Derivation of viscous Burgers equations from weakly asymmetric simple exclusion processes

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Outline of the talk

1. Overview and Related Works

Overview

- The macroscopic density of the $(\mathrm{WASEP})_n$ evolves according to the nonlinear heat eq. as the system size n grows to infinity (Hydrodynamic limit):

$$\partial_t u = \nabla \cdot [D(u)\nabla u] + \nabla \cdot [\sigma(u)\mathbf{m}] \ , \qquad (1)$$

where D, σ are $d \times d$-matrices (Diffusivity and Mobility).

- For small $\varepsilon > 0$, let us consider the first order correction to (1) around a constant profile $\alpha_0 \in (0,1)$:

$$\begin{cases} \partial_t u^\varepsilon = \nabla \cdot [D(u^\varepsilon)\nabla u^\varepsilon] + \varepsilon^{-1}\nabla \cdot [\sigma(u^\varepsilon)\mathbf{m}] \ , \\ u^\varepsilon(0,\cdot) = \alpha_0 + \varepsilon v_0 \ , \end{cases} \qquad (2)$$

for some smooth function v_0.

- The solution u^ε should evolve as $u^\varepsilon_t \sim \alpha_0 + \varepsilon v_t$.
- Indeed, if $\sigma'(\alpha_0) = 0$, the sequence $\{\varepsilon^{-1}(u^\varepsilon - \alpha_0)\}_{\varepsilon>0}$ converges to the solution to the Burgers eq. as $\varepsilon \downarrow 0$ (Incompressible limit):

$$\begin{cases} \partial_t v \ = \ \nabla \cdot [D(\alpha_0)\nabla v] + (1/2)\nabla \cdot [v^2 \sigma''(\alpha_0)\mathbf{m}] \ , \\ v(0,\cdot) \ = \ v_0(\cdot) \ . \end{cases}$$

- Main Result (rough version):
  Taking $\varepsilon = \varepsilon_n \downarrow 0$ $(n \to \infty)$, the correctly scaled density of the WASEP evolves according to the Burgers eq.

Related Works

- Many results on hydrodynamic limits.
  e.g. Guo-Papanicolaou-Varadhan, 88, Yau, 91.
- Esposito-Marra-Yau, 94, 96 $\cdots$ Derivation of Burgers equation and Navier-Stokes equation ($d \geq 3$).
- Quastel-Yau, 98 $\cdots$ Large deviations for the incompressible limits ($d = 3$).
- Beltán-Landim, 08 $\cdots$ Derivation of Burgers equation and Navier-Stokes equation in any dimensions but with (meso-scopically) big jumps.
- Jara-Menezes, 19+ $\cdots$ Sharp entropy bound.

2. Model and Main Results

Model

- Each particle moves on the d-dimensional discrete torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d = \{1, 2, \cdots, n\}^d$, $n \in \mathbb{N}$. Let $\mathbb{T}^d$ be the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0, 1)^d$.
- Denote the number of particles at site $x \in \mathbb{T}_n^d$ at time t by $\eta_t^n(x)$ ($\eta_t^n = \{\eta_t^n : x \in \mathbb{T}_n^d\} \in \{0, 1\}^{\mathbb{T}_n^d}$).
- Some parameters:
  - $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$: a sequence converging to 0.
  - $(c_j)_{j=1}^d$: nonnegative local functions.
  - $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$: a vector in $\mathbb{R}^d$.

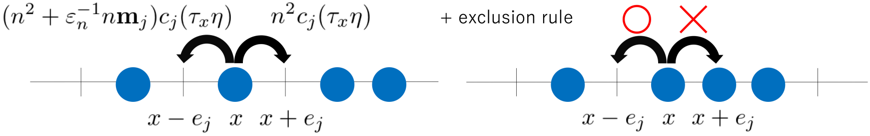
- Let η_t^n be a Markov process on $\{0,1\}^{\mathbb{T}_n^d}$ with the generator $L_n = n^2 L_n^S + \varepsilon_n^{-1} n L_n^A$ with

$$
\begin{aligned}
(L_n^S f)(\eta) &= \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} c_j(\tau_x \eta) \left\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \right\} , \\
(L_n^A f)(\eta) &= \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} \mathbf{m}_j \, c_j(\tau_x \eta) \, \eta_{x+e_j} \, (1 - \eta_x) \\
&\qquad \times \left\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \right\} .
\end{aligned}
$$

for a function $f : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$.

The dynamics of our particle system is as follows:

- Each particle can jump from x to $x+1$ or $x-1$ at given rates only if the site x is occupied and the site $x+1$ or $x-1$ is vacant.



- For a continuous function $u_0 : \mathbb{T}^d \to [0,1]$, let $\nu^n_{u_0}$ be the product Bernoulli measure on $\{0,1\}^{\mathbb{T}^d_n}$:

$$\nu^n_{u_0}(\eta : \eta(x) = 1) \;=\; u_0(x/n)\,, \quad x \in \mathbb{T}^d_n\,.$$

- Gradient condition: For each j, there exist finitely supported signed measures $m_{j,p}$, $p = 1, \ldots, n_j$ and local functions $g_{j,p}$ such that

$$c_j(\eta)[\eta_0 - \eta_{e_j}] \;=\; \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) g_{j,p}(\tau_y \eta)\,,$$

$$\sum_{y \in \mathbb{Z}^d} m_{j,p}(y) \;=\; 0\,.$$

Classical case ($\varepsilon_n = 1$)

- Assume that $\eta_0^n \overset{\mathrm{d}}{=} \nu_{u_0}^n$ for some continuous function $u_0 : \mathbb{T}^d \to [0,1]$.
- Hydrodynamic limit: For any $t \geq 0$ and any smooth function $H : \mathbb{T}^d \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}^n \left[\left| \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} H(x/n)\eta_t^n(x) - \int_{\mathbb{T}^d} H(x)u(t,x)dx \right| \right] = 0 ,$$

where $u : [0,\infty) \times \mathbb{T}^d \to [0,1]$ is the unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t u = \nabla \cdot [D(u)\nabla u] + \nabla \cdot [\sigma(u)\mathbf{m}] , \\ u(0,\cdot) = u_0(\cdot) . \end{cases}$$

Incompressible case ($\varepsilon_n \downarrow 0$)

- Fix $\alpha_0 \in (0,1)$ with $\sigma'(\alpha_0) = 0$ and assume that $\eta_0^n \overset{\mathrm{d}}{=} \nu^n_{\alpha_0 + \varepsilon_n v_0}$ for some function $v_0 \in C^{3+}(\mathbb{T}^d)$.
- Let $v : [0, \infty) \times \mathbb{T}^d \to \mathbb{R}$ be the unique weak (classical) solution of the Burgers eq.:

$$\begin{cases} \partial_t v = \nabla \cdot [D(\alpha_0) \nabla v] + (1/2) \nabla \cdot [v^2 \sigma''(\alpha_0) \mathbf{m}] \ , \\ v(0, \cdot) = v_0(\cdot) \ . \end{cases}$$

- For each $t \geq 0$, let $u_t^n = \alpha_0 + \varepsilon_n v_t, \nu_t^n = \nu^n_{u_t^n}$ and let μ_t^n be the distribution of η_t^n.

Main Results

Theorem 1 (Jara-Landim-T., 19+)

Assume that $n^2\varepsilon_n^4 \le C_0 g_d(n)$ *for some constant* C_0*, where* $g_d(n) = n, \log n, 1$ *if* $d = 1, d = 2, d \ge 3$*, respectively. Then, for any* $T > 0$*, there exists a constant* $C_1 = C_1(T, v_0, C_0)$ *such that for any* $0 \le t \le T$,

$$H(\mu_t^n | \nu_t^n) \le C_1 n^{d-2} g_d(n) ,$$

where $H(\mu|\nu)$ *is the relative entropy of* μ *w.r.t.* ν.

Corollary 2 (Jara-Landim-T., 19+)

Assume that $n^2\varepsilon_n^4 \leq C_0 g_d(n)$ *and* $\varepsilon_n^2 n^2 g_d(n)^{-1} \uparrow \infty$*. For any* $t \geq 0$ *and any smooth function* $H : \mathbb{T}^d \to \mathbb{R}$*,*

$$\lim_{n\to\infty} \mathbb{E}^n \left[\left| \frac{1}{\varepsilon_n n^d} \sum_{x\in\mathbb{T}_n^d} H(x/n)[\eta_t^n(x) - \alpha_0] - \int_{\mathbb{T}^d} H(x) v(t,x) dx \right| \right] = 0 \, ,$$

Remarks

- Initial distribution: The assumption $\eta_0^n \overset{\mathrm{d}}{=} \nu^n_{\alpha_0+\varepsilon_n v_0}$ can be replaced with the entropy bound at time 0.
- $\sigma'(\alpha_0) = 0$: In the case of general $\alpha \in (0,1)$, introducing the Galilean transformation $\alpha + \varepsilon_n v(t, x - \varepsilon_n^{-1}\sigma'(\alpha)\mathbf{m}t)$, we can obtain a similar result.

3. Comments on the Proof

- Following [Jara-Menezes, 19+], we shall compute the entropy production. Let $H_t = H(\mu_t^n|\nu_t^n)$. Then, we have

$$\frac{d}{dt}H_t \ \leq \ -n^2 D(g_t^n, L_n^S, \nu_t^n) + \int \left\{ L_n^{*,\nu_t^n}\mathbf{1} - \partial_t \log \nu_t^n \right\} d\mu_t^n \ .$$

- We need to compute the integrand $L_n^{*,\nu_t^n}\mathbf{1} - \partial_t \log \nu_t^n$ explicitly. Indeed, it can be expressed in terms of the "Fourier coefficients" of $g_{j,p}$ (but quite messy...).
- We also need to expand several terms in ε_n properly: e.g. $E_{\nu_t^n}[\tau_{-e_j} g_{j,p} - g_{j,p}]$.
- Together with these calculations, we shall apply techniques developed in [Jara-Menezes, 19+].

Thank you for your attention.