# Cayley Transforms

# of

# a Homogeneous Siegel Domain

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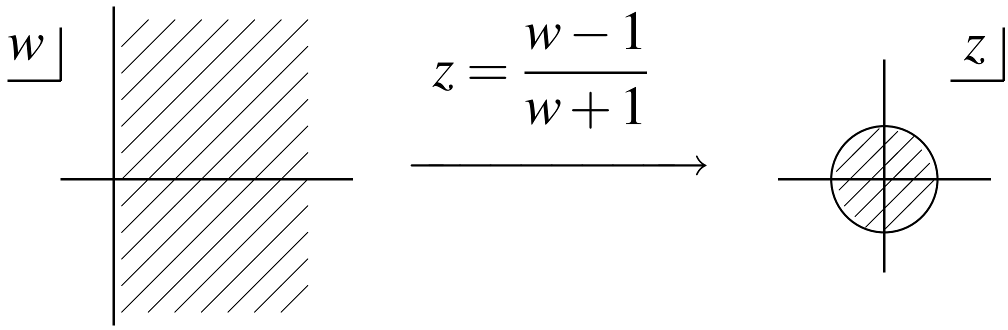
# Examples of Cayley transform:

(I)




(II) $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Pos}(r, \mathbb{R})$, $\Omega + iV$

$z = (w - E)(w + E)^{-1}$ ($E$ : unit matrix)

| Siegel right half space $\ni w \mapsto z \in$ Siegel disk |
|---|

$\{z \in \mathrm{Sym}(r, \mathbb{C})\ ;\ E - zz^* \gg 0\}$.

(III) General symmetric tube domain

$\Omega + iV$ ($\Omega$: a selfdual open convex cone in $V$)

$V$ (hence $V_{\mathbb{C}}$) can be equipped with JA structure.

$z = (w - e)(w + e)^{-1}$ ($e$: the unit elemt. in $V$)

| Symm. tube domain $\longrightarrow$ Open unit ball (w.r.t some norm) |
|---|

Siegel disk $= \{z \in \mathrm{Sym}(r, \mathbb{C})\ ;\ \|z\|_{\mathrm{op}} < 1\}$.

For $\mathrm{Sym}(r, \mathbb{C})$, JA product is:

$$\begin{cases} A \circ B = \frac{1}{2}(AB + BA), \\ \text{JA inverse} = \text{inverse matrix.} \end{cases}$$

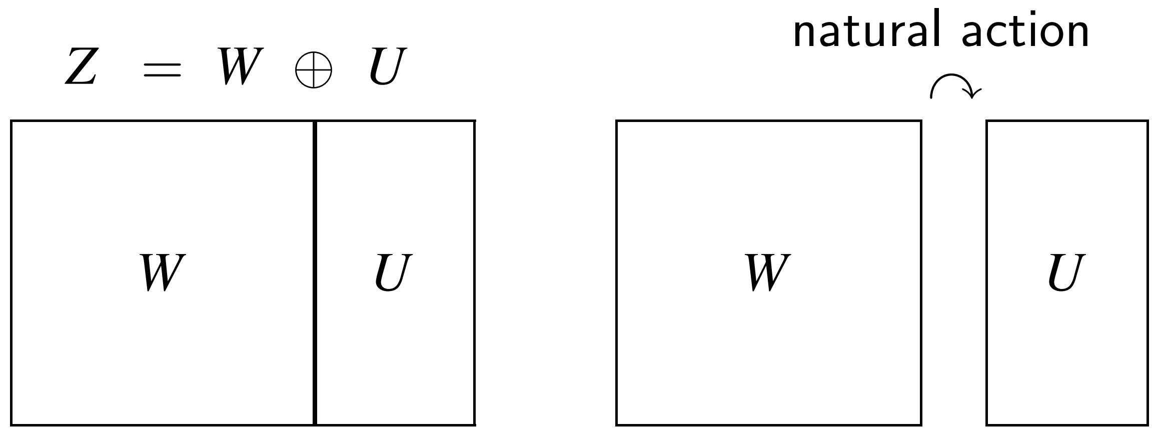
Thus $(w - e) \circ (w + e)^{-1} = (w - e)(w + e)^{-1}$.

(IV) $D := \{(u,w) \in \mathbb{C}^m \times \mathbb{C} \,;\, w + \overline{w} - 2\|u\|^2 > 0\}$.
(rank 1 (symmetric) Siegel domain)

$$\mathscr{C}(u,w) := \left( \frac{2u}{w+1}, \frac{w-1}{w+1} \right)$$

$\mathscr{C} : D \longrightarrow$ open unit ball in $\mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}$

(V) <u>General symmetric Siegel domain $D \subset Z$</u>




complex semisimple Jordan algebra $W = V_{\mathbb{C}}$ with $V$ Euclidean JA

Jordan algebra $*$-repre. $\varphi$ of $W$

$$\mathscr{C}(u,w) := (2\varphi(w+e)^{-1}u,\, (w-e)(w+e)^{-1})$$

$D \longrightarrow$ Open unit ball (w.r.t spectral norm)

↼ Harish–Chandra realization of a non-cpt Hermitian symm. space

For general Siegel domain,

one first needs something like $(w+e)^{-1}$. Then

$$(w-e)(w+e)^{-1} = e - 2(w+e)^{-1}.$$

Recall

- $-\dfrac{d}{dt}\log\det(x+tv)^{-1}\Big|_{t=0} = \mathrm{tr}(x^{-1}v)$

$$(x \in \mathrm{Pos}(r,\mathbb{R}),\ v \in \mathrm{Sym}(r,\mathbb{R}))$$

- $\mathrm{tr}(xy)$ is the inner product to identify $\mathrm{Sym}(r,\mathbb{R})$ with the dual vector space $\mathrm{Sym}(r,\mathbb{R})^*$ so that the cone $\mathrm{Pos}(r,\mathbb{R})$ coincides with its dual cone.

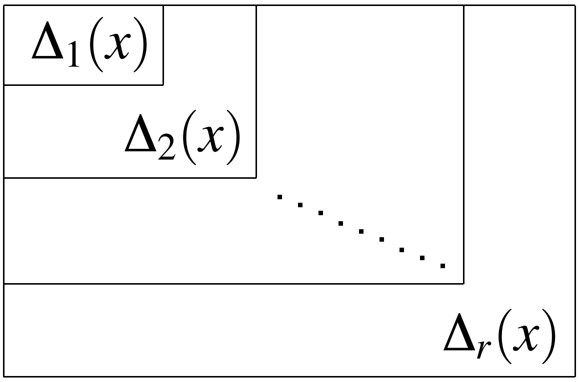
# Generalization of determinant functions

(compound power functions by Gindikin)

**Example.** $\mathrm{Pos}(r,\mathbb{R})$: For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$

$\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\cdots\Delta_r(x)^{s_r}$, where

$x \mapsto$




$\Delta_1(x) := x_{11}$

$\Delta_2(x) := x_{11}x_{22} - x_{21}^2$

$\vdots$

$\Delta_r(x) := \det x$

$\Omega$: a homogeneous regular open convex cone $\subset V$

- homogeneous $\overset{\text{def}}{\Longleftrightarrow}$ $G(\Omega) := \{g \in GL(V)\,;\, g\Omega = \Omega\} \curvearrowright \Omega$ transitively.
- regular $\overset{\text{def}}{\Longleftrightarrow}$ $\Omega$ contains no entire line.

$\exists H$ : split solvable $\subset G(\Omega)$ such that
$H \curvearrowright \Omega$ simply transitively.

Fix $E \in \Omega$. Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.
For each group hom. $\chi : H \longrightarrow \mathbb{R}_{>0}$, define a function $\Delta_\chi$ on $\Omega$ by

$$\Delta_\chi(hE) := \chi(h) \qquad (h \in H).$$

Recall $\det x \to 0$ if $\mathrm{Pos}(r,\mathbb{R}) \ni x \to x_0 \in \partial\mathrm{Pos}(r,\mathbb{R})$.

> **Definition.** $\chi : H \to \mathbb{R}_{>0}$ is <u>admissible</u>
> $\overset{\text{def}}{\iff}$ $\Delta_\chi(x) \to 0$ if $\Omega \ni x \to x_0 \in \partial\Omega$.

**<u>Example.</u>** $\Omega = \mathrm{Pos}(2,\mathbb{R})$, $\Delta_{\mathbf{s}}(x) = x_{11}^{s_1-s_2}(\det x)^{s_2}$.
If $\mathbf{s}$ is admissible, then considering $x = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$,
we have necessarily $s_1 > 0$, $s_2 > 0$.

Conversely, suppose $s_1 > 0$, $s_2 > 0$. Then, from

$$x = \begin{pmatrix} x_{11} & x_{21} \\ x_{21} & x_{22} \end{pmatrix} \in \Omega, \quad h = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad x = h\,{}^t h,$$

we get $x_{11} = a^2$, $x_{21} = ab$, $x_{22} = b^2 + c^2$, so that

$$\Delta_{\mathbf{s}}(x) = (a^2)^{s_1-s_2}(a^2c^2)^{s_2} = a^{2s_1}c^{2s_2}.$$

If $x \to x_0 \in \partial\Omega$, then $x_{11} = a^2$ and $x_{22} = b^2 + c^2$ (hence $c$) remain bounded. Then, by

$$\Delta_{\mathbf{s}}(x) = \begin{cases} a^{2(s_1-s_2)}(\det x)^{s_2} & (s_1 \geqq s_2), \\ c^{2(s_2-s_1)}(\det x)^{s_1} & (s_1 < s_2), \end{cases}$$

we see that $\mathbf{s} = (s_1, s_2)$ is admissible.

For each admissible $\chi$, define $I_\chi(x) \in V^*$ $(x \in \Omega)$ by

$$\langle v, I_\chi(x) \rangle = -\frac{d}{dt} \log \Delta_\chi(x+tv)^{-1}\Big|_{t=0} \quad (v \in V).$$

**Example.** $\phi$: the characteristic ftn of $\Omega$, i.e.,

$$\phi(x) := \int_{\Omega^*} e^{-\langle x, \lambda \rangle} \, d\lambda \quad (x \in \Omega),$$

where $\Omega^*$ is the dual cone of $\Omega$:

$$\Omega^* := \left\{ \lambda \in V^* \,;\, \langle x, \lambda \rangle > 0 \ \forall x \in \overline{\Omega} \setminus \{0\} \right\}.$$

Then, $\phi$ is of the form

$$\phi(x) = C \Delta_{\chi_0}(x)^{-1} \quad (C > 0 : \text{const.})$$

for some admissible $\chi_0$. (Indeed $\chi_0(h) = \mathrm{Det}_V h$.)
Then $I_{\chi_0}(x) = x^*$, the $*$-map defined by Vinberg. $\square$

**Facts.** (1) $I_\chi : \Omega \to \Omega^*$ is a bijection.
(2) $I_\chi$ has an analytic continuation to a birational map $V_{\mathbb{C}} =: W \to W^*$.
(3) $I_\chi$ is holomorphic on $\Omega + iV$.

**Theorem** (Kai-N. 2005).
$I_\chi(\Omega + iV) = \Omega^* + iV^*$
$\iff$ $\Omega$ is selfdual and $\Delta_\chi(x) = \det(x)^\lambda$.
($\lambda > 0$ and det is the JA alg. determinant ftn.)

$\Omega$ is selfdual $\overset{\text{def}}{\iff}$ $\Omega = \Omega^*$ by some inner product through which we identify $V^*$ with $V$.

**Example.**

$$\big(\mathrm{Pos}(r,\mathbb{R}) + i\,\mathrm{Sym}(r,\mathbb{R})\big)^{-1} = \mathrm{Pos}(r,\mathbb{R}) + i\,\mathrm{Sym}(r,\mathbb{R}).$$

**Cayley transform for tube domains:**

For admissible $\chi$, we define

$$C_\chi(w) := I_\chi(E) - 2I_\chi(w+E) \quad (w \in \Omega + iV).$$

**Remark.** If $\chi$ is admissible, then one can define an inner product in $V$ by

$$\langle v_1 \,|\, v_2 \rangle_\chi := D_{v_1} D_{v_2} \log \Delta_\chi^{-1}(E).$$

($D_v f(x) := \frac{d}{dt} f(x+tv)\big|_{t=0}$). If we use this $\langle \cdot \,|\, \cdot \rangle_\chi$ to identify $V^*$ with $V$, then $I_\chi(E) = E$.

## **Siegel domains.** — Definition —

$V$ : a real vector space ($\dim V < \infty$)
$\cup$
$\Omega$ : a regular open convex cone
$W := V_{\mathbb{C}}$ $\quad$ ($w \mapsto w^*$ : conjugation w.r.t. $V$)

$U$ : another complex vector space ($\dim U < \infty$)

$Q : U \times U \to W$, Hermitian sesquilinear $\Omega$-positive

*i.e.,* $\begin{cases} Q(u',u) = Q(u,u')^* \\ Q(u,u) \in \overline{\Omega} \setminus \{0\} \ \ (0 \neq \forall u \in U) \end{cases}$

**Siegel domain (of type II)**

$$D := \left\{ (u,w) \in U \times W \; ; \; w + w^* - Q(u,u) \in \Omega \right\}$$

• $U = \{0\}$ is allowed. In this case $D = \Omega + iV$.

Assume that $D$ is homogeneous

*i.e.,* $\mathrm{Hol}(D) \curvearrowright D$ transitively.

Then $\Omega$ is also homogeneous.

## Cayley transform for homogeneous Siegel domains

For admissible $\chi$,

$$\mathscr{C}_\chi(u,w) := 2\langle Q(u,\cdot), I_\chi(w+E)\rangle \oplus C_\chi(w) \quad ((u,w)\in D).$$

Note $U \ni u' \mapsto \langle Q(u,u'), I_\chi(w+E)\rangle$ is $\mathbb{C}$ anti-linear.

Thus $\mathscr{C}_\chi(u,w) \in U^\dagger \oplus W^*$.

($U^\dagger$ : the space of anti-linear forms on $U$)

**Theorem** (N. 2003). $\mathscr{C}_\chi(D)$ is bounded.

To give examples we need:

**Lemma** (Gindikin 1975, Ishi 2000). For any $\chi$, the function $\Delta_\chi$ extends holomorphically to $\Omega + iV$.

(Laplace trsfm of a distribution supported by $\overline{\Omega^*}$.)

**Examples.** $K$: Bergman kernel, $S$: Szegö kernel
Then for $z_j = (u_j, w_j) \in D$ $(j = 1, 2)$, we have

$$K(z_1, z_2) = \Delta_{\chi_1}(w_1 + w_2^* - Q(u_1, u_2))^{-1},$$
$$S(z_1, z_2) = \Delta_{\chi_2}(w_1 + w_2^* - Q(u_1, u_2))^{-1},$$

upto positive # multiples for some admissible $\chi_1$, $\chi_2$.
Then we have the corresponding Cayley transforms:

$$\mathscr{C}_{\chi_1}, \quad \mathscr{C}_{\chi_2}.$$

If $\chi = \chi_0$ is s.t. $\phi = C\Delta_{\chi_0}^{-1}$, then the Cayley transform $\mathscr{C}_{\chi_0}$ is the one introduced by Penney in 1996.

**Remark.** $\langle \cdot | \cdot \rangle_\chi$: inner product in $V$ defined earlier
$\rightsquigarrow \mathbb{C}$ bilinear form on $W$.
$(u_1 \,|\, u_2)_\chi := \langle Q(u_1, u_2) \,|\, E \rangle_\chi$ defines a Hermitian inner product in $U$.
Let us use these to identify $U^\dagger \oplus W^*$ with $U \oplus W$.
Moreover, for $w \in W$, define an operator $\varphi_\chi(w)$ by

$$(\varphi_\chi(w)u_1 \,|\, u_2)_\chi = \langle Q(u_1, u_2) \,|\, w \rangle_\chi.$$

Then, our Cayley transform $\mathscr{C}_\chi$ can be rewritten as

$$\mathscr{C}_\chi(u, w) = \big(2\varphi_\chi(I_\chi(w + E))u, C_\chi(w)\big).$$

**Example.** <u>The case of quasisymmetric Siegel domains</u>
Suppose $\chi = \chi_1$ : the Bergman parameter.
Then, $D$ is <u>quasisymmetric</u>

$$\overset{\text{def}}{\Longleftrightarrow} \ \Omega \text{ is selfdual w.r.t } \langle \cdot | \cdot \rangle_{\chi_1}.$$

Suppose $D$ quasisymmetric.
Then $V$ has a structure of Euclidean Jordan algebra ($\langle \cdot | \cdot \rangle_{\chi_1}$ is an associative inner product), so that we have $I_{\chi_1}(w) = w^{-1}$, the JA inverse. Moreover

$$\varphi_{\chi_1} : W \ni w \mapsto \varphi_{\chi_1}(w) \in \mathrm{End}_{\mathbb{C}}(U)$$

is a $*$ JA representation of $W$:

$$\begin{cases} \varphi_{\chi_1}(w^*) = \varphi_{\chi_1}(w)^*, \\ \varphi_{\chi_1}(w_1 w_2) = \frac{1}{2}\big(\varphi_{\chi_1}(w_1)\varphi_{\chi_1}(w_2) + \varphi_{\chi_1}(w_2)\varphi_{\chi_1}(w_1)\big). \end{cases}$$

Then

$$\mathscr{C}_{\chi_1}(u, w) = (2\varphi_{\chi_1}(w+E)^{-1}u, \, (w-E)(w+E)^{-1}).$$

This coincides with Dorfmeister's Cayley transform for quasisymmetric Siegel domains introduced in 1980.

Moreover, if $D$ is symmetric, then $\mathscr{C}_{\chi_1}$ coincides with (the inverse map) of the Cayley transform defined by Korány–Wolf in 1965. $\square$

**Theorem** (Kai, preprint 2006)**.** $\mathscr{C}_\chi(D)$ is convex

$$\iff \begin{cases} D \text{ is symmetric,} \\ \chi \text{ is a positive power of } \chi_1. \end{cases}$$

**Remark.** If $D$ is symmetric, then $C_{\chi_1}(D)$ is the Harish-Chandra model of a noncompact Hermitian symmetric space. Thus it is the open unit ball w.r.t a certain norm (the spectral norm of the underlying JTS), so that it is convex.

# Application to Poisson–Hua kernel

$S$ : the Szegö kernel of $D$,
$\Sigma$ : the Shilov boundary of $D$.
One knows

$$\Sigma = \{(u,w) \; ; \; w + w^* - Q(u,u) = 0\}$$

The explicit expression through $\Delta_{\chi_2}$ assures us that $S(z,\zeta)$ $(z \in D, \, \zeta \in \Sigma)$ still has a meaning.

The Poisson–Hua kernel is defined to be

$$P(z,\zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in D, \, \zeta \in \Sigma).$$

**Theorem** (Hua–Look–Korányi–Xu)**.**
$\mathscr{L}$ : Laplace–Beltrami operator on $D$
(w.r.t. the Bergman metric)
Then

$$\mathscr{L}P(\cdot,\zeta) = 0 \;\; (\forall \zeta \in \Sigma) \Longleftrightarrow D \text{ is symmetric.}$$

**Remark.** $\Longleftarrow$ Hua–Look (classical domains, 1959),
Korányi (general 1965).
$\Longrightarrow$ Xu (difficult computation, 1979).

## Interpretation through Cayley transform

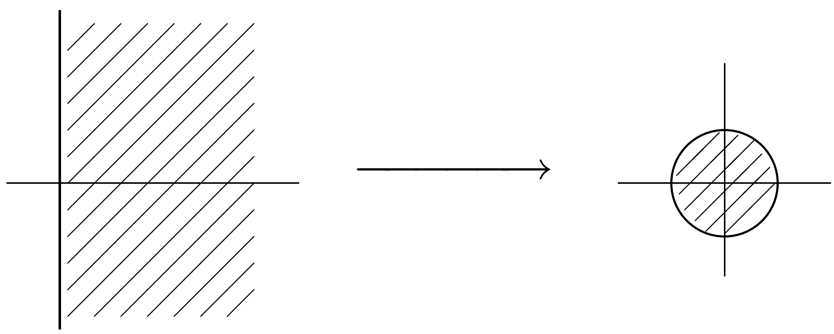
$\exists G \subset \mathrm{Aff}_{\mathbb{C}}(D)$ s.t. $G \curvearrowright D$ simply transitively.
Fix $\mathsf{e} := (0, E) \in D$.
Then $G \approx D$ (diffeo) by $g \mapsto g \cdot \mathsf{e}$.
$p_\zeta(g) := P(g \cdot \mathsf{e}, \zeta)$ $(g \in G)$.
$\mathscr{L}^G$ : the L.–B. operator on $G$ corresponding to $\mathscr{L}$.

**Key formula.**

$$\mathscr{L}^G p_\zeta = \left(-\|\mathscr{C}_{\chi_2}(\zeta)\|^2 + c_2\right) p_\zeta.$$

$\mathscr{C}_{\chi_2}$ : the Cayley transform with Szegö parameter $\chi_2$.
$\|\cdot\|$ : the norm on the tangent space $T_0(\mathscr{C}_{\chi_2}(D))$
def'd by the Bergman metric (note $\mathscr{C}_{\chi_2}(\mathsf{e}) = 0$).
$c_2$ : explicitly given const.$(> 0)$ independent of $\zeta$.

Therefore
$\mathscr{L}^G p_\zeta = 0$ $(\forall \zeta \in \Sigma) \iff \|\mathscr{C}_{\chi_2}(\zeta)\|^2 = c_2$ $(\forall \zeta \in \Sigma)$.



**Theorem** (N. 2003)**.**
$\|\mathscr{C}_{\chi_2}(\zeta)\|^2 = c_2\, (\forall \zeta \in \Sigma) \iff D$ is symmetric.

**Remark.** If $D$ is symmetric, then $\chi_2$ is a positive power of $\chi_1$, so that $\mathscr{D} := \mathscr{C}_{\chi_1}(D)$ is essentially the Harish-Chandra model of a Hermitian symmetric space. Since

$$\mathscr{C}_{\chi_2}(\Sigma) \subset \text{Shilov boundary } \Sigma_{\mathscr{D}} \text{ of } \mathscr{D}$$

and since

$$\Sigma_{\mathscr{D}} \text{ lies on the sphere mentioned in Theorem,}$$

we have $\Longleftarrow$ .

**Example.** Non-symmetric 4-dim. Siegel domain

$V = \mathrm{Sym}(2,\mathbb{R})$, $\Omega := \mathrm{Pos}(2,\mathbb{R})$.
Then $W := V_{\mathbb{C}} = \mathrm{Sym}(2,\mathbb{C})$. $U = \mathbb{C}$.

$$Q(u_1,u_2) := 2\begin{pmatrix} 0 & 0 \\ 0 & u_1\overline{u}_2 \end{pmatrix}.$$

Then $Q : U \times U \to W$, Hermitian and $\Omega$-positive.

$$\begin{aligned} D &= \{(u,w)\ ;\ w+w^* - Q(u,u) \in \Omega\} \\ &= \left\{ (u,w_1,w_2,w_3) \in \mathbb{C}^4\ ;\ \begin{matrix} v_1(v_3 - |u|^2) - v_2^2 > 0, \\ v_3 > |u|^2 \end{matrix} \right\} \\ &\qquad\qquad\qquad\qquad (v_j := \mathrm{Re}\, w_j). \end{aligned}$$

$\chi$ : admissible positive character of

$$H := \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\ ;\ a>0, c>0, b \in \mathbb{R} \right\}.$$

The inner product $\langle \cdot | \cdot \rangle_\chi$ in $V$ is expressed as

$$\langle v | v' \rangle_{\mathbf{s}} := s_1 v_{11} v'_{11} + 2 s_2 v_{21} v'_{21} + s_2 v_{22} v'_{22} \quad (s_j > 0).$$

Using $\langle \cdot | \cdot \rangle_{\mathbf{s}}$ to identify $V^*$ with $V$, we get

$$\Omega^* = \left\{ \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}\ ;\ y_1 y_3 - \frac{s_2}{s_1} y_2^2 > 0,\ y_3 > 0 \right\},$$

$$I_{\mathbf{s}}(w) = \frac{1}{\det w} \begin{pmatrix} w_3 - s_1^{-1}(s_1 - s_2) w_1^{-1} w_2^2 & -w_2 \\ -w_2 & w_1 \end{pmatrix}.$$

If one uses $\operatorname{tr}(vv')$ to identify $V^*$ with $V$, then $\Omega^* = \Omega$ and

$$I^0_{\mathbf{s}}(w) = \frac{1}{\det w}\begin{pmatrix} s_1 w_3 - (s_1 - s_2) w_1^{-1} w_2^2 & -s_2 w_2 \\ -s_2 w_2 & s_2 w_1 \end{pmatrix}.$$

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Geatti's parameter (1982): $s_1 = s_2 \rightsquigarrow I_{\mathbf{s}}(w) = w^{-1}$,

Bergman parameter: $s_1 = 3$, $s_2 = 4$,

($\rightsquigarrow D$ is not even quasisymmetric).

Szegö parameter: $s_1 = \frac{3}{2}$, $s_2 = \frac{5}{2}$,

Penney's parameter: $s_1 = s_2 = \frac{3}{2}$.

Geatti's Cayley transform ($\equiv$ Penney's)

$$\mathscr{C}(u,w) = \left(\frac{2(w_1+1)}{\det(w+E)}u,\, (w-E)(w+E)^{-1}\right).$$

But CT with Bergman param. and Szegö param. both look strange:

$$\mathscr{C}_\chi(u,w) = \left(\frac{2(w_1+1)}{\det(w+E)}u,\, E - 2I_{\mathbf{s}}(w+E)\right).$$

Nevertheless CT with Szegö parameter appeared in my analysis of Poisson–Hua kernel.