Homogeneous Convex Cones Associated to Representations of Euclidean Jordan Algebras

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Euclidean Jordan Algebras

- V with a bilinear product xy is called a **Jordan algebra** if for all $x, y \in V$
  (1) $xy = yx$,
  (2) $x^2(xy) = x(x^2y)$.

- A real Jordan algebra with e_0 is said to be **Euclidean** if $\exists\langle \cdot \,|\, \cdot \rangle$ s.t.
$$\langle\, xy \,|\, z \,\rangle = \langle\, x \,|\, yz \,\rangle \qquad (\forall x, y)$$

- Euclidean Jordan algebras V $\rightleftarrows$ symmetric cones $\Omega = \mathrm{Int}\{x^2\ ;\ x \in V\}$

Example:

$V = \mathrm{Sym}(r, \mathbb{R}) \supset \Omega := \mathrm{Sym}(r, \mathbb{R})^{++}$
Jordan product $\circ$ of V: $x \circ y := \frac{1}{2}(xy + yx)$; note $x \circ x = x^2$.
$GL(r, \mathbb{R}) \curvearrowright \Omega$ by $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx\,{}^tg \in \Omega$ transitively

- More generally

  V: a real vector space with an inner product $\langle \cdot \,|\, \cdot \rangle$ $(\dim V < \infty)$

  $V \supset \Omega$: a regular open convex cone (contains no entire line)

  $G(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}$: linear automorphism group of Ω

  (a Lie group as a closed subgroup of $GL(V)$)

  Ω is **homogeneous** $\overset{\text{def}}{\Longleftrightarrow}$ $G(\Omega) \curvearrowright \Omega$ is transitive

- **dual cone** Ω^* of Ω (w.r.t $\langle \cdot \,|\, \cdot \rangle$)

  $$\overset{\text{def}}{\Longleftrightarrow} \quad \Omega^* := \left\{ y \in V\ ;\ \langle x \,|\, y \rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \right\}$$

- Ω is **selfdual** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \langle \cdot \,|\, \cdot \rangle$ s.t. $\Omega = \Omega^*$
- **symmetric cone** $\overset{\text{def}}{\Longleftrightarrow}$ homogeneous selfdual open convex cone

List of irreducible symmetric cones and Eulidean Jordan algebras:

- $\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$
- $\Omega = \mathrm{Herm}(r, \mathbb{C})^{++} \subset V = \mathrm{Herm}(r, \mathbb{C})$
- $\Omega = \mathrm{Herm}(r, \mathbb{H})^{++} \subset V = \mathrm{Herm}(r, \mathbb{H})$
- $\Omega = \mathrm{Herm}(3, \mathbb{O})^{++} \subset V = \mathrm{Herm}(3, \mathbb{O})$
- $\Omega = \Lambda_n$ (n-dimensional Lorentz cone) $\subset V = \mathbb{R}^n$

Selfadjoint representations of Euclidean Jordan algebras

V: a Euclidean Jordan algebra with unit element e_0
E: a real vector space with $\langle \cdot | \cdot \rangle_E$

- linear map $\varphi : V \to \mathrm{End}(E)$ is a **selfadjoint representation** of V

$$\overset{\text{def}}{\iff} \begin{cases} (1)\ \varphi(x) \in \mathrm{Sym}(E) \quad \text{for } \forall x \in V, \\ (2)\ \varphi(xy) = \frac{1}{2}\big(\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\big), \quad \varphi(e_0) = I \ (\text{if } \varphi \neq 0) \end{cases}$$

- $V = \mathrm{Herm}(3, \mathbb{O}) \implies \varphi = 0$
- $V = \mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

  $\implies E = \mathrm{Mat}(r \times p, \mathbb{K})$ and $\varphi(x)\xi = x\xi$ $(x \in V,\ \xi \in E)$
- V: Lorentzian type $\implies V = \mathbb{R}e_0 \oplus W$ with (W, B): Euclidean VS.

  Jordan algebra representation of $V \rightleftarrows$ Clifford algebra represnetation of $\mathrm{Cl}(W)$

  $w^2 = B(w, w)$

  In fact $V \hookrightarrow \mathrm{Cl}(W)$

Basic relative invariants

Ω: a reguler homogeneous open convex cone $\subset V$,
$G(\Omega)$: the linear automorphism group of Ω,
$\exists H$: a split solvable subgroup of $G(\Omega)$ s.t. $H \curvearrowright \Omega$ simply transitively.
• a function f on Ω is **relatively invariant** (w.r.t. H)
$\overset{\text{def}}{\iff}$ $\exists \chi$: 1-dim. rep. of H s.t. $f(hx) = \chi(h) f(x)$ (for all $h \in H$, $x \in \Omega$).

Theorem [Ishi 2001].
$\exists \Delta_1, \ldots, \Delta_r$ $(r := \mathrm{rank}(\Omega))$: relat. inv. irred. polynomial functions on V s.t.
any relat. inv. polynomial function P on V is written as

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \text{const.},\ (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

• $\Delta_1(x), \ldots, \Delta_r(x)$: the **basic relative invariants** associated to Ω

Algebras for general homogeneous convex domains (Vinberg 1963)

• V with a bilinear product $x \triangle y = L(x)y = R(y)x$ is called a **Clan** if
(1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ (**l**eft symmetric **a**lgebra),
(2) $\exists s \in V^*$ (called admissible) s.t. $s(x \triangle y)$ defines an inner product (**c**ompact),
(3) Each $L(x)$ has only real eigenvalues (**n**ormal).

Affine homogeneous open convex domains $\rightleftarrows$ Clans
Homogeneous open convex cones $\rightleftarrows$ Clans with unit element

$\Omega \rightleftarrows V$: algebraic structure in the ambient VS ($\equiv$ tangent space at a ref. pt.)

- Case of homogeneous convex cones Ω:

  Fix $E \in \Omega$ and H: simply transitive on $\Omega \rightsquigarrow H \approx HE = \Omega$ (diffeo)
  $\rightsquigarrow \mathfrak{h} := \mathrm{Lie}(H) \cong T_E(\Omega) \equiv V$ (linear isomorphism)
  $\rightsquigarrow \forall x \in V$, $\exists 1 X \in \mathfrak{h}$ s.t. $XE = x$.
  $\rightsquigarrow$ Write $X = L(x)$ and define $x \triangle y := L(x)y$
  (The clan product is non-commutative, in general.)

Theorem [Ishi–N. 2008].
$R(x)$: the right multiplication operator by x in the clan V: $R(x)y := yx$
$\Longrightarrow$ the irreducible factors of $\det R(x)$ are just $\Delta_1(x), \ldots, \Delta_r(x)$.

Flowchart of this work

V: a simple Euclidean Jordan algebra
(φ, E): a selfadjoint representation of V
$\rightsquigarrow$ Define a clan structure in $V_E := E \oplus V$
(V_E does not have a unit element unless $E = \{0\}$.)
$\rightsquigarrow$ Adjoin a unit element to V_E, and get a clan V_E^0 with unit element
$\rightsquigarrow$ Get the corresponding homogeneous open convex cone Ω^0

$\rightsquigarrow$
- Express the basic relative invariants associated to Ω^0 in terms of JA principal minors of V and stuffs related to (φ, E).
- Get the dual cone $(\Omega^0)^*$ of Ω^0 (w.r.t. an appropriate inner product)
  $\rightsquigarrow$ Express the basic relative invariants associated to $(\Omega^0)^*$ in terms of JA principal minors of V and stuffs related to (φ, E).

- Ω^0 is not a symmetric cone in general.
- The degrees of basic realtive invariants associated to $(\Omega^0)^*$ are always $1, 2, \ldots, r$ $(r = \mathrm{rank}(\Omega^0))$ whatever the representation φ.

The case of zero representation

V: a simple Euclidean Jordan algebra of rank r, e_0: the unit element
$\Omega = \mathrm{Int}\{x^2 \,;\, x \in V\}$: the corresponding symmetric cone
Fix a Jordan frame $c_1, \ldots, c_r$
$\rightsquigarrow$ H: the corresponding Iwasawa solvable subgroup of $G(\Omega)$ (reductive Lie group)
$\rightsquigarrow$ Introduce a canionical clan product $\triangle$ in V

Example: $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Sym}(r, \mathbb{R})^{++}$.
$GL(r, \mathbb{R})$-action on Ω: $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx\,{}^tg \in \Omega$
Product in V as a clan: $x \triangle y = \underline{x}\, y + y\, {}^t(\underline{x})$, where for $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$,

we put $$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & & & 0 \\ x_{21} & \frac{1}{2}x_{22} & & \\ \vdots & \ddots & \ddots & \\ x_{r1} & \cdots & x_{r,r-1} & \frac{1}{2}x_{rr} \end{pmatrix}.$$ Thus $x = \underline{x} + {}^t(\underline{x})$.

$L(x)y = R(y)x = \underline{x}\, y + y\, {}^t(\underline{x})$

General symmetric cone:

Fix $\langle\, x \,|\, y \,\rangle := \mathrm{tr}(xy)$: the trace inner product of V,
$\mathfrak{g} := \mathrm{Lie}(G(\Omega))$, $\mathfrak{k} := \mathrm{Der}(V)$,
$\mathfrak{p} := \{M(x)\ ;\ x \in V\}$: Jordan multiplication operators
$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with $\theta X = -{}^tX$,
$V = \bigoplus_{1 \le j \le k \le r} V_{kj}$: the Peirce decomposition for $c_1, \ldots, c_r$, where

$$V_{jj} := \mathbb{R}c_j \quad (j = 1, \ldots, r),$$
$$V_{kj} := \left\{x \in V\ ;\ M(c_i)x = \tfrac{1}{2}(\delta_{ij} + \delta_{ik})x \quad (i = 1, 2, \ldots, r)\right\} \quad (1 \le j < k \le r).$$

- $\mathfrak{a} := \mathbb{R}M(c_1) \oplus \cdots \oplus \mathbb{R}M(c_r)$: maximal abelian in $\mathfrak{p}$,
- $\alpha_1, \ldots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $M(c_1), \ldots, M(c_r)$.
  Then the positve $\mathfrak{a}$-roots are $\frac{1}{2}(\alpha_k - \alpha_j)$ $(j < k)$, and the corresponding root spaces are described as

$$\mathfrak{n}_{kj} := \mathfrak{g}_{(\alpha_k - \alpha_j)/2} = \left\{z \,\Box\, c_j\ ;\ z \in V_{kj}\right\} \quad (a \,\Box\, b := M(ab) + [M(a), M(b)]).$$

With $\mathfrak{n} := \sum_{j<k} \mathfrak{n}_{kj}$, we get Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Let $A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$. Then $H := N \rtimes A$ acts on Ω simply transitively.
H gives a clan structure to V, so that $\mathfrak{h} := \mathrm{Lie}(H) = \{L(v) \,;\, v \in V\}$.

Lemma. (1) $v \in \mathbb{R}c_1 \oplus \cdots \oplus \mathbb{R}c_r \implies L(v) = M(v)\,(\in \mathfrak{a})$.
(2) $v \in V_{kj} \implies L(v) = 2(v \,\square\, c_j)\,(\in \mathfrak{n}_{kj})$.

We now consider $R(x)y := L(y)x$.
$\Delta_1(x), \ldots, \Delta_r(x)$: JA principal minors of x associated to $c_1, \ldots, c_r$
(basic relative invariants of V)

Theorem. $\det R(x) = \Delta_1(x)^d \cdots \Delta_{r-1}(x)^d \Delta_r(x)$, where
d := the common dimension of V_{kj} $(j < k)$.

$d = 1$ for $\mathrm{Sym}(r, \mathbb{R})$,
$d = \dim_{\mathbb{R}} \mathbb{K}$ for $\mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$,
$r = 2$, $d = n - 2$ for $\Omega = \Lambda_n$ $(n \geq 3)$.

The case of non-trivial representations

(φ, E): a selfadjoint representation of φ, $\dim E > 0$

$\rightsquigarrow$ $\varphi(c_1), \ldots, \varphi(c_r)$: mutually orthogonal projection operators with equal rank

- Define the lower triangular part $\underline{\varphi}(x)$ of $\varphi(x)$ $(x \in V)$ by

$$\underline{\varphi}(x) := \frac{1}{2} \sum_i \lambda_i \varphi(c_i) + \sum_{j<k} \varphi(c_k)\varphi(x_{kj})\varphi(c_j) \quad (x = \sum_i \lambda_i c_j + \sum_{j<k} x_{kj}).$$

We have $\underline{\varphi}(x) + \underline{\varphi}(x)^* = \varphi(x)$.

> **Proposition.** φ is a representation of the clan $(V, \triangle)$:
> $$\varphi(x \triangle y) = \underline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x)^* \qquad (x, y \in V).$$

- Define a bilinear map $Q : E \times E \to V$ by

$$\langle\, \varphi(x)\xi \,|\, \eta \,\rangle_E = \langle\, Q(\xi, \eta) \,|\, x \,\rangle \; (x \in V,\, \xi, \eta \in E).$$

- Introduce a product $\triangle$ in $V_E := E \oplus V$ by

$$(\xi + x) \triangle (\eta + y) := \underline{\varphi}(x)\eta + (Q(\xi, \eta) + x \triangle y) \qquad (x, y \in V,\ \xi, \eta \in E).$$

Theorem. $(V_E, \triangle)$ is a clan with $s'(\xi + x) := \operatorname{Tr} L(x)$ $(\xi \in E,\ x \in V)$.

V_E does not have a unit element (because $\dim E > 0$).
Adjoin e to V_E, so that $V_E^0 := \mathbb{R}e \oplus V_E$ is a clan with unit element e.
Put $u := e - e_0$, and we use $V_E^0 = \mathbb{R}u \oplus E \oplus V$. Then

$$(\lambda u + \xi + x) \triangle (\mu u + \eta + y) = (\lambda\mu)u + (\mu\xi + \tfrac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi, \eta) + x \triangle y)$$

$(\lambda, \mu \in \mathbb{R},\ \xi, \eta \in E$ and $x, y \in V)$

Ω^0: the homogeneous convex cone associated to V_E^0

- To describe Ω^0, we introduce (note Q is Ω-positive)

$D(\Omega, Q) := \{\xi + x \in V_E\ ;\ x - \frac{1}{2}Q(\xi, \xi) \in \Omega\}$: real homogeneous Siegel domain

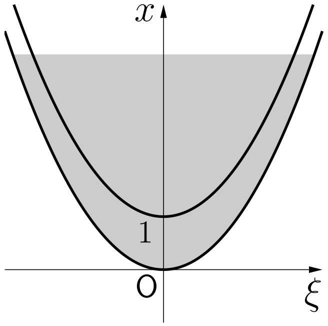
Then

$$\begin{aligned}\Omega^0 &= \{\lambda u + \lambda\xi + \lambda x \in V_E^0\ ;\ \lambda > 0,\ \xi + x \in D(\Omega, Q)\} \\ &= \{\lambda u + \xi + x \in V_E^0\ ;\ \lambda > 0,\ \lambda x - \tfrac{1}{2}Q(\xi, \xi) \in \Omega\}\end{aligned}$$

Example:

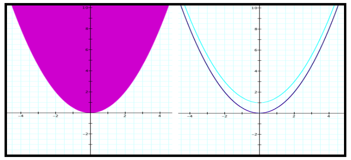
$V = \mathbb{R}$, $\Omega = \mathbb{R}_{>0}$, $E = \mathbb{R}$ and $\varphi(x)\xi = x\xi$ $(x \in \mathbb{R},\ \xi \in \mathbb{R})$.
Clearly $Q(\xi, \eta) = \xi\eta$.

$$D(\Omega, Q) = \{(\xi, x) \in \mathbb{R}^2\ ;\ x - \tfrac{1}{2}\xi^2 > 0\}$$

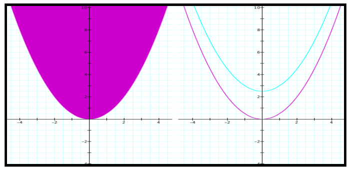


- $\forall t \in \mathbb{R},\ (\xi, x) \mapsto (e^{t/2}\xi, e^t x)$: translation of the basic parabola

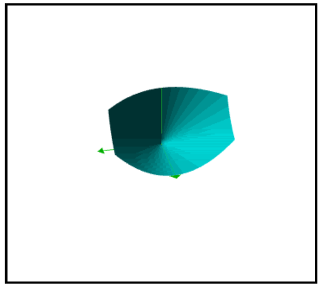
$$x = \tfrac{1}{2}\xi^2 + 1 \mapsto x = \tfrac{1}{2}\xi^2 + e^t$$

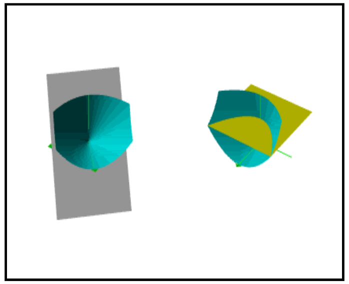


- $\forall \xi_0 \in \mathbb{R},\ (\xi, x) \mapsto (\xi + \xi_0,\ x + \xi\xi_0 + \tfrac{1}{2}\xi_0^2)$:
  movement on each of the parabolas $x = \tfrac{1}{2}\xi^2 + a$ $(a > 0)$



$\Omega^0 = \{(\lambda, \xi, x)\ ;\ \lambda > 0,\ \lambda x - \tfrac{1}{2}\xi^2 > 0\}$ $\rightsquigarrow$ See movie.





Basic relative invariants associated to Ω^0

V: Euclidean JA, $\varphi : V \to \mathrm{Sym}(E)$: selfadjoint JA representation

- φ is regular $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \xi_0 \in E$ s.t. $Q(\xi_0, \xi_0) = e_0$

(1) The Hermitian cases: $V = \mathrm{Herm}(r, \mathbb{K})$ $(r \geq 3;\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$
$E = \mathrm{Mat}(r \times p,\ \mathbb{K})$, $\varphi(x)\xi = x\xi$ $(x \in V,\ \xi \in E)$, $Q(\xi, \eta) = \frac{1}{2}(\xi\eta^* + \eta\xi^*)$

> **Fact:** φ is regular $\iff$ $p \geq r$ (that is, $E = \square$ or $E = \square$)

$V_E^0 = \mathbb{R}u \oplus E \oplus V \ni \lambda u + \xi + x =: v$
$\Delta_k(x)$: k-th principal minor of $x \in V$ (left upper corner)

> **Theorem**. If φ is regular, the basic relative invarinats associated to Ω^0 are
> $$\Delta_0^0(v) = \lambda, \quad \Delta_j^0(v) = \Delta_j(\lambda x - \tfrac{1}{2}\xi\xi^*)\ (j = 1, \ldots, r).$$

- If φ is *not* regular (i.e., if $p < r$), then $\Delta_j(\xi\xi^*) = 0$ for $p + 1 \leq j \leq r$

$\Longrightarrow$ $\Delta_j(\lambda x - \frac{1}{2}\xi\xi^*)$ is *not* irreducible for such j

$\Longrightarrow$ should be $\lambda^{-(j-p)}\Delta_j(\lambda x - \frac{1}{2}\xi\xi^*) \longleftarrow$ I do *not* like this expression.

Proposition. The following map is an injective clan isomorphism:

$$V_E^0 \ni \lambda u + \xi + x \mapsto \begin{pmatrix} \lambda I_p & \frac{1}{\sqrt{2}}\xi^* \\ \frac{1}{\sqrt{2}}\xi & x \end{pmatrix} =: X(\lambda, \tfrac{\xi}{\sqrt{2}}, x) \in \mathrm{Herm}(r+p,\, \mathbb{K})$$

Theorem. If $p < r$, then the basic relative invariants associated to Ω^0 are

$$\begin{cases} \Delta_0^0(v) = \lambda, \\ \Delta_j^0(v) = \Delta_j(\lambda x - \frac{1}{2}\xi\xi^*) & (j = 1, \dots, p), \\ \Delta_j^0(v) = \det^{(p+j)} \begin{pmatrix} \lambda I_p & \frac{1}{\sqrt{2}}\xi^* \\ \frac{1}{\sqrt{2}}\xi & x \end{pmatrix} & (j = p+1, \dots, r). \end{cases}$$

Here $\det^{(p+j)} X$ stands for the detgermionant of the left upper matrix of size $p+j$ from X, and if $\mathbb{K} = \mathbb{H}$, then determinant should be taken in the JA sense.

Proposition. Ω^0 is isomorphic to $\{X(\lambda, \xi, x) \gg 0\}$.

(2) The Lorentzian case

(W, B): a Euclidean VS, and $V = \mathbb{R}e_0 \oplus W$ with
$(\lambda e_0 + v) \circ (\mu e_0 + w) = (\lambda u + B(v, w))e_0 + (\lambda w + \mu v) \quad (\lambda, \mu \in \mathbb{R},\ v, w \in W)$.
$\varphi : V \to \mathrm{Sym}(E)$: JA representation, $e_1, \dots, e_n$: ONB of W w.r.t. B.
Then $c_1 := \frac{1}{2}(e_0 + e_n)$, $c_1 := \frac{1}{2}(e_0 - e_n)$: Jordan frame of $V \quad \rightsquigarrow \Delta_1, \Delta_2$.
$Q(\xi, \xi) = \frac{1}{2}\|\xi\|^2 e_0 + \frac{1}{2}\sum_{j=1}^{n} \langle\, \varphi(e_j)\xi \mid \xi \,\rangle e_j$ is the expansion of $Q(\xi, \xi)$ w.r.t. $\{e_j\}_{j=0}^{n}$.

Fact [Clerc, 1992]: φ is *not* regular
$\iff$ φ is irreducible and $\dim E = 2, 4, 8, 16$.

Theorem. The basic relative invariants associated to $\Omega^0 \subset V_E^0$ are given by

$$\Delta_0^0(\lambda u + \xi + x) = \lambda, \qquad \Delta_1^0(\lambda u + \xi + x) = \Delta_1(\lambda x - \tfrac{1}{2}Q(\xi, \xi))$$

$$\Delta_2^0(\lambda u + \xi + x) = \begin{cases} \Delta_1(\lambda x - \frac{1}{2}Q(\xi, \xi) & (\varphi \text{ is regular}) \\ \lambda\Delta_2(x) - \langle x, Q(\xi, \xi)\rangle_{1,n} & (\varphi \text{ is not regular}) \end{cases}$$

$\langle x_0 e_0 + w,\ x_0' e_0 + w' \rangle_{1,n} := x_0 x_0' - B(w, w')$.

Remarks about the non-regular Lorentzian cases

φ is irreducible and $\dim E = 2, 4, 8, 16$
$\implies \dim W = \dim V_{21} + 1 = \frac{1}{2}\dim E + 1 = 2, 3, 5, 9.$
Then $V = \mathbb{R}e_0 \oplus W \cong \mathrm{Herm}(2, \mathbb{K})$ $(\mathbb{K} = \mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{O})$.

Put $v(\alpha, \beta, z) := \begin{pmatrix} \alpha & z \\ \overline{z} & \beta \end{pmatrix} \in \mathrm{Herm}(2, \mathbb{K})$ with $\alpha, \beta \in \mathbb{R},\ z \in \mathbb{K}$.

With $E = \mathbb{K}^2$, JA represenation φ is realized as φ_1 or φ_2:

$$\varphi_1(v(\alpha, \beta, z)) := \begin{pmatrix} \alpha I & L_z \\ L_{\overline{z}} & \beta I \end{pmatrix} \in \mathrm{End}_{\mathbb{R}}(\mathbb{K}^2), \quad \varphi_2(v) = \varphi_1({}^t v)$$

- If $\mathbb{K} = \mathbb{R}$, then φ_1 and φ_2 are identical.
  This is *the* unique irreducible representation of $\mathrm{Sym}(2, \mathbb{R})$.
- If $\mathbb{K} = \mathbb{H}$ or $\mathbb{O}$, then φ_1 and φ_2 are *not* equivalent.
  These two are the irreducible represenations of $\mathrm{Herm}(2, \mathbb{K})$ $(\mathbb{K} = \mathbb{H}, \mathbb{O})$.
- If $\mathbb{K} = \mathbb{C}$, then φ_2 is conjugate to φ_1.
  These two are the inequivalent complex represenations of $\mathrm{Herm}(2, \mathbb{C})$.

$E = \mathbb{K}^2$ and $(\varphi_j, E) \rightsquigarrow$ clan $(V_E^0, \triangle_j)$ $(j = 1, 2)$.

Proposition. $(V_E^0, \triangle_1) \cong (V_E^0, \triangle_2) \cong \mathrm{Herm}(3, \mathbb{K})$ (even if $\varphi_1 \not\cong \varphi_2$).

- $(V_E^0, \triangle_1) \cong (V_E^0, \triangle_2)$ is given by $\lambda u + \xi + x \mapsto \lambda u + \xi + {}^t x$,
- The map $x \mapsto {}^t x$ is a Jordan and clan isomorphism of $\mathrm{Herm}(2, \mathbb{K})$.
- $V_E^0 \cong \mathrm{Herm}(3, \mathbb{K})$ is given by $\lambda u + \xi + x \mapsto \begin{pmatrix} \lambda & \frac{1}{\sqrt{2}}\xi^* \\ \frac{1}{\sqrt{2}}\xi & x \end{pmatrix}$,

  where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in E = \mathbb{K}^2$.

- In $\mathrm{Herm}(2, \mathbb{K})$ we have for $v = \begin{pmatrix} \alpha & z \\ \bar{z} & \beta \end{pmatrix}$

$$\Delta_1(v) = \alpha, \quad \Delta_2(v) = \alpha\beta - |z|^2$$

Then,

$$\lambda, \quad \Delta_1(\lambda x - \frac{1}{2}\xi\xi^*), \quad \frac{1}{\lambda}\Delta_2(\lambda x - \frac{1}{2}\xi\xi^*)$$

are the principal monors of $\begin{pmatrix} \lambda & \frac{1}{\sqrt{2}}\xi^* \\ \frac{1}{\sqrt{2}}\xi & x \end{pmatrix} \in \mathrm{Herm}(3, \mathbb{K})$.

Dual clan of V_E^0

Define an inner product in $V_E^0 = \mathbb{R}u \oplus E \oplus V$ by

$$\langle\, \lambda u + \xi + x \,|\, \lambda' u + \xi' + x' \,\rangle^0 = \lambda\lambda' + \langle\, \xi \,|\, \xi' \,\rangle + \langle\, x \,|\, x' \,\rangle$$

$(\Omega^0)^* := \big\{ v \in V \;;\; \langle\, v \,|\, v' \,\rangle > 0 \;$ for all $v' \in \overline{(\Omega^0)} \setminus \{0\} \big\}$.

Then, the clan product $\triangledown$ in V_E^0 associated to $(\Omega^0)^*$ is given by $v \triangledown v' = {}^t L_v^0 v'$.

> **Proposition**. Let $v = \lambda u + \xi + x \in V_E^0$. Then
>
> $$v \in (\Omega^0)^* \iff x \in \Omega \;\text{ and }\; \lambda > \tfrac{1}{2}\langle\, \varphi(x)^{-1}\xi \,|\, \xi \,\rangle.$$

Remark. The condition corresponds to what Rothaus called *the extension of Ω by the representation φ*.

Here, (representation R) $\equiv$ (representation $R : V \to \mathrm{Sym}(E)$ of a cone Ω)

$$\overset{\text{def}}{\iff} \begin{cases} (1)\; R(x) \gg 0 \quad (\forall x \in \Omega), \\ (2)\; \exists H \;(\curvearrowright \Omega \text{ transitively) s.t.} \forall h \in H,\ \exists h' \in GL(E) \\ \qquad\qquad\qquad \text{with } R(hv) = h' R(v)\, {}^t h' \;\; (\forall v \in V). \end{cases}$$

- JA representation $\implies$ representation of the corresponding symmetric cone

$\Delta_1^*(x), \ldots, \Delta_r^*(x)$: JA principal minors of $x \in V$ associated to $c_r, \ldots, c_1$.

Theorem. The basic relative invariants associated to $(\Omega^0)^*$ are given by

$$P_j(\lambda u + \xi + x) = \Delta_j^*(x) \qquad (j = 1, \ldots, r),$$
$$P_r(\lambda u + \xi + x) = \lambda \det x - \tfrac{1}{2}\langle \varphi({}^{\mathrm{co}}x)\xi \,|\, \xi \rangle$$

If $x \in V$ is invertible, ${}^{\mathrm{co}}x := (\det x)x^{-1}$.
We know that $x \mapsto {}^{\mathrm{co}}x$ is a polynomial map of degree $r - 1$.
In particular, $\deg P_j = j$ $(j = 1, \ldots, r, r + 1)$.

(1) The Hermitian cases. $V = \mathrm{Herm}(r, \mathbb{K})$ $(r \geq 3,\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$.

Theorem. $(\Omega^0)^*$ is linearly isomorphic to

$$\Omega' := \left\{ Y = \begin{pmatrix} \mu & \eta^* \\ \eta & y \otimes I_p \end{pmatrix} \gg 0\ ;\ \begin{matrix} \mu \in \mathbb{R},\ \eta \in \mathbb{K}^{rp} \\ y \in V \end{matrix} \right\} \subset \mathrm{Herm}(rp+1,\ \mathbb{K})$$

$\Delta_k^*(y)$: the k-th principal minor (right lower corner) of $y \in V$ $(k = 1, \ldots, r)$.

Theorem The basic relative invariants associated to Ω' are given by

$$P_j(Y) = \Delta_j^*(y)\ (j = 1, \ldots, r), \quad P_{r+1}(Y) = \mu \det y - \eta^* ({}^{\mathrm{co}}y \otimes I_p)\eta$$

${}^{\mathrm{co}}y$: the cofactor matrix of y. Thus ${}^{\mathrm{co}}y = (\mathrm{Det}\, y) y^{-1}$ if y is invertible.

Remark. $\deg P_j = j$ for $\forall j$. But if $p > 1$, then Ω' is not a symmetric cone.

(2) The Lorentzian case.

We have fixed the Jordan frame $c_1 := \frac{1}{2}(e_0 + e_n)$, $c_2 = \frac{1}{2}(e_0 - e_n)$.
$\Delta_1^*(x), \Delta_2^*(x)$: JA principal minors of $x \in V$ associated to c_2, c_1.

Theorem. The basic relative invariants associated to $(\Omega^0)^*$ are given by $P_j(\lambda u+\xi+x) = \Delta_j^*(x)\ (j=1,2), \quad P_3(\lambda u+\xi+x) = \lambda \det x - \frac{1}{2}\langle\, \varphi(\widetilde{x})\xi \,|\, \xi\,\rangle$

$x \mapsto \widetilde{x}$: restriction to $V = \mathbb{R}e_0 \oplus W$ of the $\mathrm{Cl}(W)$-automorphism that extends the isometry $w \mapsto -w$ of W.

Remark. $\deg P_j = j$ for $\forall j$. But if φ is regular, then Ω^0 is not a symmetric cone.