Some Symmetry Conditions

of

Homogeneous Siegel Domains

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Siegel Domains — Introduction —

Introduced by Piatetski-Shapiro (1957), holomorphically equivalent to bounded domains

Motivation for the introduction

- Description of Hermitian symmetric spaces (HSS) by upper-half plane type domains
- There are HSSs that cannot be realized as tube domains $V + i\Omega$
  (V: real VS, Ω: open convex cone in V)
- Application to the theory of automorphic functions

The most unexpected application

$\cdots\cdots$ Discovery of "many" non-symmetric homogeneous bounded domains (HBD) (1959)

Earlier study of HBD

É. Cartan, Abh. Math. Sem. Univ. Hamburg, **11** (1935)

$\cdots\cdots$ HBDs in $\mathbb{C}^2$ and $\mathbb{C}^3$ are all symmetric.

Problem : What happens in $\mathbb{C}^n$ for $n \geqq 4$?

Remark. Cartan did *not* make the conjecture that all HBDs are symmetric. What Cartan actually wrote is:

"..., il semble que là, comme dans beaucoups d'autres problèmes, il faille s'appuyer sur une idée nouvelle."

$\mathscr{D}$: HBD

Armand Borel (1954), Jean-Louis Koszul (1955)
$\mathscr{D}$ is a hom. space of ss Lie gr. $\Longrightarrow$ $\mathscr{D}$ is symmetric.

Jun-ichi Hano (1957)
weaken the assumption of ss to unimodular
(unimodular $\underset{\text{def}}{\Longleftrightarrow}$ left Haar measure is right invariant)

Piatetski-Shapiro (1959)
Examples of non-symm. homogeneous Siegel domains (type II domains = non-tube domains)

• Gindikin wrote: [Israel Math. Conf. Proc.]
"It is funny to remember now, how suspiciously we listened for the first time to the proof that this domain is nonsymmetric."

Vinberg (1960)
Non-symm. homogeneous tube domain
$\leftrightsquigarrow$ Non-selfdual homogenous open convex cone
Min. dimension = 5

Natural Question. How do we characterize symmetric Siegel domains (among homogeneous Siegel domains)?

Siegel Domains — Definition —

V : a real vector space ($\dim V < \infty$)

$\cup$

Ω : a *regular* open convex cone

($\underset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)

U : another *complex* vector space ($\dim U < \infty$)

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \ (0 \neq \forall u \in U) \end{cases}$

Siegel domain (of type II)

$$\boxed{D := \{(u, w) \in U \times W \ ; \ w + w^* - Q(u, u) \in \Omega\}}$$

- $U = \{0\}$ is allowed. In this case $D = \Omega + iV$.
  (tube domain or type I domain)

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively

D: a homogeneous Siegel domain

- D is *symmetric*

$$\underset{\text{def}}{\Longleftrightarrow} \forall z \in D,\ \exists \sigma_z \in \mathrm{Hol}(D) \text{ s.t.}$$

$$\begin{cases} \sigma_z^2 = \text{identity}, \\ z \text{ is an isolated fixed point of } \sigma_z. \end{cases}$$

Siegel domain of rank 1 (symmetric)

$V = \mathbb{R}, \quad \Omega = \{x \in \mathbb{R}\ ;\ x > 0\},$
$W = \mathbb{C}, \quad U = \mathbb{C}^m \ (m = 0, 1, 2, \dots),$
$Q(u_1, u_2) := \frac{1}{2} u_1 \overline{u}_2 \quad (u_1, u_2 \in \mathbb{C}).$

$D = \{(u, w)\ ;\ \mathrm{Re}\, w > \frac{1}{4}|u|^2\} \approx B^{m+1} \subset \mathbb{C}^{m+1} = U \times W$

by $\mathscr{C}(u, w) = \left(\dfrac{u}{w+1}, \dfrac{w-1}{w+1} \right)$: Cayley transform

- B^{m+1} is symmetric

$\cdots\cdots \quad z \mapsto -z$ is the symmetry around $0 \in B^{m+1}$.

Via $\mathscr{C}$, the symmetry around $(0, 1)$ is given by
$$(u, w) \mapsto (-w^{-1}u,\ w^{-1})$$

Non-quasisymmetric Siegel domain

$$V = \mathrm{Sym}(2,\mathbb{R}), \quad \Omega = \mathrm{Sym}^{++}(2,\mathbb{R}),$$
$$W = \mathrm{Sym}(2,\mathbb{C}), \quad U = \mathbb{C},$$
$$Q(u_1,u_2) = \begin{pmatrix} 0 & 0 \\ 0 & 2u_1\overline{u}_2 \end{pmatrix} \quad (u_1,u_2 \in \mathbb{C}).$$

By $W \ni w = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \longleftrightarrow (w_1,w_2,w_3) \in \mathbb{C}^3$

$$D = \{(u,w)\,;\, 2\,\mathrm{Re}\,w - Q(u,u) \in \Omega\}$$
$$= \left\{ (u,w_1,w_2,w_3) \in \mathbb{C}^4\,;\, \begin{pmatrix} v_1 & v_2 \\ v_2 & v_3 - |u|^2 \end{pmatrix} \gg 0 \right\}.$$

Non-symmetric quasisymmetric Siegel domain

V, Ω, W : as above, $\quad U = \mathbb{C}^2$,

$$Q(u,u') = \begin{pmatrix} 2u_1\overline{u}'_1 & u_1\overline{u}'_2 + u_2\overline{u}'_1 \\ u_1\overline{u}'_2 + u_2\overline{u}'_1 & 2u_2\overline{u}'_2 \end{pmatrix} \in W.$$

$$Q(u,u) = 2\begin{pmatrix} |u_1|^2 & \mathrm{Re}\,u_1\overline{u}_2 \\ \mathrm{Re}\,u_1\overline{u}_2 & |u_2|^2 \end{pmatrix} \in \overline{\Omega}.$$

$$D = \left\{ (u,w)\,;\, \begin{pmatrix} v_1 - |u_1|^2 & v_2 - \mathrm{Re}\,u_1\overline{u}_2 \\ v_2 - \mathrm{Re}\,u_1\overline{u}_2 & v_2 - |u_2|^2 \end{pmatrix} \gg 0 \right\}.$$

Characterizations of symmetric Siegel domains

Late 1970's : Satake (book published in 1980)
Dorfmeister (Habilitationsschrift, 1978)
$\cdots$ In terms of defining data

D'Atri (1979) $\cdots$ Diff. Geometric (curvature cond.)

D'Atri, Dorfmeister and Y. Zhao [DDZ] (1985)
$\cdots$ Study of isotropy representation

One of DDZ's results

$\mathbf{D}(D)^{\mathbf{G}}$ is commutative $\iff$ D is symmetric

$\mathbf{G} := \mathrm{Hol}(D)^\circ$: identity component
$\mathbf{D}(D)^{\mathbf{G}}$: algebra of $\mathbf{G}$-inv. differential operators on D

Remark.

- If D is symmetric, then $\mathbf{D}(D)^{\mathbf{G}}$ is better:

$$\mathbf{D}(D)^{\mathbf{G}} \cong \mathbb{C}[t_1, \ldots, t_r] \quad (r := \mathrm{rank}(D))$$

- If D is non-symmetric, then $\mathbf{D}(D)^{\mathbf{G}}$ is worse [DDZ]:

$$\exists \text{ first order } T \in \mathbf{D}(D)^{\mathbf{G}}$$

Today's talk

$\mathscr{L}$: Laplace–Beltrami operator
(w.r.t. a standard Kähler metric of D)

Theorem A. [N, 2001]
$\mathscr{L}$ commutes with the Berezin transforms
$\iff$ D is symmetric and
the metric considered is Bergman
(up to const. multiple > 0).

Theorem B. [N, 2003]
The Poisson–Hua kernel is annihilated by $\mathscr{L}$
$\iff$ D is symmetric and
the metric considered is Bergman
(up to const. multiple > 0).

Remark. If one takes the Bergman metric from the beginning in Theorem B, then the theorem is due to
Hua–Look ('59), Korányi ('65) for $\Leftarrow$
Xu ('79) for $\Rightarrow$

However, I think very few people traced Xu's proof (required to understand his own theory of N-Siegel domains, and to read some of his papers written in Chinese that are not available in English translation).

Piatetski-Shapiro algebras – normal j-algebras –

$\exists G \subset \mathrm{Hol}_{\mathrm{Aff}}(D)$: split solvable $\curvearrowright D$
simply transitively

$\mathfrak{g} := \mathrm{Lie}(G)$ has a str. of Piatetski-Shapiro algebra.
(normal j-algebra)

$$\begin{cases} \exists J : \text{ integrable almost complex structure on } \mathfrak{g}, \\ \exists \omega : \text{ admissible linear form on } \mathfrak{g}, \textit{ i.e.,} \\ \qquad \langle x | y \rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines a } J\text{-invariant} \\ \qquad \text{(pos. def.) inner product on } \mathfrak{g}. \end{cases}$$

Example (Koszul '55). Koszul form.

$$\langle x, \beta \rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

This β is admissible

- In fact, $\langle x | y \rangle_\beta$ is the real part of the Hermitian inner product on $\mathfrak{g} \equiv T_e(D)$ defined by the Bergman metric on $D \approx G$ (up to a positive scalar multiple).

Structure of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n} \qquad \begin{cases} \mathfrak{a} : \text{ abelian,} \\ \mathfrak{n} : \text{ sum of } \mathfrak{a}\text{-root spaces (positive roots only)} \end{cases}$$

Always contains a product of $ax+b$ algebra:
$\exists H_1, \ldots, H_r$: a basis of $\mathfrak{a}$ $(r := \operatorname{rank} \mathfrak{g})$ s.t.
if one puts $E_j := -JH_j \in \mathfrak{n}$, then $[H_j, E_k] = \delta_{jk} E_k$.

Possible forms of roots:

$$\tfrac{1}{2}(\alpha_k \pm \alpha_j)\ (j < k),\ \ \alpha_1, \ldots, \alpha_r,\ \ \tfrac{1}{2}\alpha_1, \ldots, \tfrac{1}{2}\alpha_r$$

$\alpha_1, \ldots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$.

- $\mathfrak{g}_{\alpha_k} = \mathbb{R} E_k$ $(k = 1, \ldots, r)$.
- $\mathfrak{g}_\alpha$ are mutually orthogonal w.r.t. $\langle \cdot | \cdot \rangle_\omega$ ($\forall \omega$: adm.)

$E_k^* \in \mathfrak{g}^*$: $\langle E_k, E_k^* \rangle = 1$ and $= 0$ on $\mathfrak{a}$ and $\mathfrak{g}_\alpha$ $(\alpha \neq \alpha_k)$.

- Admissible linear forms are $\mathfrak{a}^* \oplus \{0\} \oplus \sum_{k=1}^{r} \mathbb{R}_{>0} E_k^*$.

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we put $E_\mathbf{s}^* := \sum_{k=1}^{r} s_k E_k^* \in \mathfrak{g}^*$.
If $s_1 > 0, \ldots, s_r > 0$ (we'll write $\mathbf{s} > 0$), then
$\langle x | y \rangle_\mathbf{s} := \langle [Jx, y], E_\mathbf{s}^* \rangle$ is a J-inv. inner product on $\mathfrak{g}$
$\rightsquigarrow$ left invariant Riemannian metric on G
$\rightsquigarrow$ $\mathscr{L}_\mathbf{s}$: the corresponding L-B operator on G.

Berezin transforms

κ : the Bergman kernel of D

the Berezin kernel

$$A_\lambda(z_1,z_2) := \left(\frac{|\kappa(z_1,z_2)|^2}{\kappa(z_1,z_1)\kappa(z_2,z_2)}\right)^\lambda \quad (z_j \in D;\ \lambda \in \mathbb{R})$$

- A_λ is G-invariant: $A_\lambda(g\cdot z_1,\ g\cdot z_2) = A_\lambda(z_1,z_2)$.

Since $D \approx G$, we work on G:

$$a_\lambda(g) := A_\lambda(g\cdot \mathrm{e}, \mathrm{e}) \quad (g \in G,\ \mathrm{e} \in D : \text{fixed ref. pt.})$$

- $a_\lambda \in L^1(G)$ if $\lambda > \lambda_0$ $(0 < \lambda_0 < 1$: explicitly calculated$)$.

(non-vanishing condition for Hilbert spaces of holomorphic functions on D, in which κ^λ is the reproducing kernel.)

Berezin transform

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x)$$

$B_\lambda \in \mathbf{B}(L^2(G))$: selfadjoint, positive.

Recall $\beta \in \mathfrak{g}$: Koszul form. $\boxed{\beta|_{\mathfrak{n}} = E^*_{\mathbf{c}}|_{\mathfrak{n}}}$ with $\mathbf{c} > 0$.

Theorem A. $\lambda > \lambda_0$: fixed.
B_λ commutes with $\mathscr{L}_{\mathbf{s}}$
$\iff$ D is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

Poisson–Hua kernel

$S(z_1, z_2)$: the Szegö kernel of D
(= reprod. kernel of the Hardy space)

- Hardy space

Hilbert space of holomorphic functions F on D s.t.

$$\boxed{\sup_{t\in\Omega}\int_U dm(u)\int_V \left|F\left(u, t+\tfrac{1}{2}Q(u,u)+ix\right)\right|^2 dx < \infty}$$

Σ : the Shilov boundary of D
Then, $\Sigma = \left\{(u,w) \in U\times W\,;\, 2\operatorname{Re} w = Q(u,u)\right\}$.

$S(z,\zeta)$ for $z \in D$ and $\zeta \in \Sigma$ still has a meaning.

$$P(z,\zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in D,\ \zeta \in \Sigma) :$$

the Poisson kernel of D

$$P_\zeta^G(g) := P(g\cdot \mathrm{e},\ \zeta) \quad (g \in G).$$

Theorem B. $\mathscr{L}_{\mathbf{s}} P_\zeta^G = 0$ for $\forall\zeta \in \Sigma$
$\Longleftrightarrow$ D is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

Geometric backgrounds

Geometric reason that Theorems A and B are true ?

— Connection with a geometry of bounded models of homogeneous Siegel domains —

geometry $\leftrightsquigarrow$ geometric norm equality

- Validity of norm equality $\iff$ Symmetry of the domain

Specialists' folklore

There is *no* (most) canonical bounded model for non-(quasi)symmetric Siegel domains.

My standpoint

Appropriate bounded model varies with problems one treats.

• Canonical bounded model for symmetric Siegel domains

$\cdots\cdots$ Harish-Chandra model

of non-cpt Hermitian symmetric spaces

$\left(\begin{array}{l}\text{Open unit ball of a positive Hermitian JTS} \\ \text{w.r.t the spectral norm}\end{array}\right)$

• Canonical bounded model for quasisymmetric Siegel domains

$\cdots\cdots$ by Dorfmeister (1980)

Image of Siegel domain under the Cayley transform

naturally defined in terms of Jordan algebra structure

(but non-convex unless symmetric. By C. Kai, in preparation)

• For general homogeneous Siegel domains

We can consider

• Cayley transf. assoc. to the Szegö kernel
• Cayley transf. assoc. to the Bergman kernel
• Cayley transf. assoc. to the char. ftn of the cone

etc. . .

More generally, we can define Cayley transforms associated to the admissible linear forms $E^*_{\mathbf{s}}$ $(\mathbf{s} > 0)$.
[N, 2003]

Compound power functions (After Gindikin)

$\exists H \subset G$: s.t. $H \curvearrowright \Omega$ simply transitively

$E \in \Omega$ (canonically fixed base point)

Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

- Note $G = N \rtimes A$, $H = N_0 \rtimes A$ with $A := \exp \mathfrak{a}$

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, put $\alpha_{\mathbf{s}} := \sum_{j=1}^{r} s_j \alpha_j \in \mathfrak{a}^*$

($\alpha_1, \ldots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$).

$\chi_{\mathbf{s}}(\exp x) := \exp\langle x, \alpha_{\mathbf{s}} \rangle$ $(x \in \mathfrak{a})$:

1-dim. representation of A, hence of H.

$\rightsquigarrow$ function on Ω by $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$

Example : If $\Omega = \mathrm{Sym}^{++}(r, \mathbb{R})$, then

$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$.

$\Delta_1(x), \ldots, \Delta_r(x)$: principal minors of x

> $\Delta_{\mathbf{s}}$ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^* (Gindikin, Ishi (2000)), where
>
> $$\Omega^* := \{\xi \in V^* ; \langle x, \xi \rangle > 0 \ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

Pseudoinverse map associated to $E_{\mathbf{s}}^*$

For each $x \in \Omega$, define $\mathscr{I}_{\mathbf{s}}(x) \in V^*$ by

$$\langle v, \mathscr{I}_{\mathbf{s}}(x) \rangle := -D_v \log \Delta_{-\mathbf{s}}(x) \quad (v \in V).$$

$$\left(D_v f(x) := \frac{d}{dt} f(x+tv)\Big|_{t=0}\right)$$

- $\mathscr{I}_{\mathbf{s}}(\lambda x) = \lambda^{-1} \mathscr{I}_{\mathbf{s}}(x) \qquad (\lambda > 0)$

Proposition. Suppose $E_{\mathbf{s}}^*$ is admissible.

(1) $\mathscr{I}_{\mathbf{s}}(x) \in \Omega^*$ and $\mathscr{I}_{\mathbf{s}} : \Omega \twoheadrightarrow \Omega^*$ is bijective.

(2) $\mathscr{I}_{\mathbf{s}}$ extends analytically to a rational map $W \to W^*$.

(3) One also has an explicit formula for $\mathscr{I}_{\mathbf{s}}^{-1} : \Omega^* \twoheadrightarrow \Omega$, which continues analytically to a rational map $W^* \to W$.
Thus $\mathscr{I}_{\mathbf{s}}$ is birational.

(4) $\mathscr{I}_{\mathbf{s}} : \Omega + iV \twoheadrightarrow \mathscr{I}_{\mathbf{s}}(\Omega + iV)$ is biholomorphic.

Remark. Bergman kernel and Szegö kernel are of the form (up to positive const.)
$\eta(z_1, z_2) = \Delta_{-\mathbf{s}}\big(w_1 + w_2^* - Q(u_1, u_2)\big)$ $(z_j = (u_j, w_j))$,
and the characteristic function of Ω is $\Delta_{-\mathbf{s}}$ for some $\mathbf{s} > 0$ (up to positive const.).

- $\mathscr{I}_{\mathbf{s}}(\Omega + iV) = \Omega^* + iV^*$

$$\iff s_1 = \cdots = s_r \text{ and } \Omega \text{ is selfdual.}$$

[Kai-N, preprint, 2003]

Cayley transform

One has $E_{\mathbf{s}}^* = \mathscr{I}_{\mathbf{s}}(E) \in \Omega^*$. $\left(1 - \frac{2}{w+1} = \frac{w-1}{w+1}\right)$

$C_{\mathbf{s}}(w) := E_{\mathbf{s}}^* - 2\,\mathscr{I}_{\mathbf{s}}(w+E)$ for tube domains

$$\mathscr{C}_{\mathbf{s}}(u,w) := \underbrace{2\,\langle Q(u,\cdot),\, \mathscr{I}_{\mathbf{s}}(w+E)\rangle}_{\in U^\dagger} \oplus \underbrace{C_{\mathbf{s}}(w)}_{\in W^*}$$

$U^\dagger$: the space of antilinear forms on U

Proposition.
(1) $\mathscr{C}_{\mathbf{s}} : D \twoheadrightarrow \mathscr{C}_{\mathbf{s}}(D)$ is birat. and biholomorphic.
(2) $\mathscr{C}_{\mathbf{s}}^{-1}$ can be written explicitly.

Theorem. [N, 2003] $\mathscr{C}_{\mathbf{s}}(D)$ is bounded (in $U^\dagger \oplus W^*$).

Remark. For general $\mathbf{s} > 0$, $\mathscr{C}_{\mathbf{s}}(D)$ for symmetric D is *not* the standard Harish-Chandra model of a non-compact Hermitian symmetric space (can be even non-convex, for example).

- $C_{\mathbf{s}}(\Omega + iV)$ is convex
  $\iff s_1 = \cdots = s_r$ and Ω is selfdual.
  [Kai-N, in writing]

Norm equality I

$\langle x|y\rangle_{\mathbf{s}}$: J-invariant inner product on $\mathfrak{g}$

$\rightsquigarrow$ Upon $G \equiv D$ by $g \mapsto g\cdot \mathrm{e}$, we have
Hermitian inner prod. on $T_{\mathrm{e}}(D) \equiv U \oplus W$

$\rightsquigarrow$ Hermitian inner product $(\cdot|\cdot)_{\mathbf{s}}$ and norm $\|\cdot\|_{\mathbf{s}}$ on the "dual' vector space $U^{\dagger} \oplus W^{*}$.

Take $\Psi_{\mathbf{s}} \in \mathfrak{g}$ so that $\mathrm{trad}\,(x) = \langle x|\Psi_{\mathbf{s}}\rangle_{\mathbf{s}}$ $(\forall x \in \mathfrak{g})$.
Then we know $\Psi_{\mathbf{s}} \in \mathfrak{a}$.

Recall that $\beta|_{\mathfrak{n}} = E^{*}_{\mathbf{c}}|_{\mathfrak{n}}$ for some $\mathbf{c} > 0$, so that $\Delta_{-\mathbf{c}}(w_1 + w_2^{*} - Q(u_1, u_2))$ is the Bergman kernel of D (up to positive const.).

Proposition. For any $g \in G$
$$\mathscr{L}_{\mathbf{s}} a_{\lambda}(g) = \lambda a_{\lambda}(g)\bigl(-\lambda\|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|^2_{\mathbf{s}} + \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{c}}\rangle\bigr).$$

Observations. (1) $a_{\lambda}(g) = a_{\lambda}(g^{-1})$ for $\forall g \in G$.
(2) B_{λ} commutes with $\mathscr{L}_{\mathbf{s}}$
$$\Longleftrightarrow \mathscr{L}_{\mathbf{s}} a_{\lambda}(g) = \mathscr{L}_{\mathbf{s}} a_{\lambda}(g^{-1}) \text{ for } \forall g \in G.$$

Therefore:
B_{λ} commutes with $\mathscr{L}_{\mathbf{s}}$
$$\Longleftrightarrow \|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|_{\mathbf{s}} = \|\mathscr{C}_{\mathbf{c}}(g^{-1}\cdot \mathrm{e})\|_{\mathbf{s}} \quad (\forall g \in G).$$

Theorem. [N, 2001]
$\|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|_{\mathbf{s}} = \|\mathscr{C}_{\mathbf{c}}(g^{-1}\cdot \mathrm{e})\|_{\mathbf{s}}$ for $\forall g \in G$
$\iff$ D is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

Since $\mathscr{C}_{\mathbf{c}}(\mathrm{e}) = 0$, this can be rephrased as:

Theorem.
$\|h\cdot 0\|_{\mathbf{s}} = \|h^{-1}\cdot 0\|_{\mathbf{s}}$ for $\forall h \in \mathscr{C}_{\mathbf{c}}\circ G\circ \mathscr{C}_{\mathbf{c}}^{-1} \iff$
$\mathscr{D} := \mathscr{C}_{\mathbf{c}}(D)$ is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

If D is symmetric, $\mathscr{D}$ is essentially the Harish-Chandra model of a non-cpt Hermitian symmetric space.

$\mathsf{G} := \mathrm{Hol}(\mathscr{D})^{\circ}$: semisimple Lie group
$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgroup of G.

Using $\mathsf{G} = \mathsf{KAK}$ with $\mathsf{A} := \mathscr{C}_{\mathbf{c}}\circ A\circ \mathscr{C}_{\mathbf{c}}^{-1}$, one can prove easily that $\|h\cdot 0\|_{\mathbf{c}} = \|h^{-1}\cdot 0\|_{\mathbf{c}}$ for any $h \in \mathsf{G}$.

The case of unit disk $\mathbb{D} \subset \mathbb{C}$

$$\mathsf{G} = SU(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \; ; \; |\alpha|^2 - |\beta|^2 = 1 \right\}$$

with $\quad g \cdot z = \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \quad (z \in \mathbb{D}).$

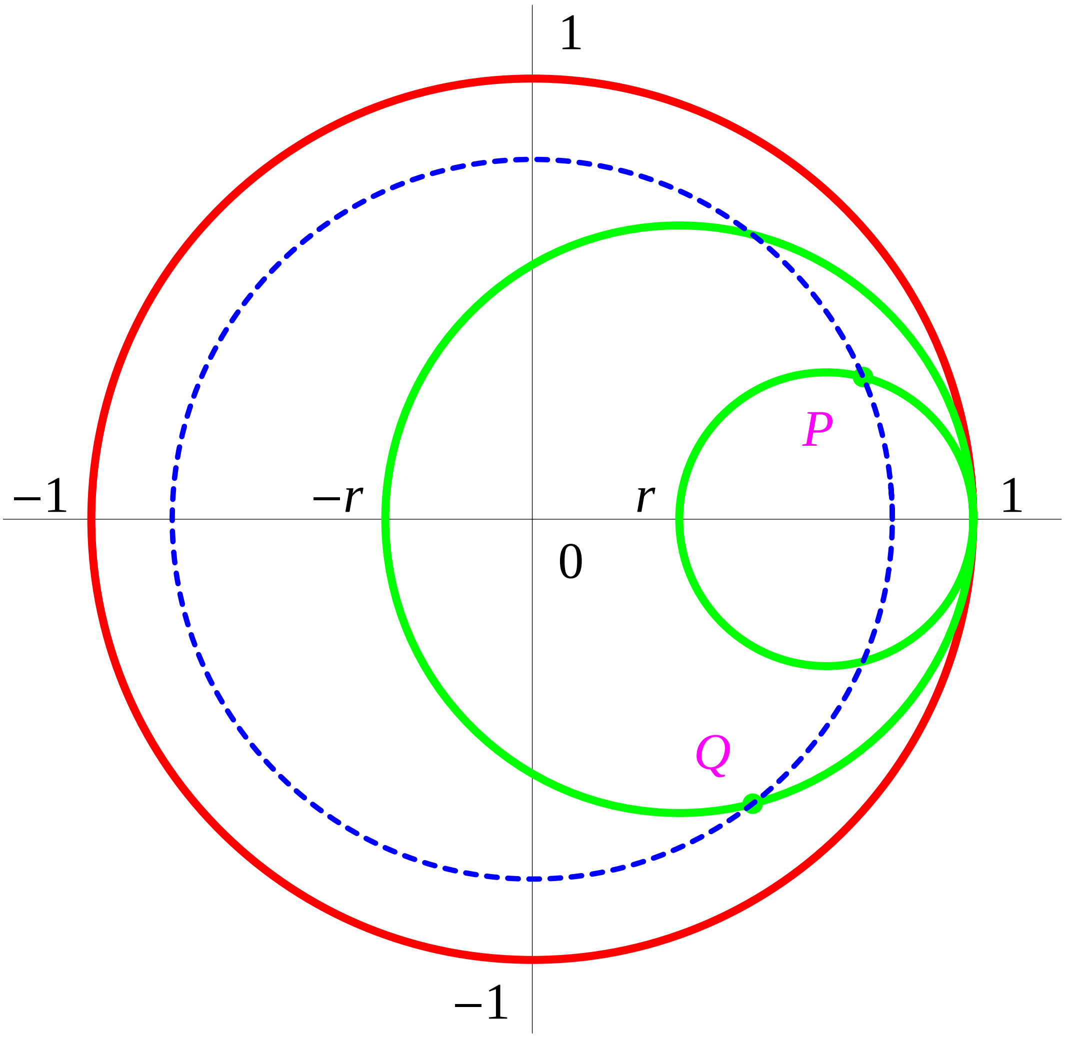
$$\begin{cases} g \cdot 0 = \dfrac{\beta}{\overline{\alpha}} \\ g^{-1} \cdot 0 = -\dfrac{\beta}{\alpha} \end{cases} \implies |g \cdot 0| = |g^{-1} \cdot 0|.$$

However, if one stays within the Iwasawa solvable subgroup, we have an interesting picture.

$$\mathsf{A} := \left\{ a_t := \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \; ; \; t \in \mathbb{R} \right\},$$

$$\mathsf{N} := \left\{ n_\xi := \begin{pmatrix} 1 - \frac{i}{2}\xi & \frac{i}{2}\xi \\ -\frac{i}{2}\xi & 1 + \frac{i}{2}\xi \end{pmatrix} \; ; \; \xi \in \mathbb{R} \right\}.$$

Then $\mathscr{C}_{\mathbf{c}} \circ G \circ \mathscr{C}_{\mathbf{c}}^{-1} = \mathsf{NA}$.



$$r := a_t \cdot 0 = \tanh(t/2)$$

P : $n_\xi a_t \cdot 0 = n_\xi \cdot r \in \mathsf{N} \cdot r$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at r.

Q : $(n_\xi a_t)^{-1} \cdot 0 = n_{-e^{-t}\xi} a_{-t} \cdot 0 = n_{-e^{-t}\xi} \cdot (-r) \in \mathsf{N} \cdot (-r)$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at $-r$.

Norm equality II

Take $\mathbf{b} > 0$ so that $\Delta_{-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2))$ is the Szegö kernel of D (up to positive const.).

Proposition.

$$\mathscr{L}_{\mathbf{s}} P_\zeta^G(e) = (-\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 + \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle) P_\zeta^G(e).$$

Remark. By $P(g \cdot z,\ \zeta) = \chi_{-\mathbf{b}}(g) P(z,\ g^{-1} \cdot \zeta)$ $(g \in G)$,

$$\mathscr{L}_{\mathbf{s}} P_\zeta^G = 0 \ \ \forall \zeta \in \Sigma \iff \mathscr{L}_{\mathbf{s}} P_\zeta^G(e) = 0 \ \ \forall \zeta \in \Sigma.$$

Therefore:

$$\mathscr{L}_{\mathbf{s}} P_\zeta^G = 0 \ \forall \zeta \in \Sigma \iff \|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle \ \forall \zeta \in \Sigma.$$

Theorem. [N, 2003]

$\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle$ for $\forall \zeta \in \Sigma$

$\iff$ D is symmetric and $\mathbf{s} = \gamma \mathbf{b}$ with $\gamma > 0$.

In this case we also have $\mathbf{s} = \gamma' \mathbf{c}$ with $\gamma' > 0$.

Recall $\mathbf{c} > 0$ is taken so that $\beta|_{\mathfrak{n}} = E_{\mathbf{c}}^*|_{\mathfrak{n}}$, where β is the Koszul form.

Validity of NE for symmetric D $(\mathbf{s} = \mathbf{c})$

D : symmetric $\Longrightarrow$ $\mathscr{D} := \mathscr{C}_{\mathbf{c}}(D)$ is the Harish-Chandra model of a Hermitian symmetric space

In particular, $\mathscr{D}$ is circular (Note $\mathscr{C}_{\mathbf{c}}(\mathrm{e}) = 0$).

$\mathsf{G} := \mathrm{Hol}(\mathscr{D})^{\circ}$: semisimple Lie gr. (with trivial center)

$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgr. of G

Circularity of $\mathscr{D}$ ($\Longrightarrow$ K is linear)
$+$ K-invariance of the Bergman metric
$\Longrightarrow$ K $\subset$ Unitary group

$$\begin{cases} \mathscr{C}_{\mathbf{c}} : \Sigma \ni 0 \mapsto -E_{\mathbf{c}}^{*}, \\ \text{Shilov boundary } \Sigma_{\mathscr{D}} \text{ of } \mathscr{D} = \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}). \end{cases}$$

Since $\Sigma_{\mathscr{D}}$ is also a G-orbit $\Sigma_{\mathscr{D}} = \mathsf{G} \cdot (-E_{\mathbf{c}}^{*})$ and since Σ is an orbit of a nilpotent subgroup of $G \subset \mathrm{Hol}(D)^{\circ}$, we get

$$\begin{aligned} \mathscr{C}_{\mathbf{c}}(\Sigma) \subset \mathsf{G} \cdot (-E_{\mathbf{c}}^{*}) &= \Sigma_{\mathscr{D}} \\ &= \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}) \\ &\subset \{z \,;\, \|z\|_{\mathbf{c}} = \|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}\}. \end{aligned}$$

We see easily that $\|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}^{2} = \langle \Psi_{\mathbf{c}}, \alpha_{\mathbf{b}} \rangle$ in this case (because $\mathbf{b}$ is a multiple of $\mathbf{c}$).

Norm equality $\Longrightarrow$ symmetry of D

Assumption :

(i) $\|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|_{\mathbf{s}} = \|\mathscr{C}_{\mathbf{c}}(g^{-1}\cdot \mathrm{e})\|_{\mathbf{s}}$ for $\forall g \in G$.
or
(ii) $\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}}\rangle$ for $\forall \zeta \in \Sigma$.

What we do is substitute specific $g \in G$ in (i) (resp. $\zeta \in \Sigma$ in (ii)) and extract informations.

(1) Reduction to a quasisymmetric domain

κ : the Bergman kernel of D
Recall that $\kappa(z_1, z_2) = \Delta_{-\mathbf{c}}(w_1 + w_2^* - Q(u_1, u_2))$
(up to positive const.).

If $x, y \in V$, define $\langle x | y \rangle_\kappa := D_x D_y \log \Delta_{-\mathbf{c}}(E)$.

> **Definition.** $D = D(\Omega, Q)$ is *quasisymmetric*
>
> $\underset{\text{def}}{\Longleftrightarrow}$ Ω is selfdual w.r.t. $\langle \cdot | \cdot \rangle_\kappa$.

Define a non-associative product xy in V by

$$\langle xy | z \rangle_\kappa = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{-\mathbf{c}}(E).$$

Prop. (Dorfmeister-D'Atri-Dotti-Vinberg)

D is quasisymmetric $\iff$ product xy is Jordan.

In this case, V is a Euclidean Jordan algebra.

My tool is the following

Proposition. (D'Atri-Dotti) D : irreducible.

D is quasisymmetric

$$\iff \begin{cases} (1) \ \ \dim \mathfrak{g}_{(\alpha_k+\alpha_j)/2} \text{ is indep. of } j,k, \\ (2) \ \ \dim \mathfrak{g}_{\alpha_k/2} \text{ is indep. of } k. \end{cases}$$

Extend $\langle \cdot \,|\, \cdot \rangle_\kappa$ to a $\mathbb{C}$-bilinear form on $W \times W$.

$(u_1 \,|\, u_2)_\kappa := \langle Q(u_1, u_2) \,|\, E \rangle_\kappa$
defines a Hermitian inner product on U.

For each $w \in W$, define $\varphi(w) \in \mathrm{End}_{\mathbb{C}}(U)$ by

$$(\varphi(w)u_1 \,|\, u_2)_\kappa = \langle Q(u_1, u_2) \,|\, w \rangle_\kappa.$$

Clearly $\varphi(E) =$ identity operator on U.

Proposition. (Dorfmeister). D is quasisymm.
$\Longrightarrow$ $w \mapsto \varphi(w)$ is a Jordan $*$-repre. of $W = V_{\mathbb{C}}$

$$\begin{cases} \varphi(w^*) = \varphi(w)^*, \\ \varphi(w_1 w_2) = \frac{1}{2}\big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\big). \end{cases}$$

(2) Reduction : quasisymmetric $\Longrightarrow$ symmetric

Quasisymmetric Siegel domain

$$\leftrightarrow \begin{cases} \text{Euclidean Jordan algebra } V \text{ and} \\ \text{Jordan } *\text{-representation } \varphi \text{ of } W = V_{\mathbb{C}}. \end{cases}$$

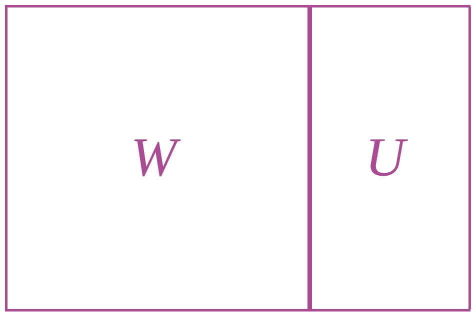
Symmetric Siegel domain
$\leftrightarrow$ Positive Hermitian JTS

The following strange formula fills the gap:

$$\varphi(w)\varphi(Q(u,u'))u = \varphi(Q(\varphi(w)u,u'))u,$$

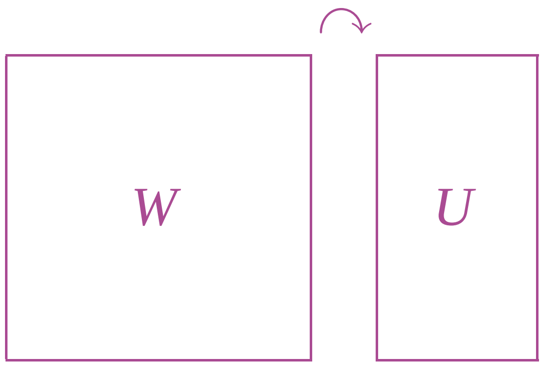
where $u, u' \in U$ and $w \in W$.

$Z = W \oplus U$



complex semisimple Jordan algebra
$W = V_{\mathbb{C}}$
with V Euclidean JA

natural action



Jordan algebra $*$-repre. of W

> **Proposition.** (Satake) Quasisymm. D is symm.
> $\Longleftrightarrow$ V and φ come from a positive Hermitian JTS this way.

Definition of triple product: $z_j = (u_j, w_j)$ $(j = 1, 2, 3)$,
$\{z_1, z_2, z_3\} := (u, w)$, where

$$\begin{aligned}
u &:= \tfrac{1}{2}\varphi(w_3)\varphi(w_2^*)u_1 + \tfrac{1}{2}\varphi(w_1)\varphi(w_2^*)u_3 \\
&\quad + \tfrac{1}{2}\varphi(Q(u_1, u_2))u_3 + \tfrac{1}{2}\varphi(Q(u_3, u_2))u_1, \\
w &:= (w_1 w_2^*)w_3 + w_1(w_2^* w_3) - w_2^*(w_1 w_3) \\
&\quad + \tfrac{1}{2}Q(u_1, \varphi(w_3^*)u_2) + \tfrac{1}{2}Q(u_3, \varphi(w_1^*)u_2).
\end{aligned}$$

Proposition. (Dorfmeister)

Irreducible quasisymmetric D is symmetric

$\iff$ $\exists f_1, \ldots, f_r$: Jordan frame of V s.t.

with $U_k := \varphi(f_k)U$ we have

$$\varphi(Q(u_1, u_2))u_1 = 0$$

for $\forall u_1 \in U_1$ and $\forall u_2 \in U_2$.

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