Paracontrolled calculus and regularity structures

Masato Hoshino

Kyushu University

July 31, 2019

Joint work with Ismaël Bailleul (Université de Rennes 1)

RS vs PC

Regularity structure (RS, Hairer (2014)) and paracontrolled calculus (PC, Gubinelli-Imkeller-Perkowski (2015)) both solve many singular (semilinear or quasilinear) PDEs. They are believed to be equivalent theories, but there are some gaps.

- PC is less general than RS. For example, the general KPZ equation

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + g(h)\xi$$

  cannot be solved within PC.
- No systematic theory in PC ($\leftrightarrow$ "Black box" in RS).
- In PC, solutions are written by existing analytic tools. More informations for specific SPDEs were obtained.

$\Rightarrow$ Can we solve general SPDEs (including general KPZ) within PC?

Rough description of the main result

The first step to implant the algebraic structure of RS to PC. We obtained the equivalence between the two kinds of definitions of the solutions.

Theorem (Bailleul-H. (2018))

Rough path theory	*RS*		*PC*
Rough path	*Model*	$\xleftrightarrow{\textit{equivalence}}$	*Paracontrolled remainders*
Controlled path	*Modelled distr.*	$\xleftrightarrow{\textit{equivalence}}$	*Paracontrolled distr.*

Algebraic renormalization is a future work.

Micro vs Macro

Both of RS and PC are extensions of the rough path theory for SDEs

$$dX = F(X)dB.$$

- RS: microscopic

$$X_t - X_s = F(X_s)(B_t - B_s) + O(|t-s|^{1-}).$$

$\Rightarrow$ Modelled distribution.

- PC: macroscopic

$$X = F(X) \varolessthan B + (C^{1-}), \quad \varolessthan\text{: Bony's paraproduct.}$$

$\Rightarrow$ Paracontrolled distribution.

All we need is to show

microscopic definition $\Leftrightarrow$ macroscopic definition

Related researches

- Gubinelli-Imkeller-Perkowski (2015):

$$\mathcal{R}f \text{ (reconstruction)} = Pf + (C^\gamma), \quad f \in \mathcal{D}^\gamma(\Pi, \Gamma),$$

  where

$$Pf(x) := \sum_{i \ll j} \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_i(x-y) K_j(x-z)(\Pi_y f(y))(z) dydz.$$

  K_i: kernel of the Littlewood-Paley block Δ_i.
- Martin-Perkowski (2018) rewrote the definition of "modelled distribution" by "paramodelled distribution", using the operator P, but the model (microscopic object) is still needed.
- Our result implies that the nonlinear space of all models is topologically isomorphic to the direct product of Banach spaces. Tapia-Zambotti (2018) showed a similar result for the space of branched rough paths.

1. Introduction

2. Regularity structures $\Rightarrow$ Paracontrolled calculus

3. Paracontrolled calculus $\Rightarrow$ Regularity structures

Settings

Let (T^+, T) be a (concrete) RS, i.e.,

- $T^+ = \bigoplus_{\alpha \in A^+} T_\alpha$ is a graded Hopf algebra with unit $\mathbf{1}$ and the coproduct
$$\Delta^+ : T^+ \to T^+ \otimes T^+.$$
- $T = \bigoplus_{\alpha \in A} T_\alpha$ is a graded linear space with the comodule structure over T^+ by the coproduct
$$\Delta : T \to T \otimes T^+.$$

Structural conditions

- For $\sigma \in T_\alpha^+$ with $\alpha > 0$, one has
$$\Delta^+ \sigma \in \sigma \otimes \mathbf{1} + \mathbf{1} \otimes \sigma + \bigoplus_{0<\beta<\alpha} (T_\beta^+ \otimes T_{\alpha-\beta}^+).$$
- For $\tau \in T_\alpha$, one has
$$\Delta \tau \in \tau \otimes \mathbf{1} + \bigoplus_{\beta<\alpha} (T_\beta \otimes T_{\alpha-\beta}^+).$$

Notations

- Each $\tau \in T_\alpha^+$ (or T_α) is said to be have the homogeneity α. We write
$$|\tau| = \alpha.$$
- Fix a homogeneous basis $\mathcal{F}^+$ (or $\mathcal{F}$) of T^+ (or T). For any $\tau, \sigma \in \mathcal{F}^+$ (or $\mathcal{F}$), we define the element $\tau/\sigma \in T^+$ by
$$\Delta^+\tau \text{ (or } \Delta\tau) = \sum_{\sigma \in \mathcal{F}^+ \text{ (or } \mathcal{F})} \sigma \otimes (\tau/\sigma).$$

Model

Let $\mathcal{M}$ be the set of all models (Π, g) for (T^+, T) on $\mathbb{R}^d$, i.e.

- Π is a continuous linear map from T to $S'(\mathbb{R}^d)$. If we define

$$\Pi_x = (\Pi \otimes g_x^{-1})\Delta,$$

  then $\Pi_x \tau \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $C^{|\tau|}$ at the point x, for any $\tau \in \mathcal{F}$.

- $g : x \mapsto g_x$ is a map from $\mathbb{R}^d$ to the character group G of T^+. If we define

$$g_{yx} = (g_y \otimes g_x^{-1})\Delta^+,$$

  then

$$|g_{yx}(\sigma)| \lesssim |y - x|^{|\sigma|},$$

  for any $\sigma \in \mathcal{F}^+$.

In general, $\mathcal{M}$ is not a linear space.

Modelled distribution

Let $\gamma \in \mathbb{R}$ and $Z \in \mathcal{M}$. Denote by $\mathcal{D}^\gamma(Z)$ the set of all T-valued γ-class functions

$$f(x) = \sum_{|\tau|<\gamma} f_\tau(x)\tau, \quad x \in \mathbb{R}^d$$

modelled by Z, that is,

$$f_\tau(y) = \sum_{|\sigma|<|\tau|} f_\sigma(x) g_{yx}(\tau/\sigma) + O(|y-x|^{\gamma-|\tau|}), \quad \tau \in \mathcal{F}.$$

In general, $\mathcal{D}^\gamma(Z)$ is not a direct product of the Hölder spaces.

Model $\Rightarrow$ Paracontrolled remainders

Proposition (Bailleul-H. (2018))

Let $Z = (\Pi, g) \in \mathcal{M}$. There exist continuous linear maps $[\cdot]^Z : T \to \mathcal{S}'(\mathbb{R}^d)$ and $[\cdot]^g : T^+ \to C(\mathbb{R}^d)$ with the following properties.

- *For any $\tau \in T_\alpha$, one has $[\tau]^Z \in C^\alpha$, and*

$$\Pi\tau = \sum_{\eta \in \mathcal{F}, |\eta| < \alpha} g(\tau/\eta) \otimes [\eta]^Z + [\tau]^Z.$$

- *For any $\sigma \in T_\alpha^+$, one has $[\sigma]^g \in C^\alpha$, and*

$$g(\sigma) = \sum_{\zeta \in \mathcal{F}^+, |\zeta| < \alpha} g(\sigma/\zeta) \otimes [\zeta]^g + [\sigma]^g.$$

In general, $\Pi\tau \notin C^{|\tau|}$ nor $g(\sigma) \notin C^{|\sigma|}$.

Modelled distr. $\Rightarrow$ Paracontrolled distr.

Proposition (Bailleul-H. (2018))

Let $\gamma \in \mathbb{R}$. *For any modelled distribution* $f = \sum_{|\tau|<\gamma} f_\tau \tau \in \mathcal{D}^\gamma(Z)$, *one has*

$$f_\sigma = \sum_{|\sigma|<|\tau|<\gamma} f_\tau \varolessthan [\tau/\sigma]^g + [f]_\sigma^Z, \quad \sigma \in \mathcal{F},$$

with $[f]_\sigma^Z \in C^{\gamma-|\sigma|}$. *Moreover, the reconstruction* $\mathcal{R}^Z f$ *has the form*

$$\mathcal{R}^Z f = \sum_{|\tau|<\gamma} f_\tau \varolessthan [\tau]^Z + [f]^Z,$$

where $[f]^Z \in C^\gamma$.

Paracontrolled remainders $\Rightarrow$ Model

Can we recover the model (Π, g) from the paracontrolled remainders $\{[\tau]^Z\}_{\tau\in\mathcal{F}}$ and $\{[\sigma]^g\}_{\sigma\in\mathcal{F}^+}$?

Proposition (Bailleul-H. (2018))

Assume that the map $g : \mathbb{R}^d \to G$ is given and satisfies the continuity condition in the definition of models. Then for a given family

$$\{[\tau] \in C^{|\tau|}\}_{\tau\in\mathcal{F};|\tau|\le 0},$$

there exists a unique model $Z = (\Pi, g)$ such that $[\tau]^Z = [\tau]$.

cf. Lyons' extension theorem and Hairer's reconstruction theorem.

Assume that T^+ is constructed from

- a finite set S of generating symbols,
- polynomials $\{X_i\}_{i=1}^d$,
- derivatives $\{\partial_i\}_{i=1}^d$.

Additionally assume that T^+ is "well-ordered", i.e., any basis τ is generated from the "simpler" bases σ than τ.

Proposition (Bailleul-H., in preparation)

For any given family

$$\{[\sigma] \in C^{|\sigma|}\}_{\sigma \in S},$$

there exists a unique model g such that $[\sigma]^g = [\sigma]$ and $g_x(X_i) = x_i$.

e.g. Hopf algebra of rooted trees (Bruned-Hairer-Zambotti (2016)).

Conclusion

Proposition (Bailleul-H., in preparation)

The following topological isomorphisms hold.

$$\mathcal{M} \simeq \prod_{\tau\in\mathcal{F},|\tau|\le 0} C^{|\tau|} \times \prod_{\sigma\in\mathcal{F}^+,\sigma\in S} C^{|\sigma|},$$

$$\mathcal{D}^\gamma(Z) \simeq \prod_{\tau\in\mathcal{F},|\tau|<\gamma} C^{\gamma-|\tau|}, \quad \gamma\in\mathbb{R}.$$

(cf. Tapia-Zambotti (2018) for the case of rough paths.)
It should be possible to solve general SPDEs (including general KPZ) by using only the right hand sides and the classical Bony's paraproduct.