COMPLEX NETWORKS:
STRUCTURE AND FUNCTIONALITY
I. SPECTRA

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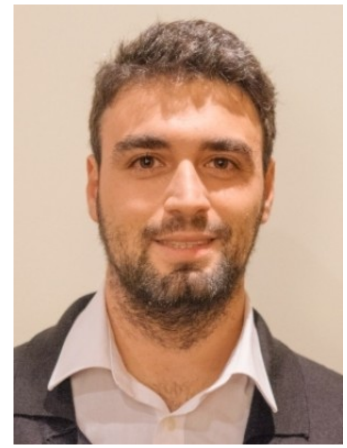
Spectra of random matrices have been analysed for almost a century. In recent years, many interesting results have been derived for spectra of random matrices associated with networks.

QUESTION ADDRESSED IN THIS TALK:

What can be said about the spectrum of the adjacency matrix of a large inhomogeneous Erdős-Rényi random graph?





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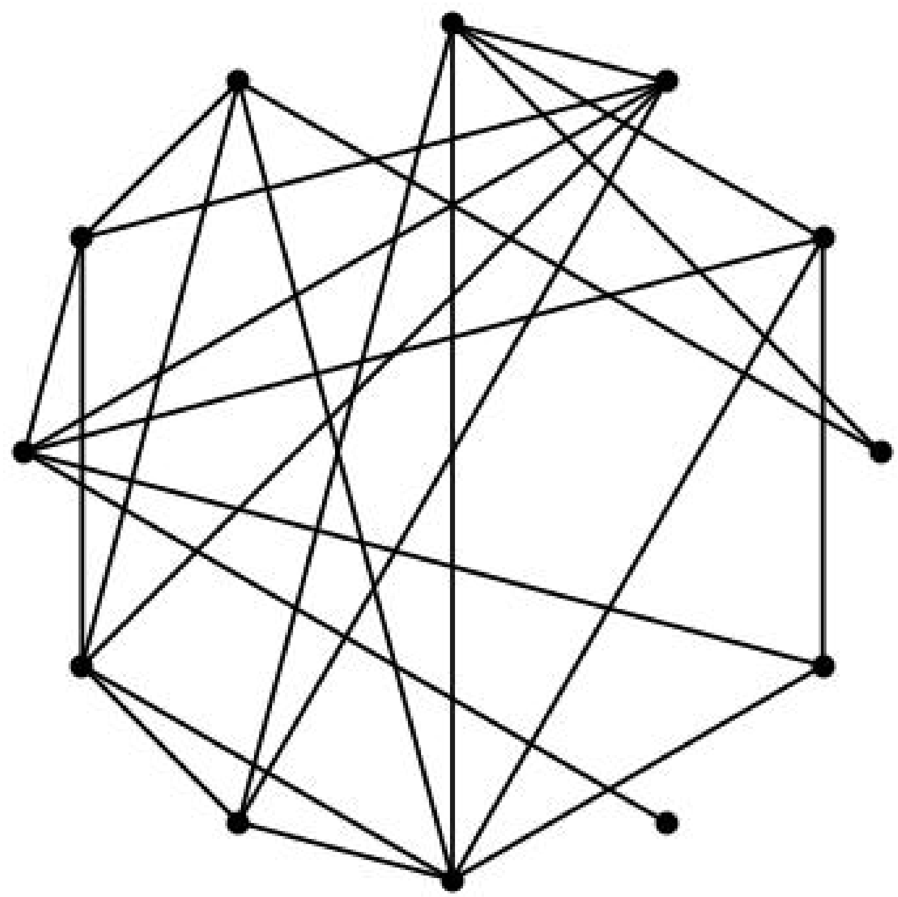
Submitted to *Random Matrices: Theory and Applications*



Erdős-Rényi random graph

§ SETTING

1. Let $(\varepsilon_N)_{N\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\lim_{N\to\infty} \varepsilon_N = 0, \qquad \lim_{N\to\infty} N\varepsilon_N = \infty.$$

Let $f\colon [0,1]\times[0,1] \to [0,\infty)$ be a continuous function such that $f(x,y) = f(y,x)$ for all $x,y \in [0,1]$.

2. Fix $N \in \mathbb{N}$, and consider the inhomogeneous Erdös-Rényi random graph ER_N on N vertices where an edge is placed between the pair of vertices $\{i,j\}$ with probability

$$\varepsilon_N\, f\left(\tfrac{i}{N}, \tfrac{j}{N}\right), \quad 1 \le i,j \le N,$$

independently for different edges. Write $\mathbb{P}$ for the law of ER_N.

3. Let A_N be the adjacency matrix of ER_N. Write

$$\lambda_i(A_N), \qquad 1 \leq i \leq N,$$

for the real eigenvalues of A_N. The empirical spectral distribution of A_N is defined as

$$\mathsf{ESD}(A_N) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(A_N)},$$

which is a random probability distribution on $\mathbb{R}$.

§ SCALING

THEOREM 1:

There exists a compactly supported symmetric probability measure μ *on* $\mathbb{R}$ *such that, weakly in* $\mathbb{P}$*-probability,*

$$\lim_{N\to\infty} \mathrm{ESD}\left(A_N \Big/ \sqrt{N\varepsilon_N}\right) = \mu.$$

Furthermore, if

$$\min_{x,y\in[0,1]} f(x,y) > 0,$$

then μ *is absolutely continuous with respect to Lebesgue measure. The density of* μ *can be characterised implicitly via an integral equation for its Stieltjes transform.*

Zhu 2018

It is possible to identify μ when

$$f(x,y) = r(x)r(y), \qquad x, y \in [0,1],$$

for some continuous function $r\colon [0,1] \to [0,\infty)$.

THEOREM 2:

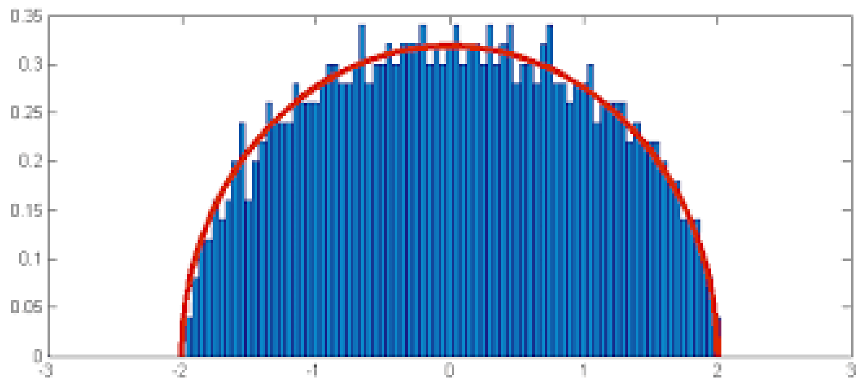
If f is of product form, then

$$\mu = \mu_r \boxtimes \mu_s,$$

where

$$\mu_r = \text{LAW}[r(U)], \quad U = \text{UNIF}[0,1],$$
$$\mu_s = \text{standard Wigner semicircle law},$$

and $\boxtimes$ denotes free multiplicative convolution.



Wigner 1955

In free probability, the Wigner semicircle law takes over the role of the normal law in classical probability. The so-called free cumulants replace the classical cumulants, in the sense that partitions are replaced by non-crossing partitions.

Just as the cumulants of degree ≥ 2 are all zero if and only if the distribution is normal, the free cumulants of degree ≥ 2 are all zero if and only if the distribution is the Wigner semicircle law.

THEOREM 3:

Theorems 1–2 can be generalized to the situation where the function f is random, depends on N and converges to a deterministic limit as $N \to \infty$.

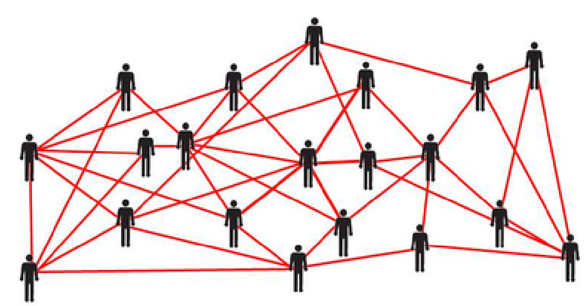
Key ingredients of the proof are:

centering, Gaussianisation, perturbation, decoupling, combinatorics from free probability, ...

§ APPLICATION 1 statistics for social networks

Consider a community of N individuals, represented by the vertices in ER_N. Data is available about which individuals are acquainted. Based on this data, the sociability pattern of the community has to be inferred statistically.

Let ρ denote a probability measure on $[0, \infty)$ with bounded support. Let $(R_i)_{1 \leq i \leq N}$ be i.i.d. random variables drawn from ρ. Think of R_i as the sociability index of individual i.



Pick N so large that

$$0 \le \varepsilon_N R_i R_j \le 1 \qquad \forall\, 1 \le i, j \le N.$$

Suppose that i, j are acquainted with probability $\varepsilon_N R_i R_j$, which is represented by an edge in ER_N between vertices i, j. The data that is available is the adjacency matrix A_N.

The statistical inference problem is to estimate ρ from A_N. To standardise ρ, we assume that

$$\int_0^\infty x \rho(dx) = 1.$$

Since, weakly $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{R_i} = \rho,$$

Theorem 3 gives that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\left(A_N \Big/ \sqrt{N\varepsilon_N}\right) = \rho \boxtimes \mu_s.$$

In practice, ε_N is unknown, which can be worked around by arguing that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\left(\sqrt{\frac{N}{\mathrm{Tr}(A_N^2)}}\, A_N\right) = \rho \boxtimes \mu_s.$$

The procedure is that $\rho \boxtimes \mu_s$ can be statistically estimated from A_N. Subsequently, ρ can be estimated because the moments of $\rho \boxtimes \mu_s$ are functions of the moments of ρ and μ_s.

Indeed, since the moments of μ_s are known, the moments of ρ can be recursively computed from the moments of $\rho \boxtimes \mu_s$. Since ρ is compactly supported, it can in turn be computed via its moments.

§ APPLICATION 2 soft configuration model

Let $\mathcal{S}_N$ be the set of simple graphs on N vertices. We fix the degrees of all the vertices, namely, vertex i has degree d_i^*, where

$$\vec{d}_N^* = \{d_i^*\}_{1 \le i \le N}$$

is a sequence of positive integers of which we only require that it is graphical, i.e., there is at least one simple graph matching these degrees.

hard configuration model

The Gibbs canonical ensemble P_N is the unique probability distribution on $\mathcal{S}_N$ with the following two properties:

(I) The average degree of vertex i, defined by

$$\sum_{G \in \mathcal{S}_N} d_i(G) P_N(G),$$

equals d_i^* for all $i \leq i \leq N$.

(II) The entropy of P_N, defined by

$$-\sum_{G \in \mathcal{S}_N} P_N(G) \log P_N(G),$$

is maximal.

P_N models a random graph of which we have no prior information other than the average degrees.

soft configuration model

Property (II) forces P_N to take the form Jaynes 1957

$$P_N(G) = \frac{1}{Z_N(\vec{\theta}^*)} \exp\left[-\sum_{i=1}^{N} \theta_i^* d_i(G)\right], \qquad G \in \mathcal{S}_N,$$

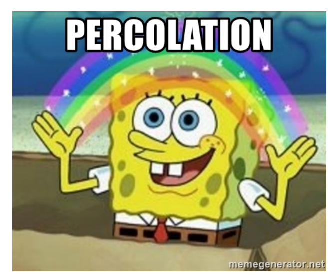
where $\vec{\theta}_N^* = \{\theta_i^*\}_{1 \le i \le N}$ is the unique sequence of Lagrange multipliers such that property (I) is satisfied.

Reparametrisation yields

$$P_N(G) = \prod_{1 \le i < j \le N} (p_{ij}^*)^{A_N[G](i,j)} \, (1 - p_{ij}^*)^{1 - A_N[G](i,j)}, \; G \in \mathcal{S}_N,$$

where $A_N[G]$ is the adjacency matrix of G, and

$$p_{ij}^* = \frac{x_i^* x_j^*}{1 + x_i^* x_j^*}, \quad x_i^* = e^{-\theta_i^*}, \qquad 1 \le i \ne j \le N.$$



Property (I) requires that

$$d_i^* = \sum_{\substack{1 \le j \le N \\ j \neq i}} p_{ij}^*, \qquad 1 \le i \le N,$$

which constitutes a set of N equations for N unknowns.

Abbreviate

$$m_N = \max_{1 \le i \le N} d_i^*.$$

We focus on the regime

$$\boxed{\lim_{N \to \infty} m_N = \infty, \quad \lim_{N \to \infty} m_N / \sqrt{N} = 0.}$$

Squartini, de Mol, den Hollander, Garlaschelli 2015

It turns out that in this regime

$$p^*_{ij} = [1 + o(1)]\,\frac{d^*_i d^*_j}{\sigma_N}, \qquad N \to \infty,$$

with

$$\sigma_N = \sum_{1 \le i \le N} d^*_i.$$

Pick

$$\varepsilon_N = m_N^2 / \sigma_N.$$

Then

$$\lim_{N\to\infty} \varepsilon_N = 0, \qquad \lim_{N\to\infty} N\varepsilon_N = \infty,$$

and

$$p^*_{ij} = [1 + o(1)]\,\varepsilon_N\,(d^*_i/m_N)(d^*_j/m_N).$$

Under the assumption that

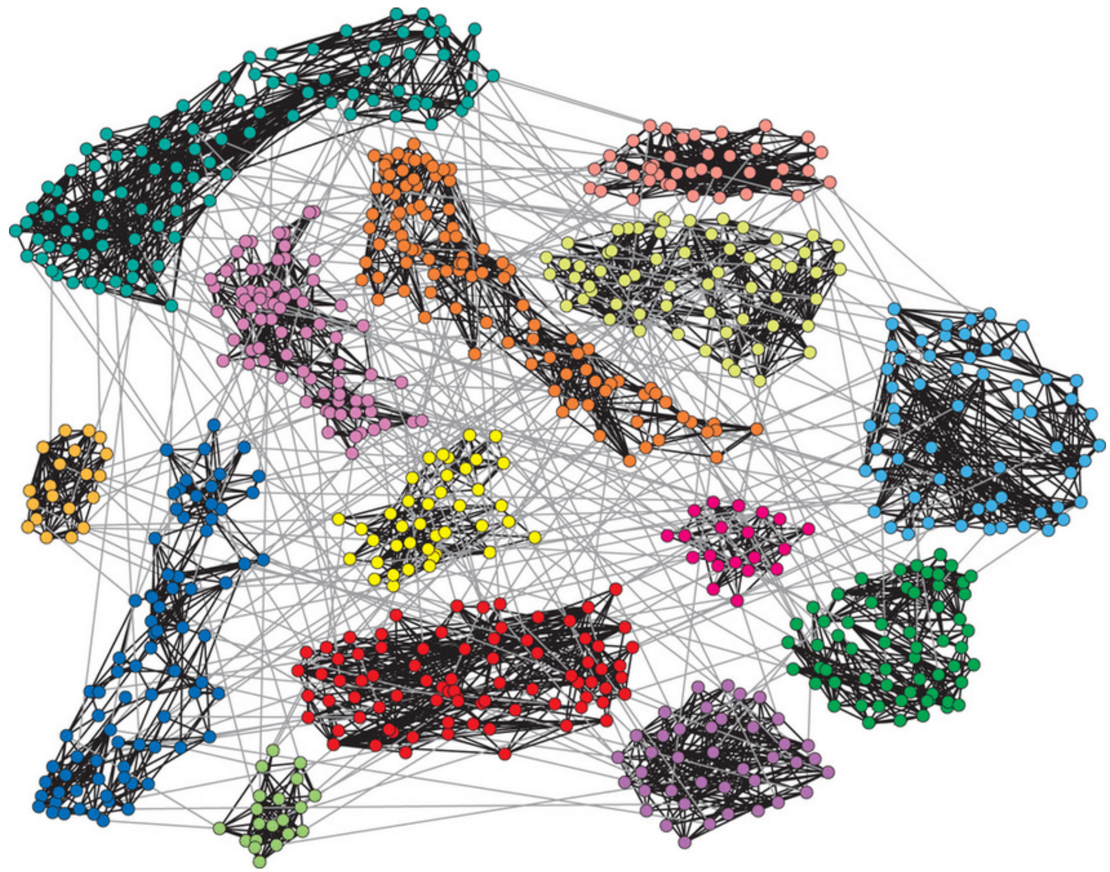
$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{d_i^*/m_N} = \rho$$

for some probability measure ρ, Theorem 3 gives that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\Big(A_N \Big/ \sqrt{N\varepsilon_N}\Big) = \rho \boxtimes \mu_s.$$

This identifies the scaling of the ESD for the network that is modeled by the soft configuration model as a function of the imposed average degrees.

CHALLENGE:



Networks with a community structure