Multiplicity for irreducible representations and strength of convergence in duals of nilpotent Lie groups

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The *upper multiplicity*, $M_U(\pi)$, was introduced by Archbold for irreducible representations π of an arbitrary C^*-algebra A in order to study the properties of trace functions $\pi \to \mathrm{tr}(a), a \in A$, on the dual space $\widehat{A}$ of A. $M_U(\pi)$ counts the maximal number of nets of orthogonal equivalent pure states which can converge to a given pure state associated to π.

On the other hand, in the dual $\widehat{G}$ of a simply connected nilpotent Lie group G, there is the notion, due to Ludwig, of *strength of convergence* of a sequence to a limit point. Denoting by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$, strength of convergence in $\widehat{G}$ reflects the convergence of the associated Kirillov orbits in $\mathfrak{g}^*$ in a natural geometric sense .

The link between these two concepts is provided by trace formulae in which upper multiplicities and strength of convergence play analogous roles. It turns out that $M_U(\pi)$ is the greatest strength with which a sequence in $\widehat{G}$ can converge to π. Establishing this involves the concept of variable simply connected nilpotent Lie groups.

Upper multiplicities have been thoroughly investigated in joint work with Archbold, Ludwig, Schlichting and Somerset. It is known that $M_U(\pi) < \infty$ if and only if the associated orbit in $\mathfrak{g}^*$ has maximal dimension. On the other hand, the representations corresponding to so-called generic functionals in $\mathfrak{g}^*$ have upper multiplicity equal to one. However, the situation for non-generic functionals with maximal orbit dimension is much more complicated. For the groups G_N $(n \geq 3)$ in the sequence of threadlike generalisations of the Heisenberg group, upper multiplicities of all irreducible representations can be calculated by combining detailed analysis of the convergence of coadjoint orbits with some new C^*-theoretic methods. The results show that every positive integer occurs for this class of groups. Moreover, $M_U(\pi) = 1$ if and only if π has a Hausdorff neighbourhood in $\widehat{G}_N$. In particular, in addition to the generic functionals in $\mathfrak{g}_N^*$, there are many other functionals giving rise to irreducible representations with upper multiplicity equal to one.