On a certain 8-dimensional non-symmetric homogeneous convex cone

Takaaki Nomura

Faculty of Mathematics
Kyushu University

於：表現論シンポジウム

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Subgroups of $GL(r,\mathbb{C})$:

$$A_{\mathbb{C}} := \left\{ a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix} \; ; \; a_1 \in \mathbb{C}^\times, \ldots, a_r \in \mathbb{C}^\times \right\},$$

$$N_{\mathbb{C}} := \left\{ n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ n_{21} & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ n_{r-1,1} & n_{r-1,2} & & 1 & 0 \\ n_{r1} & n_{r2} & \cdots & n_{r,r-1} & 1 \end{pmatrix} \; ; \; n_{ij} \in \mathbb{C} \right\}.$$

$V := \mathrm{Sym}(r,\mathbb{R})$, $\quad \Omega := \{x \in V \; ; \; x \gg 0\}$.
$\Omega + iV$: the corresponding tube domain in $W := V_{\mathbb{C}} = \mathrm{Sym}(r,\mathbb{C})$.
$GL(r,\mathbb{C})$ acts on W by $(g,w) \mapsto gw{}^tg$.

Fact: $\quad \Omega + iV \subset N_{\mathbb{C}} A_C \cdot E \quad$ (E: the $r \times r$ identity matrix).

For $w = (w_{ij}) \in \mathrm{Mat}(r,\mathbb{C})$ we set for $k = 1, 2, \ldots, r$,

$$\Delta_k(w) := \det \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & & \vdots \\ w_{k1} & \cdots & w_{kk} \end{pmatrix} \quad \text{(the k-th principal minor).}$$

• By the above Fact, $\Delta_k(w) \neq 0$ for $w \in \Omega + iV$.

We also set $\Delta_0(w) \equiv 1$.

Lemma 1.1. *Let* $w \in \mathrm{Sym}(r, \mathbb{C})$ *and* $\mathrm{Re}\, w \in \Omega$.
If $w = na\,{}^t n$ *with* $a = \mathrm{diag}[a_1, \dots, a_r] \in A_{\mathbb{C}}$ *and* $n \in N_C$, *then*

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1, \dots, r).$$

Proof. Let us write n, a as follows:

$$\begin{array}{ccc} n = \left(\begin{array}{c|c} n' & 0 \\ \hline \Xi & n'' \end{array}\right), & a = \left(\begin{array}{c|c} a' & 0 \\ \hline 0 & a'' \end{array}\right) & \begin{array}{l} \updownarrow k \\ \updownarrow r-k \end{array} \\ \overset{k}{\longleftrightarrow} \big| \overset{r-k}{\longleftrightarrow} & \overset{k}{\longleftrightarrow} \big| \overset{r-k}{\longleftrightarrow} & \end{array}$$

Then

$$\begin{aligned} na\,{}^t n &= \left(\begin{array}{c|c} n' & 0 \\ \hline \Xi & n'' \end{array}\right)\left(\begin{array}{c|c} a' & 0 \\ \hline 0 & a'' \end{array}\right)\left(\begin{array}{c|c} {}^t n' & {}^t\Xi \\ \hline 0 & {}^t n'' \end{array}\right) \\ &= \left(\begin{array}{c|c} n'a'\,{}^t n' & n'a'\,{}^t\Xi \\ \hline \Xi\, a'\,{}^t n' & \Xi\, a'\,{}^t\Xi + n''a''\,{}^t n'' \end{array}\right). \end{aligned}$$

Writing $w = \left(\begin{array}{c|c} w' & z \\ \hline {}^t z & w'' \end{array}\right)$ in the same way, we obtain $w' = n'a'\,{}^t n'$. Hence

$$\Delta_k(w) = \det w' = \det(n'a'\,{}^t n') = \det a' = a_1 \cdots a_k.$$

The lemma follows from this immediately. $\square$

Lemma 1.2. *Suppose* $\mathrm{Re}(na{}^tn) \gg 0$ *for* $n \in N_{\mathbb{C}}$ *and* $a = \mathrm{diag}[a_1, \ldots, a_r] \in A_{\mathbb{C}}$. *Then*

$$\mathrm{Re}\, a_1 > 0, \ldots, \mathrm{Re}\, a_r > 0.$$

Proof. Induction on r. Clear for $r = 1$. Let $r > 1$ and assume the truth of the lemma for $r - 1$. Writing

$$n = \begin{array}{c} \left(\begin{array}{c|c} n' & 0 \\ \hline {}^t\xi & 1 \end{array}\right) \\ \scriptstyle \overset{r-1}{\longleftrightarrow}\,|\,\overset{1}{\longleftrightarrow} \end{array}, \quad a = \begin{array}{c} \left(\begin{array}{c|c} a' & 0 \\ \hline 0 & a_r \end{array}\right) \\ \scriptstyle \overset{r-1}{\longleftrightarrow}\,|\,\overset{1}{\longleftrightarrow} \end{array}, \qquad \begin{array}{l} \scriptstyle \updownarrow\, r-1 \\ \scriptstyle \updownarrow\, 1 \end{array}$$

we obtain

$$na{}^tn = \left(\begin{array}{c|c} n'a'\,{}^tn' & n'a'\xi \\ \hline {}^t\xi\, a'\,{}^tn' & {}^t\xi\, a'\xi + a_r \end{array}\right).$$

Since $\mathrm{Re}(na{}^tn) \gg 0$, we have $\mathrm{Re}(n'a'\,{}^tn') \gg 0$. Then induction hypothesis yields $\mathrm{Re}\, a_1 > 0, \ldots, \mathrm{Re}\, a_{r-1} > 0$. To get $\mathrm{Re}\, a_r > 0$, we note $\mathrm{Re}({}^tn^{-1}a^{-1}n^{-1}) \gg 0$. Then we compute:

$$\begin{aligned} {}^tn^{-1}a^{-1}n^{-1} &= \left(\begin{array}{c|c} {}^tn'^{-1} & \eta \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} a'^{-1} & 0 \\ \hline 0 & a_r^{-1} \end{array}\right)\left(\begin{array}{c|c} n'^{-1} & 0 \\ \hline {}^t\eta & 1 \end{array}\right) \\ &= \left(\begin{array}{c|c} {}^tn'^{-1}a'^{-1}n'^{-1} + \eta\, a_r^{-1}\,{}^t\eta & \eta\, a_r^{-1} \\ \hline a_r^{-1}\,{}^t\eta & a_r^{-1} \end{array}\right). \end{aligned}$$

Thus $\mathrm{Re}\, a_r^{-1} > 0$. Obviously this implies $\mathrm{Re}\, a_r > 0$. □

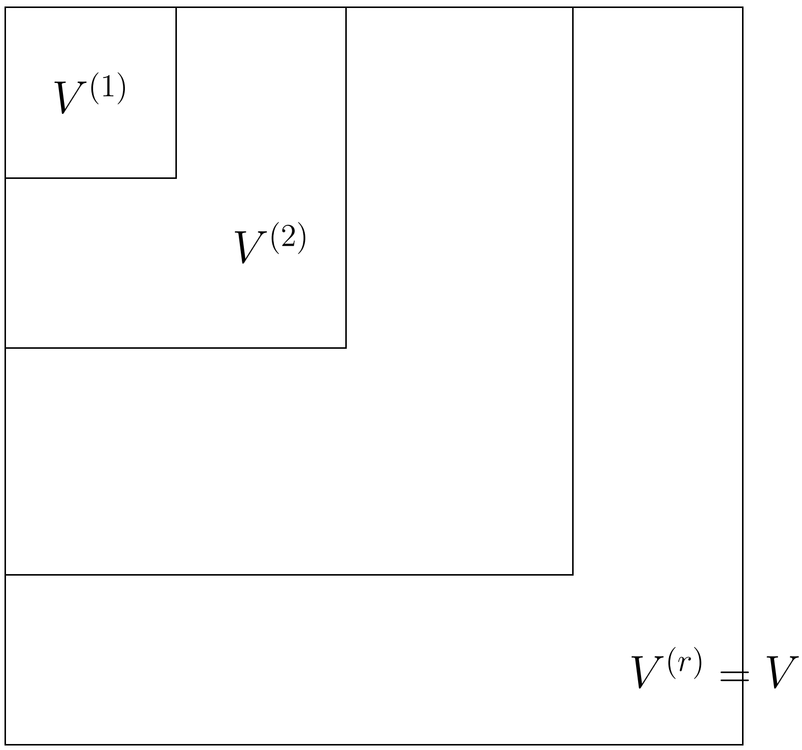
Proposition 1.3. *Let* $w \in \mathrm{Sym}(r, \mathbb{C})$ *and suppose that* $\mathrm{Re}\, w \in \Omega$. *Then*

$$\mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, \ldots, r).$$

Proof. Immediate from Lemmas 1.1 and 1.2. □

The case of general symmetric cone

V : a simple Euclidean Jordan algebra of rank r with e.
$L(x)$: the multiplication operator by $x \in V$.
$\langle\, x \,|\, y \,\rangle = \mathrm{tr}(xy)$: inner product of V.
$c_1, \ldots, c_r$: Jordan frame (so that $e = c_1 + \cdots + c_r$).
$V^{(k)} := V(c_1 + \cdots + c_k; 1)$: Peirce 1-space for $c_1 + \cdots + c_k$.
We have $V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(r)} = V$.
P_k : the orthogonal projector $V \to V^{(k)}$.
$\Delta_k(x) := \det^{(k)}(P_k x)$: the k-th principal minor of x.
($\det^{(k)}$ is the determinant function of the Jordan algebra $V^{(k)}$.)



Jordan frame $c_1, \dots, c_r$ gives $V = \bigoplus_{j<k} V_{jk}$.

V_{11}	V_{12}	$\cdots$	V_{1r}
V_{12}	V_{22}	$\cdots$	V_{2r}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{1r}	V_{2r}	$\cdots$	V_{rr}

$\Omega := \mathrm{Int}\{x^2 \ ; \ x \in V\}$: the symmetric cone of V.
$\mathfrak{g} := \mathrm{Lie}\, G(\Omega)$, $\quad \mathfrak{k} := \mathrm{Der}\, V$, $\quad \mathfrak{p} := \{L(x) \ ; \ x \in V\}$.
Then, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (Cartan decomposition). $\quad \theta X = - {}^t X$.
$R := \mathbb{R}c_1 \oplus \cdots \oplus \mathbb{R}c_r$.
$\mathfrak{a} := \{L(a) \ ; \ a \in R\}$: abelian and maximal in $\mathfrak{p}$.
$\alpha_1, \dots, \alpha_r$: the basis of $\mathfrak{a}^*$ dual to $L(c_1), \dots, L(c_r)$.
The positive $\mathfrak{a}$-roots of $\mathfrak{g}$ are $\frac{1}{2}(\alpha_k - \alpha_j)$ $(k > j)$ and

$$\mathfrak{g}_{(\alpha_k - \alpha_j)/2} = \{z \,\square\, c_j \ ; \ z \in V_{jk}\} =: \mathfrak{n}_{kj},$$

where $a \,\square\, b := L(ab) + [\, L(a), L(b)\,]$.
Putting $\mathfrak{n} := \sum_{j<k} \mathfrak{n}_{kj}$, we get

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \quad \text{(Iwasawa decomposition)}.$$

$A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$.
$W := V_{\mathbb{C}}$. $\quad$ Then, $W = \bigoplus_{j<k} W_{jk}$ with $W_{jk} := (V_{jk})_{\mathbb{C}}$.
$A_{\mathbb{C}}$, $N_{\mathbb{C}}$: complexifications of A, N. Note $A_{\mathbb{C}}, N_{\mathbb{C}} \subset GL(W)$.
Δ_k : considered as holomorphic polynomial fctn on W.

If c is an idempotent, we put

$$\tau_c(z) := \exp(2z \,\square\, c) \quad (z \in W(c; \tfrac{1}{2})).$$

The operators $\tau_c(z)$ are called the Frobenius operators. They are unipotent.

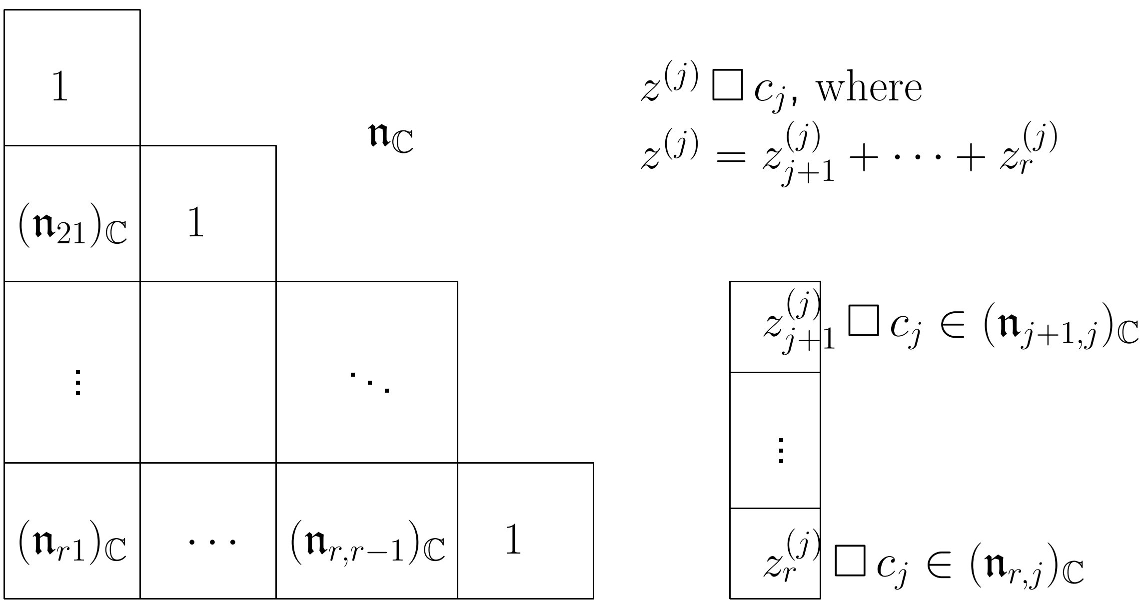
Lemma 1.4. *Suppose $w \in W$ satisfies $\Delta_k(w) \neq 0$ for any $k = 1, \ldots, r$. Then*

$$\exists 1\ z^{(j)} \in \bigoplus_{m=j+1}^{r} W_{jm}\ (1 \leqq j \leqq r-1),\ a_1 \in \mathbb{C}^\times, \ldots, a_r \in \mathbb{C}^\times$$

such that

$$w = \tau_{c_1}(z^{(1)})\tau_{c_2}(z^{(2)}) \cdots \tau_{c_{r-1}}(z^{(r-1)})(a_1c_1 + \cdots + a_rc_r).$$

Note that $\tau_{c_j}(z^{(j)}) \in N_{\mathbb{C}}$ for $j = 1, \ldots, r-1$.



Lemma 1.5. *In Lemma 1.4, the numbers $a_1, \dots, a_r$ are given by*

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1, \dots, r),$$

where $\Delta_0(w) \equiv 1$.

Proof. Consequence of

$$P_k\big(\tau_{c_1}(z^{(1)})\tau_{c_2}(z^{(2)}) \cdots \tau_{c_{r-1}}(z^{(r-1)})(a_1 c_1 + a_2 c_2 + \cdots + a_r c_r)\big)$$
$$= \tau_{c_1}^{(k)}\big(P_k(z^{(1)})\big)\tau_{c_2}^{(k)}\big(P_k(z^{(2)})\big) \cdots \tau_{c_{k-1}}^{(k)}\big(P_k(z^{(k-1)})\big)\Big(\sum_{j=1}^{k} a_j c_j\Big),$$

which is just a Jordan algebra variant of the matrix computation done in Lemma 1.2. □

Lemma 1.6. *Let $n \in N_{\mathbb{C}}$ and $a_1 \in \mathbb{C}^\times, \dots, a_r \in \mathbb{C}^\times$. If $w := n \cdot (a_1 c_1 + \cdots + a_r c_r) \in \Omega + iV$, then*

$$\operatorname{Re} a_1 > 0, \dots, \operatorname{Re} a_r > 0.$$

Proof. Just in the same way as Lemma 1.2. We have

$$w^{-1} = {}^t n^{-1} \cdot (a_1^{-1} c_1 + \cdots + a_r^{-1} c_r).$$

The projector onto $W_{rr} = \mathbb{C} c_r$ is given by $Q_r = \langle \cdot \,|\, c_r \rangle c_r$. Hence

$$Q_r(w^{-1}) = \langle {}^t n^{-1} \cdot (a_1^{-1} c_1 + \cdots + a_r^{-1} c_r) \,|\, c_r \rangle c_r$$
$$= \langle a_1^{-1} c_1 + \cdots + a_r^{-1} c_r \,|\, n^{-1} c_r \rangle c_r.$$

Since $n^{-1} c_r = c_r$, we get $Q_r(w^{-1}) = a_r^{-1} c_r$. Since

$$\operatorname{Re} Q_r(w^{-1}) = Q_r(\operatorname{Re} w^{-1}) \in Q_r(\Omega) = \mathbb{R}_+^\times c_r,$$

we get $\operatorname{Re} a_r^{-1} > 0$. □

Proposition 1.7. *Let $w \in W$ and suppose $\operatorname{Re} w \in \Omega$. Then*

$$\operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, \ldots, r).$$

(1) Lemma 1.6 is proved by making use of the stability of $\Omega + iV$ under the Jordan algebra inverse map $w \mapsto w^{-1}$,
(2) this stability characterizes symmetric tube domains (Kai–N. [4])

$\rightsquigarrow$ it would be quite natural to have the following question:

Question 1.8.
Is Proposition 1.7 characteristic of symmetric cones?

Generalization of Δ_k to a homogenous convex cone $\Omega \rightsquigarrow$
$\Delta_k(x)$: basic relative invariants associated to Ω (Ishi [3]).
(polynomial functions on the ambient vector space V).
$\Delta_k(w)$: considered as holomorphic polynomial fctns on $W := V_{\mathbb{C}}$.

- If Ω is symmetric, then Δ_k are the JA principal minors.

Now Question 1.8 can be formulated in the following way:

Conjecture 1.9. *With the above notation, the implication for $w \in V_{\mathbb{C}}$ that*

$$\operatorname{Re} w \in \Omega \implies \operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r)$$

is equivalent to the symmetry of Ω.

The aim is to <u>present a counterexample to Conjecture 1.9</u>:

$$\boxed{\begin{array}{l}\exists \text{ non-symmetric homogeneous cone } \Omega \text{ for which} \\ \qquad \operatorname{Re} w \in \Omega \implies \operatorname{Re} \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r).\end{array}}$$

The cone is 8-dimensional as mentioned in the title.

I : 2×2 unit matrix

$$V := \left\{ x = \begin{pmatrix} x_{11} I & x_{21} I & \mathbf{y} \\ x_{21} I & x_{22} I & \mathbf{z} \\ {}^t\mathbf{y} & {}^t\mathbf{z} & x_{33} \end{pmatrix} \; ; \; \mathbf{y} \in \mathbb{R}^2, \; \mathbf{z} \in \mathbb{R}^2, \; x_{ij} \in \mathbb{R} \right\}.$$

Note $V \subset \mathrm{Sym}(5, \mathbb{R})$. In fact $x \in V$ is a 5×5 matrix written as

$$x = \begin{pmatrix} x_{11} & 0 & x_{21} & 0 & y_1 \\ 0 & x_{11} & 0 & x_{21} & y_2 \\ x_{21} & 0 & x_{22} & 0 & z_1 \\ 0 & x_{21} & 0 & x_{22} & z_2 \\ y_1 & y_2 & z_1 & z_2 & x_{33} \end{pmatrix}$$

We take

$$\Omega := \{ x \in V \; ; \; x \gg 0 \}.$$

Let

$$A := \left\{ a = \begin{pmatrix} a_1 I & 0 & 0 \\ 0 & a_2 I & 0 \\ 0 & 0 & a_3 \end{pmatrix} \; ; \; a_1 > 0, \; a_2 > 0, \; a_3 > 0 \right\},$$

$$N := \left\{ n = \begin{pmatrix} I & 0 & 0 \\ \xi I & I & 0 \\ {}^t\mathbf{n}_1 & {}^t\mathbf{n}_2 & 1 \end{pmatrix} \; ; \; \xi \in \mathbb{R}, \; \mathbf{n}_1 \in \mathbb{R}^2, \; \mathbf{n}_2 \in \mathbb{R}^2 \right\}.$$

Then $N \ltimes A \curvearrowright \Omega$ by $(N \rtimes A) \times \Omega \ni (h, x) \mapsto h \, x \, {}^t h \in \Omega$.

The action is simply transitive.
Indeed, if $x \in \Omega$, then the equation $x = n\, a\, {}^t n$ with $a \in A$ and $n \in N$ is solved as

$$a_1 = \Delta_1(x), \qquad a_2 = \frac{\Delta_2(x)}{\Delta_1(x)}, \qquad a_3 = \frac{\Delta_3(x)}{\Delta_2(x)},$$
$$\xi = \frac{x_{21}}{\Delta_1(x)}, \qquad \mathbf{n}_1 = \frac{\mathbf{y}}{\Delta_1(x)}, \qquad \mathbf{n}_2 = \frac{x_{11}\mathbf{z} - x_{21}\mathbf{y}}{\Delta_2(x)}.$$

Here, $\Delta_1, \Delta_2, \Delta_3$ are the polynomial functions on V given by

$$\begin{cases} \Delta_1(x) = x_{11}, \\ \Delta_2(x) = x_{11}x_{22} - x_{21}^2, \\ \Delta_3(x) = x_{11}x_{22}x_{33} + 2x_{21}\mathbf{y}\cdot\mathbf{z} - x_{33}x_{21}^2 - x_{22}\|\mathbf{y}\|^2 - x_{11}\|\mathbf{z}\|^2, \end{cases}$$

$\mathbf{y}\cdot\mathbf{z}$: the canonical inner product in $\mathbb{R}^2$,
$\|\cdot\|$: the corresponding norm.

$\Delta_1(x), \Delta_2(x), \Delta_3(x)$ are the basic relative invariants ass. to Ω.
If $\delta_k(x)$ $(k = 1, \ldots, 5)$ stands for the k-th principal minor of the 5×5 matrix $x = \begin{pmatrix} x_{11} & 0 & x_{21} & 0 & y_1 \\ 0 & x_{11} & 0 & x_{21} & y_2 \\ x_{21} & 0 & x_{22} & 0 & z_1 \\ 0 & x_{21} & 0 & x_{22} & z_2 \\ y_1 & y_2 & z_1 & z_2 & x_{33} \end{pmatrix} \in V$, then

$$\delta_1(x) = \Delta_1(x), \quad \delta_2(x) = \Delta_1(x)^2, \quad \delta_3(x) = \Delta_1(x)\Delta_2(x),$$
$$\delta_4(x) = \Delta_2(x)^2, \quad \delta_5(x) = \Delta_2(x)\Delta_3(x).$$

Therefore

$$x \in \Omega \iff \Delta_k(x) > 0 \text{ for any } k = 1, 2, 3.$$

(This is true for general homogeneous cones by Ishi [3].)

The dual cone of Ω:

$$V' := \left\{ x = \begin{pmatrix} x_{11} & x_{21} & {}^t\mathbf{y} \\ x_{21} & x_{22} & {}^t\mathbf{z} \\ \mathbf{y} & \mathbf{z} & x_{33} I \end{pmatrix} \;;\; \mathbf{y} \in \mathbb{R}^2,\ \mathbf{z} \in \mathbb{R}^2,\ x_{ij} \in \mathbb{R} \right\}$$

Note $V' \subset \mathrm{Sym}(4, \mathbb{R})$. In fact $y \in V$ is a 4×4 matrix written as

$$x = \begin{pmatrix} x_{11} & x_{21} & y_1 & y_2 \\ x_{21} & x_{22} & z_1 & z_2 \\ y_1 & z_1 & x_{33} & 0 \\ y_2 & z_2 & 0 & x_{33} \end{pmatrix}.$$

Ω' is the set of positive definite ones in V':

$$\Omega' := \{x \in V' \;;\; x \gg 0\}.$$

The duality mapping is given by

$$\langle x, x' \rangle = \sum_{j=1}^{3} x_{jj} x'_{jj} + 2\,\mathbf{y} \cdot \mathbf{y}' + 2\,\mathbf{z} \cdot \mathbf{z}' + 2 x_{21} x'_{21}$$

for $x = \begin{pmatrix} x_{11} I & x_{21} I & \mathbf{y} \\ x_{21} I & x_{22} I & \mathbf{z} \\ {}^t\mathbf{y} & {}^t\mathbf{z} & x_{33} \end{pmatrix} \in \Omega$ and $x' = \begin{pmatrix} x'_{11} & x'_{21} & {}^t\mathbf{y}' \\ x'_{21} & x'_{22} & {}^t\mathbf{z}' \\ \mathbf{y}' & \mathbf{z}' & x'_{33} I \end{pmatrix} \in \Omega'$.

$$A := \left\{ a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 I \end{pmatrix} \ ; \ a_1 > 0,\ a_2 > 0,\ a_3 > 0 \right\},$$
$$N := \left\{ n = \begin{pmatrix} 1 & \xi & {}^t\mathbf{n}_1 \\ 0 & 1 & {}^t\mathbf{n}_2 \\ 0 & 0 & I \end{pmatrix} \ ; \ \xi \in \mathbb{R},\ \mathbf{n}_1 \in \mathbb{R}^2,\ \mathbf{n}_2 \in \mathbb{R}^2 \right\}.$$

$N \rtimes A \curvearrowright \Omega'$ by $(N \rtimes A \times \Omega') \ni (h, x) \mapsto h\, x\, {}^t h \in \Omega'$.
The action is simply transitive.
The basic relative invariants $\Delta'_k(x)$ $(k = 1, 2, 3)$ are

$$\begin{cases} \Delta'_1(x) := x_{33} \\ \Delta'_2(x) := x_{22}x_{33} - \|z\|^2, \\ \Delta'_3(x) := \det x \qquad \text{(as } 4 \times 4 \text{ matrix).} \end{cases}$$

Note that $\deg \Delta'_3(x) = 4$.
If $\delta^*_k(x)$ $(k = 1, 2, 3, 4)$ denotes the k-th principal minor taken from the right-lower corner of the matrix $x = \begin{pmatrix} x_{11} & x_{21} & y_1 & y_2 \\ x_{21} & x_{22} & z_1 & z_2 \\ y_1 & z_1 & x_{33} & 0 \\ y_2 & z_2 & 0 & x_{33} \end{pmatrix}$, then

$$\delta^*_1(x) = \Delta'_1(x), \qquad \delta^*_2(x) = \Delta'_1(x)^2,$$
$$\delta^*_3(x) = \Delta'_1(x)\Delta'_2(x), \qquad \delta^*_4(x) = \Delta'_3(x).$$

Thus $\quad x \in \Omega' \iff \Delta'_k(x) > 0 \quad$ for any $k = 1, 2, 3$.
But $w = \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & (1+2i)I \end{pmatrix} \in \Omega' + iV'$ gives

$$\mathrm{Re}\, \frac{\Delta'_3(w)}{\Delta'_2(w)} = \mathrm{Re}\{(1+i)(1+2i)\} = -1 < 0.$$

Return to Ω

Regard $\mathbf{y}\cdot\mathbf{z}$ as $\mathbb{C}$-bilinear on $\mathbb{C}^2$.
Write $\nu(\mathbf{y}) := \mathbf{y}\cdot\mathbf{y}$ instead of $\|\mathbf{y}\|^2$ (and similarly for $\nu(\mathbf{z})$).

$$\begin{cases} \Delta_1(x) = x_{11}, \\ \Delta_2(x) = x_{11}x_{22} - x_{21}^2, \\ \Delta_3(x) = x_{11}x_{22}x_{33} + 2x_{21}\mathbf{y}\cdot\mathbf{z} - x_{33}x_{21}^2 - x_{22}\nu(\mathbf{y}) - x_{11}\nu(\mathbf{z}), \end{cases}$$

$A_{\mathbb{C}}, N_{\mathbb{C}}$: the complexifications of A, N, respectively.
We know

$$\Omega + iV \subset N_{\mathbb{C}}A_{\mathbb{C}}\cdot E,$$

where E is the 5×5 identity matrix. Note $E \in \Omega$.
Hence $\Delta_k(x) \neq 0$ on $\Omega + iV$ for $k = 1, 2, 3$.

Proposition 1.10. *Suppose $w \in V_{\mathbb{C}}$ and $\operatorname{Re} w \in \Omega$. Then*

$$\operatorname{Re}\frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, 2, 3).$$

Proof. Let $a \in A_{\mathbb{C}}$ and $n \in N_{\mathbb{C}}$. Then

$$n a {}^t n = \begin{pmatrix} a_1 I & a_1\xi I & a_1\mathbf{n}_1 \\ \xi a_1 I & (\xi^2 a_1 + a_2)I & \xi a_1\mathbf{n}_1 + a_2\mathbf{n}_2 \\ a_1{}^t\mathbf{n}_1 & a_1\xi{}^t\mathbf{n}_1 + a_2{}^t\mathbf{n}_2 & a_1\nu(\mathbf{n}_1) + a_2\nu(\mathbf{n}_2) + a_3 \end{pmatrix}.$$

Hence, given $w \in \Omega + iV$, we can solve the equation $w = n a {}^t n$ for $a \in A_{\mathbb{C}}$ and $n \in A_{\mathbb{C}}$, so that

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1, 2, 3).$$

Since $\Omega \subset \operatorname{Pos}(5, \mathbb{R})$, we can apply Lemma 1.2 to the present case with $r = 5$. Then we get $\operatorname{Re} a_k > 0$ $(k = 1, 2, 3)$. $\square$

Interesting cones for developing harmonic analysis

$$V := \left\{ x = \begin{pmatrix} x_{11}I & x_{21}I & \mathbf{y} \\ x_{21}I & x_{22}I & \mathbf{z} \\ {}^t\mathbf{y} & {}^t\mathbf{z} & x_{33} \end{pmatrix} \ ; \ \mathbf{y} \in \mathbb{R}^n, \ \mathbf{z} \in \mathbb{R}^n, \ x_{ij} \in \mathbb{R} \right\}$$

and $\Omega := \{x \in V \ ; \ x \gg 0\}$;

$$V := \left\{ x = \begin{pmatrix} x_{11}I & x_{21}I & \mathbf{y} \\ \overline{x}_{21}I & x_{22}I & \mathbf{z} \\ {}^t\overline{\mathbf{y}} & {}^t\overline{\mathbf{z}} & x_{33} \end{pmatrix} \ ; \ \mathbf{y} \in \mathbb{C}^n, \ \mathbf{z} \in \mathbb{C}^n, \ \begin{matrix} x_{ii} \in \mathbb{R}, \\ x_{21} \in \mathbb{C} \end{matrix} \right\}$$

and $\Omega := \{x \in V \ ; \ x \gg 0\}$.