Symmetric Submanifolds and Grassmann Geometry

Hiroo Naitoh, Yamaguchi University (e-mail: naitoh@yamaguchi-u.ac.jp)

C O N T E N T S

1. Grassmann geometry

Let M be an m-dimensional connected Riemannian manifold and s be an integer such that $1 \leq s \leq m-1$. Consider the Grassmann bundle $G_s(TM)$ over M which, for each point $p \in M$, has the Grassmann manifold $G_s(T_pM)$ of s-dimensional subspaces of T_pM as the fibre. Namely $G_s(TM) = \cup_{p\in M} G_s(T_pM)$. And take a subset $\mathcal{S}$ of $G_s(TM)$. Then a connected submanifold S of M is called an $\mathcal{S}$-submanifold if its tangent spaces belong all to $\mathcal{S}$. And the collection of $\mathcal{S}$-submanifolds is called an $\mathcal{S}$-geometry.

Grassman geometry is a collected name of such geometries. This notion of Grassmann geometry was introduced by Harvey-Lawson in their paper [Caribrated geometry; Acta Math. 1982].

In particular an $\mathcal{S}$-geometry is called of orbit type if the subset $\mathcal{S}$ is an orbit under the action of the identity conponent G of the isometry group of M on the Grassmann bundle $G_s(TM)$. This action is induced by the differentials of isometries. A Grassmann geometry of orbit type is said to be an $\mathcal{O}$-geometry. Roughly speaking, a Grassmann geometry of orbit type is a class of submanifolds with congruent tangent spaces.

In the following we consider a Grassmann geometry of orbit type over a simply connected Riemannian symmetric space.

2. Symmetric submanifolds

A symmetric submanifold S of a Riemannian manifold M is, by definition, a connected submanifold with an extrinsic symmetry t_p for each point of S. The extrinsic symmetry is an isometry of M such that

$$t_p(p) = p, \qquad t_p(S) = S, \qquad \text{and } d(t_p)_p(X) = -X, \qquad d(t_p)_p(\xi) = \xi$$

for a tangent vector X of S and a normal vector ξ of S. Since for each point p, the restriction of t_p into S defines the intrinsic symmetry of S at p, the submanifold S is also a Riemannian symmetric space with the induced metric. Moreover it is a parallel submanifold, namely, a submanifold with parallel 2nd fundamental form, since the covariant derivative of 2nd fundamental form is a normal valued tensor of S with degree 3 and the extrinsic symmetry preserves this tensor. Moreover it follows that the tangent and the normal spaces are curvature-inbariant with respect to the curvature tensor of M.

The notion of symmetric submanifold was introduced by Ferus in his paper [Symmetric submanifolds of Euclidean space; Math. Ann. 1980] for the case that the ambient space M is a Euclidean space. After that, it was extended for the general case, in particular, the case that M is a Riemannian symmetric space. When M is a Riemannan symmetric space, the theory of symmetric submanifolds has many similar results to that of symmetric spaces. For example;

(1) Characterization of symmetric submanifolds: If we moreover assume that M is simply connected, a connected submanifold S of M is a symmetric submanifold if and only if it is a complete parallel submanifold and the tangent spaces and the normal spaces are curvature-invariant subspaces. Here the curvature-invariance of a tangent and a normal spaces assures the existance of complete totally geodesic submanifolds tangent to these subspaces.

The converse of this claim is proved in the following way. First by the curvature-invariance, at each point $p \in S$, the linear isometry with the same form as the differential $d(t_p)_p$ preserves the curvature tensor of M. So, this linear isometry can be extended into a local isometry around p. By the simply connectedness of M this local isometry is extended into a global isometry t_p. Last the completeness and the parallelity of S assures that this global isometry preserves the submanifold S. Hence t_p gives the extrinsic symmetry of S at p.

This result is analogous to the fact that a simply connected, complete, Riemannian locally symmetric space is a (globally) Riemannan symmetric space. And the essential part in the proof of these results is from the extension of a special linear isometry into a local isometry.

(2) Decomposition of symmetric submanifolds: Let S be a symmetric submanifold of a simply connected Riemannian symmetric space M and assume that S is not contained in a proper product factor of M. Decompose the symmetric submanifold (S, M) into the product of irreducible submanifolds (S_i, M_i), namely,

$$S = S_1 \times S_2 \times \cdots \times S_r \subset M_1 \times M_2 \times \cdots \times M_r = M.$$

Then the submanifolds S_i are also symmetric submanifolds of M_i. Moreover M_i's are a Euclidean space or a simply connected semisimple Riemannian symmetric space of compact type or noncompact type.

Assume moreover that (S, M) is a symmetric submanifold with a semisimple Riemannian symmetric space M, and take a point p of S. Then, there exists a unique complete totally geodesic submanifold (N, M) tangent to (S, M) at p. By the above characterization of symmetric submanifolds, this submanifold (N, M) is also a totally geodesic symmetric submanifold with the same extrinsic symmetry t_p at p. Then the irreducible decomposition of the symmetric submanifold (S, M) induces the irreducible decomposition of the totally geodesic symmetric submanifld (N, M):

$$N = N_1 \times N_2 \times \cdots \times N_r \subset M_1 \times M_2 \times \cdots \times M_r = M$$

where (N_i, M_i)'s are also tangent to (S_i, M_i)'s, respectively.

These results are similar to the de Rham decomposition of simply connected Riemannian symmetric space. For the case of symmetric submanifold, we moreover need a decomposition theory of isometric immersion [Moore; Isometric immersions of riemannian products, J.D.G, 1971] , besides the de Rham decomposition.

(3) Local structures of totally geodesic symmetric submanifolds: Let (S, M) be an irreducible symmetric submanifold with a simply connected semisimple Riemannian symmetric space M of compact type. Then we can define an algebraic structure $(\mathfrak{g}, \sigma, \tau)$: $\mathfrak{g}$ is the compact-type Lie algebra of the isometry group G of M; σ and τ are involutive automorphisms of $\mathfrak{g}$ defined as differentials of the inner automorphisms of G induced by the intrinsic symmetry s_p of M and the extrinsic symmetry t_p of S at a point $p \in S$, respectively. The algebraic structure $(\mathfrak{g}, \sigma)$ determines the very local structure of the symmetric space M, but the algebraic structure $(\mathfrak{g}, \sigma, \tau)$ does not determine the local structure of (S, M). This structure determines only the local structure of the totally geodesic symmetric submanifold (N, M) tangent to (S, M).

Next let $(\hat{S}, \hat{M})$ be an irreducible symmetric submanifold with a simply connected semisimple Riemannian symmetric space M of noncompact type. In this case, by the same way as in the compact case, we can also define an algebraic structure $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{\tau})$ and then this structure determines the totally geodesic symmetric submanifold $(\hat{N}, \hat{M})$ tangent to $(\hat{S}, \hat{M})$. In this case, $\hat{\mathfrak{g}}$ is a noncompact-type Lie algebra and σ is a Cartan involution of $\hat{\mathfrak{g}}$. Moreover the duality between local structures $(\mathfrak{g}, \sigma)$ and $(\hat{\mathfrak{g}}, \hat{\sigma})$ of symmetric spaces induces the one between local structures $(\mathfrak{g}, \sigma, \tau)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{\tau})$ of totally geodesic symmetric submanifolds. Here the local structures $(\hat{\mathfrak{g}}, \hat{\tau})$ are nothing but the local structures of irreducible semisimple affine symmetric spaces except of the Riemannian cases, classified by Berger. For the classification of irreducible compact-type totally geodesic symmetric submanifolds (N, M), there are explicit ones by Leung [Indiana Univ. Math. J, 1974] or by me [Japan. J. Math, 2000 and etc.].

(4) Orbit-type Grassmann geometry and symmetric submanifolds: Let G be the identity component of the isometry group of M. We consider the orbit-type Grassmann geometry over M. For a totally geodesic symmetric submanifold (N, M), the tangent spaces of N are contained in a G-orbit. We denote this by $\mathcal{O}_{(N,M)}$. Then, any symmetric submanifold (S, M) is an $\mathcal{O}_{(N,M)}$- submanifold for the totally geodesic symmetric submanifold (N, M) tangent to (S, M), since a symmetric submanifold is a connected and equivariant submanifold. Moreover (N, M) is a unique complete totally geodesic $\mathcal{O}_{(N,M)}$-submanifold, since a totally geodesic submanifold is determined by one point of the submanifold and the tangent space at the point.

3. The classification of symmetric submanifolds

Now we give a classification of symmetric submanifolds with the ambient spaces of simply connected Riemannian symmetric spaces. The classification is essentially reduced to the following three cases.

(I) The case of irreducible totally geodesic symmetric submanifolds (N, M) : In this case the classification has been already done. For example, by Berger's classification.

(II) The case of nontotally geodesic symmetric submanifolds (S, M) with the ambient spaces M of Euclidean spaces: In this case the symmetric submanifolds (S, M) are essentially exhausted by the so-called symmetric R-spaces. Here the essentiality implies that S is not contained in a proper Euclidean subspace. A symmetric R-space is constructed in a tangent space of a simply connected semisimple Riemannian symmetric space of compact type, as an isotoropy orbit with the structure of compact symmetric space. This orbit in the tangent space gives the realization of symmetric submanifold for this case. There are 18 kinds of the irreducible symmetric R-spaces containing 6 kinds of the irreducible compact Hermitian symmetric spaces.

In this case, the classification of symmetric submanifolds was done by Ferus [5].

(III) The case of irreducible nontotally geodesic symmetric submanifolds (S, M) with the ambient spaces M of simply connected semisimmple Riemannian symmetric spaces: In this case we first have the following theorem.

Theorem.(Naitoh [21, 24, 25, 26]) Let (N, M) be an irreducible totally geodesic symmetric submanifold with an ambient space of simply connected semisimple Riemannian symmetric space. Then any $\mathcal{O}_{(N,M)}$-submanifold is totally geodesic if and only if (N, M) is other than the following 5 cases:

(1) (S^r, S^n) and its noncompact dual $(\mathbb{R}H^r, \mathbb{R}H^n)$.
(2) $(\mathbb{R}P^n, \mathbb{C}P^n)$ and its noncompact dual $(\mathbb{R}H^n, \mathbb{C}H^n)$.
(3) $(\mathbb{C}P^r, \mathbb{C}P^n)$ and its noncompact dual $(\mathbb{C}H^r, \mathbb{C}H^n)$.
(4) $(\mathbb{C}P^n, \mathbb{H}P^n)$ and its noncompact dual $(\mathbb{C}H^n, \mathbb{H}H^n)$.
(5) The following compact (N, M)'s and their noncompact duals, constructed from the irreducible symmetric R-spaces R: In this case let M be the simply connected semisimple Riemannian symmetric space of compact type such that an irreducible symmetric R-space R can be realized in a tangent space V of it. Then a tangent space W of R in V and its orthogonal compliment are curvature-invariant with respect to the curvature tensor of M. Let N be the totally geodesic symmetric submanifold of M. The totally geodesic symmetric submanifolds of this case are such (N, M)'s and their noncompact duals. In this case M has higher rank except of one case. The exceptional case is contained in the case (1).

The proof of this theoem is very long. We first classify the irreducible local structure $(\mathfrak{g}, \sigma, \tau)$ associated with (N, M), by using the representation theory. And we find out a condition for the second fundamental form of any $\mathcal{O}_{(N,M)}$-submanifold to vanish, by the words of local structure. Last we check this condition for each local structure. The above 5 cases are the ones for which this condition does not vanish. This condition is related to the injectivity of a certain module-homomorphism associated with each local structure.

Now we explain the classification of irreducible nontotally geodesic symmetric submanifolds for the above cases (1) through (5).

(1) (S^r, S^n) **and its noncompact dual** $(\mathbb{R}H^r, \mathbb{R}H^n)$**:** In this case the nontotally geodesic symmetric submanifolds are essentially exhausted in the following way:

$$S = R_1 \times \cdots \times R_k \subset S^{n_1} \times \cdots \times S^{n_k} \subset S^n = M \subset \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_k+1} = \mathbb{R}^{n+1}$$

$$S = \mathbb{R}H^{n_0} \times R_1 \times \cdots \times R_k \subset \mathbb{R}H^{n_0} \times S^{n_1} \times \cdots \times S^{n_k} \subset \mathbb{R}H^n = M$$
$$\subset \mathbb{R}_1^{n_0+1} \times \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_k+1} = \mathbb{R}_1^{n+1}$$

Here (R_i, S^{n_i})'s are irreducible symmetric R-spaces. And the essentiality implies that S is not contained in a totally umbilical hypersurface of M. We here note the following: a symmetric R-space can be constructed by another way of using the theory of Jordan triple systems and symmetric graded Lie algebras, and this way can be extended for certain indefinite cases. Then we have the notion of pseudo-Riemannian symmetric R-space. The symmetric submanifold $(\mathbb{R}H^{n_0}, \mathbb{R}_1^{n_0+1})$ is a pseudo-Riemannian symmetric R-space of this type.

Hence, roughly speaking, the symmetric submanifolds of these cases are spherical or pseudo-spherical representations of symmetric R-spaces. The classification for these cases is by Ferus and Takeuchi [5, 34 or cf. 28].

(2) $(\mathbb{R}P^n, \mathbb{C}P^n)$ **and its noncompact dual** $(\mathbb{R}H^n, \mathbb{C}H^n)$**:** In this case consider the Hopf fiberings $S^{2n+1} \to \mathbb{C}P^n$ or $L^{2n+1} \to \mathbb{C}H^n$. Here L^{2n+1} is the Lorenzian hypersphere of pseudo-Hermitian space $\mathbb{C}_1^{n+1}$ with sinature $(2, 2n)$. Then the symmetric submanifolds of these cases are exhausted by the project of $(n+1)$-dimensional totally real Riemannian (or pseudo-Riemannian) symmetric R-spaces of S^{2n+1} (or L^{2n+1}) compatible with the Hopf fiberings. The irreducible symmetric R-spaces appeared for the compact case are 5 kinds with 1-dimensional Euclidean factor. These are also realized as the Shilov boundary of irreducible symmetric bounded domains of tube type. Also, the ones for the noncompact case are the above 5 kinds and 4 kinds of pseudo-Riemannian symmetric R-spaces.

Hence, roughly speaking, the symmetric submanifolds of these cases are projective representations of symmetric R-spaces under the Hopf fiberings. The classification for these cases is by Takeuchi and Naitoh [17, 18, 19, 27 or cf. 28].

(3) $(\mathbb{C}P^r, \mathbb{C}P^n)$**,** $(\mathbb{C}H^r, \mathbb{C}H^n)$ **and (4)** $(\mathbb{C}P^n, \mathbb{H}P^n)$**,** $(\mathbb{C}H^n, \mathbb{H}H^n)$**:** For the case of $(\mathbb{C}P^r, \mathbb{C}P^n)$, the nontotally geodesic symmetric submanifolds are exhausted by 6 kinds of the Kähler imbeddings of compact Hermitian symmetric spaces with rank one or two, defined by homogeneous polynomials of degree 2. For the case of $(\mathbb{C}P^n, \mathbb{H}P^n)$, they are exhausted by 5 kinds of the totally complex imbeddings of compact Hermitian symmetric spaces with rank three .

The ways of proofs for these cases are different from the theory of symmetric R-space, and they are similar ways of using the unitary or the symplectic representations. These classifications are by Nakagawa, Takagi, and Takeuchi [16, 32, or cf. 28] for the case (3), and by Tsukada [37] for the case (4), respectively.

The noncompact cases of (3) and (4) have no non-totally geodesic symmetric submanifolds. These facts are by Kon [8] and by Tsukada [37], respectively.

(5) The following compact (N, M)'s and their noncompact duals, constructed from the irreducible symmetric R-spaces R [20, 2]: Assume that M is of higher rank. For the compact case the totally geodesic symmetric submanifolds are exhausted by the exponential image of homothetic irreducible Riemannian symmetric R-spaces R realized in a tangent space of M. For the noncompact case they are exhausted by (a) the exponential images like the campact case, moreover, (b) the homothetic imbeddings of noncompact duals of R and (c) one imbedding of Euclidean space. These imbeddings are pseudo-umbilical. We note that the noncompact duals of R are also pseudo-Riemannian symmetric R-spaces. Also, in the exceptional case $(\mathbb{R}H^{n-1}, \mathbb{R}H^n)$, the symmetric hypersurfaces constructed here are the totally umbilical hypersurfaces of $\mathbb{R}H^n$.

The crucial point of proof is to show that the second fundamental form any $\mathcal{O}_{(N,M)}$-submanifold of this type has the same form as the one of these models, except of the cases that $(N, M) = (S^{n-1}, S^n)$ or $(\mathbb{R}H^{n-1}, \mathbb{R}H^n)$.

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