **(up to positive const.), where**

$$\phi(x) := \int_{\Omega^*} e^{-\langle x, y\rangle}\, dy \qquad (x \in \Omega).$$

In this case $\mathcal{I}_{\mathrm{s}_0}(x) = x^*$: **Vinberg's $*$-map.**

(4) The condition $\mathrm{s} > 0$ **is used to show** $\Delta_{-\mathrm{s}}$ **is exploding on** $\partial\Omega$:
if $\Omega \ni x \to x_0 \in \partial\Omega$, **then** $\Delta_{-\mathrm{s}}(x) \to \infty$.

(5) $\mathcal{I}_{\mathbf{s}}(\Omega + iV) = \Omega^* + iV^* \iff s_1 = \cdots = s_r$ **and** Ω **is selfdual.**

[Kai-N, 2005]

- Ω **is selfdual**

  $\overset{\text{def}}{\iff} \exists \langle \cdot \,|\, \cdot \rangle$ **by which regarding** $V^* \equiv V$**, one has** $\Omega^* = \Omega$**.**

Example: $\big(\mathrm{Sym}(r,\mathbb{R})^{++} + i\,\mathrm{Sym}(r,\mathbb{R})\big)^{-1} = \mathrm{Sym}(r,\mathbb{R})^{++} + i\,\mathrm{Sym}(r,\mathbb{R}).$

Cayley transform for tube domains

Note

$$\frac{w-1}{w+1} = 1 - \frac{2}{w+1}.$$

For $\mathrm{s} > 0$**, we define**

$$C_{\mathbf{s}}(w) := \mathcal{I}_{\mathbf{s}}(E) - 2\, I_{\mathbf{s}}(w+E) \qquad (w \in \Omega + i\,V).$$

Remark. **If we use** $\langle\, x \,|\, y\,\rangle_{\mathbf{s}} = \langle [Jx, y], E^*_{\mathbf{s}}\rangle$ $(x, y \in V = \mathfrak{g}(1))$ **to identify** V^* **with** V**, then** $\mathcal{I}_{\mathbf{s}}(E) = E$**.**

Cayley transform for homogeneous Siegel domains

For $\mathrm{s} > 0$**, we set**

$$\mathcal{C}_{\mathbf{s}}(u, w) := 2\langle Q(u, \cdot), \mathcal{I}_{\mathbf{s}}(w + E)\rangle \oplus C_{\mathbf{s}}(w) \quad ((u, w) \in D).$$

Note $U \ni u' \mapsto \langle Q(u, u'), \mathcal{I}_{\mathrm{s}}(w + E)\rangle$ **is** $\mathbb{C}$ **anti-linear.**

Thus $\mathcal{C}_{\mathrm{s}}(u, w) \in U^{\dagger} \oplus W^{*}$ **(**$U^{\dagger} =$ **the space of anti-linear forms on** U**).**

Theorem (N. 2003). $\mathcal{C}_{\mathrm{s}}(D)$ **is bounded.**

Examples. **(1) The Bergman kernel and the Szegö kernel are of the form $C\Delta_{-\mathbf{s}}(w+w^*-Q(u_1.u_2))$ $(C>0)$. Then we have the Cayley transforms corresponding to the Bergman and the Szegö kernels.**

(2) We already know $\phi = C_0\Delta_{-\mathbf{s}_0}$ $(C_0>0)$. Then the Cayley transform $\mathcal{C}_{\mathbf{s}_0}$ is the one introduced by Penney in 1996.

Remarks. **(1) Take $\langle\,\cdot\,|\,\cdot\,\rangle_{\mathbf{s}}$ on V defined earlier. Extend it to a $\mathbb{C}$-bilinear form on $W = V_{\mathbb{C}}$. Then, $(u_1\,|\,u_2)_{\mathbf{s}} := \langle\, Q(u_1,u_2)\,|\,E\,\rangle_{\mathbf{s}}$ is a Hermitian inner product in U. Use these to identify $U^\dagger \oplus W^*$ with $U \oplus W$. Moreover, for $w \in W$, define an operator $\varphi_{\mathbf{s}}(w)$ by**

$$(\varphi_{\mathbf{s}}(w)u_1\,|\,u_2)_{\mathbf{s}} = \langle\, Q(u_1,u_2)\,|\,w\,\rangle_{\mathbf{s}}.$$

Then, our Cayley transform $\mathcal{C}_{\mathbf{s}}$ can be rewritten as

$$\mathcal{C}_{\mathbf{s}}(u,w) = \big(2\varphi_{\mathbf{s}}(I_{\mathbf{s}}(w+E))u,\ C_{\mathbf{s}}(w)\big).$$

(2) The image $\mathcal{C}_{\mathbf{s}_b}(D)$ is the *representative domain* of D (up to a positive # multiple) studied by Ishi–Kai (2010).

Example. The case of quasisymmetric Siegel domains

Suppose $\mathrm{s} = \mathrm{s}_b$ **: the Bergman parameter.**

Then, D **is quasisymmetric** $\overset{\text{def}}{\Longleftrightarrow}$ Ω **is selfdual w.r.t** $\langle \cdot \,|\, \cdot \rangle_{\mathrm{s}_b}$**.**

Suppose now that D **is quasisymmetric.**
Then V **has a structure of Euclidean Jordan algebra:**

- $xy = xy, \qquad x^2(xy) = x(x^2 y)$
- $\langle\, xy \,|\, z \,\rangle_{\mathrm{s}_b} = \langle\, x \,|\, yz \,\rangle_{\mathrm{s}_b}$

In this case $I_{\mathrm{s}_b}(w) = w^{-1}$ **(JA inverse). Moreover**

$$\varphi_{\mathrm{s}_b} : W \ni w \mapsto \varphi_{\mathrm{s}_b}(w) \in \mathrm{End}_{\mathbb{C}}(U)$$

is a JA $*$**-representation of** W**:**

$$\begin{cases} \varphi_{\mathrm{s}_b}(w^*) = \varphi_{\mathrm{s}_b}(w)^*, \\ \varphi_{\mathrm{s}_b}(w_1 w_2) = \frac{1}{2}\big(\varphi_{\mathrm{s}_b}(w_1)\varphi_{\mathrm{s}_b}(w_2) + \varphi_{\mathrm{s}_b}(w_2)\varphi_{\mathrm{s}_b}(w_1)\big). \end{cases}$$

Then

$$\mathcal{C}_{\mathbf{s}_b}(u,w) = (2\varphi_{\mathbf{s}_b}(w+E)^{-1}u,\ (w-E)(w+E)^{-1}).$$

This coincides with Dorfmeister's Cayley transform for quasisymmetric Siegel domains introduced in 1980.

Moreover, if D is symmetric, then $\mathcal{C}_{\mathbf{s}_b}$ coincides with (the inverse map) of the Cayley transform defined by Korány–Wolf in 1965.

Theorem (Kai, 2007).

$$\mathcal{C}_{\mathbf{s}}(D) \text{ is convex} \iff \begin{cases} D \text{ is symmetric}, \\ s_1 = \cdots = s_r. \end{cases}$$

Remark. If D is symmetric, then $C_{\mathbf{s}_b}(D)$ is the Harish-Chandra realization of a noncompact Hermitian symmetric space. Thus it is the open unit ball w.r.t a certain norm (the spectral norm of the underlying JTS), so that it is convex.

Application to Poisson–Hua kernel

S : the Szegö kernel of D, Σ : the Shilov boundary of D.
One knows

$$\Sigma = \{(u, w) \; ; \; w + w^* - Q(u, u) = 0\}.$$

$\mathbf{s} = \mathbf{s}_s$: Szegö parameter
The explicit expression by $\Delta_{\mathbf{s}_s}$ assures us that $S(z, \zeta)$ $(z \in D, \, \zeta \in \Sigma)$ still has a meaning.

The Poisson–Hua kernel is defined to be

$$P(z, \zeta) := \frac{|S(z, \zeta)|^2}{S(z, z)} \quad (z \in D, \, \zeta \in \Sigma).$$

$G \curvearrowright D$ **simply transitively by affine maps.**
Fix $\mathbf{e} := (0, E) \in D$. **Then** $G \approx D$ **(diffeo) by** $g \mapsto g \cdot \mathbf{e}$.
Put $p_\zeta(g) := P(g \cdot \mathbf{e}, \zeta) \quad (g \in G)$.
$\mathcal{L}_{\mathrm{s}}^G$ **: the L.–B. operator on** G **corresponding to** $\mathcal{L}_{\mathrm{s}} \leftrightarrow \langle x \mid y \rangle_{\mathrm{s}}$.

Key formula. $\mathcal{L}_{\mathrm{s}}^G p_\zeta = \big(-\|\mathcal{C}_{\mathrm{s}_s}(\zeta)\|_{\mathrm{s}}^2 + c_2\big) p_\zeta$.

c_2 **: explicitly given const.**(> 0) **independent of** ζ.

Theorem. [N, 2003]
$\mathcal{L}_{\mathrm{s}} P(\cdot, \zeta) = 0 \ \ (\forall \zeta \in \Sigma) \iff D$ **is symmetric and** $s_1 = \cdots = s_r$

Remark. For the case $\mathrm{s} = \mathrm{s}_b$,
$\Longleftarrow$ **Hua–Look (classical domains, 1959), Korányi (general 1965).**
$\Longrightarrow$ **Xu (difficult computation, 1979).**

- **application to Berezin transforms (omitted).**

Example. Non-symmetric 4-dimensional Siegel domain

$V = \mathrm{Sym}(2,\mathbb{R})$, $\Omega := \mathrm{Sym}(2,\mathbb{R})^{++}$.
Then, $W := V_{\mathbb{C}} = \mathrm{Sym}(2,\mathbb{C})$.
Take $U = \mathbb{C}$, **and define** $Q(u_1, u_2) := 2\begin{pmatrix} 0 & 0 \\ 0 & u_1\overline{u}_2 \end{pmatrix}$.
Then $Q : U \times U \to W$, **Hermitian and** Ω**-positive.**

$$D = \{(u, w) \; ; \; w + w^* - Q(u, u) \in \Omega\}$$
$$= \left\{ (u, w_1, w_2, w_3) \in \mathbb{C}^4 \; ; \; \begin{matrix} v_1(v_3 - |u|^2) - v_2^2 > 0, \\ v_3 > |u|^2 \end{matrix} \right\} \qquad (v_j := \mathrm{Re}\, w_j).$$

The inner product $\langle \cdot \,|\, \cdot \rangle_{\mathrm{s}}$ **in** V **is expressed as**

$$\langle v \,|\, v' \rangle_{\mathrm{s}} := s_1 v_{11} v'_{11} + 2 s_2 v_{21} v'_{21} + s_2 v_{22} v'_{22} \quad (s_j > 0).$$

Using $\langle \cdot \,|\, \cdot \rangle_{\mathrm{s}}$ **to identify** V^* **with** V**, we get**

$$\Omega^* = \left\{ \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \; ; \; y_1 y_3 - \frac{s_2}{s_1} y_2^2 > 0, \; y_3 > 0 \right\}.$$

$$\mathcal{I}_{\mathbf{s}}(w) = \frac{1}{\det w} \begin{pmatrix} w_3 - s_1^{-1}(s_1 - s_2)w_1^{-1}w_2^2 & -w_2 \\ -w_2 & w_1 \end{pmatrix}.$$

If one uses $\mathrm{tr}(vv')$ **to identify** V^* **with** V**, then** $\Omega^* = \Omega$ **and**

$$\mathcal{I}_{\mathbf{s}}^0(w) = \frac{1}{\det w} \begin{pmatrix} s_1 w_3 - (s_1 - s_2)w_1^{-1}w_2^2 & -s_2 w_2 \\ -s_2 w_2 & s_2 w_1 \end{pmatrix}.$$

Geatti's parameter (1982): $s_1 = s_2 \rightsquigarrow \mathcal{I}_{\mathbf{s}}(w) = w^{-1}$**,**

Bergman parameter $\mathbf{s}_b$**:** $s_1 = 3$**,** $s_2 = 4$**,**

($\rightsquigarrow$ D **is not even quasisymmetric).**

Szegö parameter $\mathbf{s}_s$**:** $s_1 = \frac{3}{2}$**,** $s_2 = \frac{5}{2}$**,**

Penney's parameter $\mathbf{s}_0$**:** $s_1 = s_2 = \frac{3}{2}$**.**

Geatti's Cayley transform ($\equiv$ **Penney's)**

$$\mathcal{C}(u, w) = \left(\frac{2(w_1 + 1)}{\det(w + E)} u,\ (w - E)(w + E)^{-1} \right).$$

But CT with Bergman param. and Szegö param. both look strange:

$$\mathcal{C}_{\mathbf{s}}(u,w) = \left(\frac{2(w_1+1)}{\det(w+E)}\, u,\ E - 2\mathcal{I}_{\mathbf{s}}(w+E) \right).$$

Nevertheless CT with Szegö parameter appeared in my analysis of Poisson–Hua kernel.