Lower-tail large deviations of the KPZ equation

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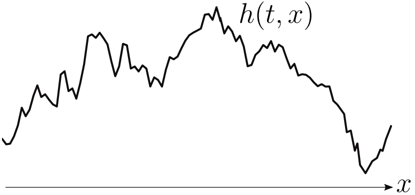
The Kardar–Parisi–Zhang (KPZ) equation

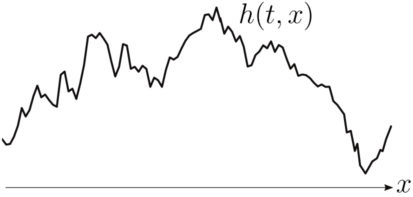


Random growth with smoothing effect and slope dependence

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \xi$$

$\xi = \xi(t,x) =$ spacetime white noise

The Kardar–Parisi–Zhang (KPZ) equation



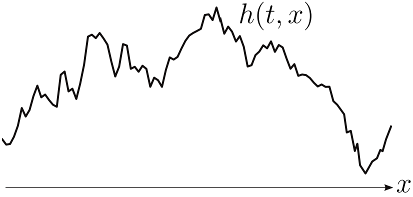
$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \xi \qquad \text{(KPZ)}$$

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \xi Z \qquad \text{(Stochastic HE)}$$

Hopf–Cole
$Z(t,x) := e^{h(t,x)}$

- Define $h(t,x) := \log Z(t,x)$.

The Kardar–Parisi–Zhang (KPZ) equation



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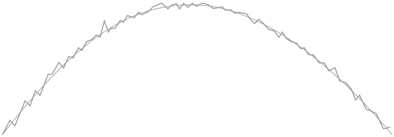
Hopf–Cole $Z(t,x) := e^{h(t,x)}$

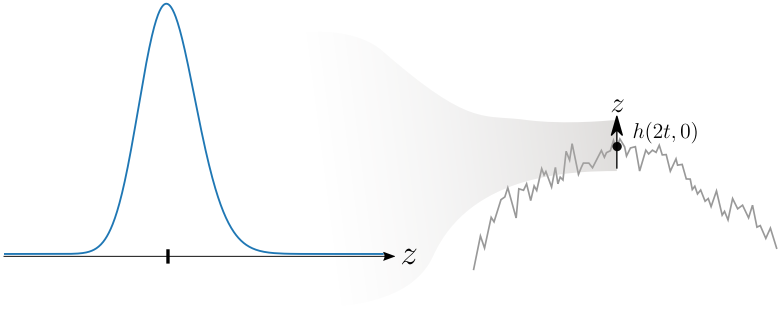
- Define $h(t,x) := \log Z(t,x)$.
- This talk: $Z(0,x) = \delta(x)$.

  For small $t \ll 1$,

  $Z(t,x) \approx \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$.



$t \to \infty$ behaviors: centering, fluctuations, and tails



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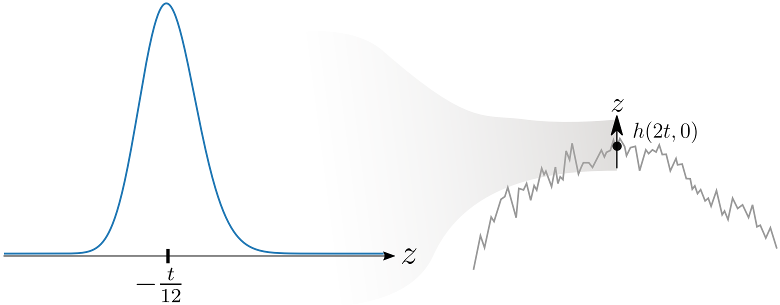
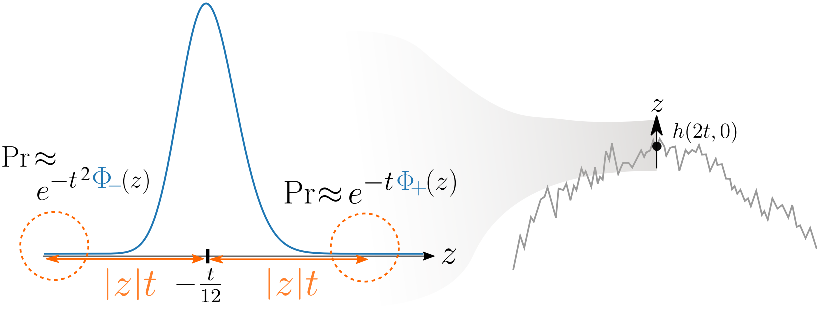


[Amir Corwin Quastel 10], [Calabrese Le Doussal Rosso 10], [Dotsenko 10], [Sasamoto Spohn 10]

For $Z(0, x) = \delta(x)$, as $t \to \infty$,

$$t^{-\frac{1}{3}}\left(h(2t, 0) + \frac{t}{12}\right) \Longrightarrow \text{GUE Tracy Widom}$$

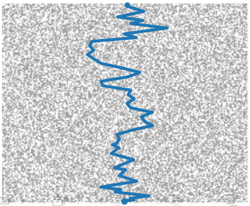
$t \to \infty$ behaviors: centering, fluctuations, and tails



$\Phi_\pm(z)$ = rate functions

Speed t v.s. t^2

$$e^{h(2t,0)} = Z(2t,0) = \mathbf{E}_{\mathsf{BB}}\big[e^{\int_0^{2t} \xi(s,b(2t-s))\mathrm{d}s}\big]$$



Perturbative versus non-perturbative

[Amir Corwin Quastel 10], [Calabrese Le Doussal Rosso 10], [Dotsenko 10], [Sasamoto Spohn 10]

$$\boxed{\mathbf{E}\Big[\exp\big(-e^{\frac{t}{12}+tz}Z(2t,0)\big)\Big] = \det\big(I - K_{t,z}\big)_{L^2(\mathbb{R}_+)}}$$

$$\det(I - K_{t,z}) := 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}_+^n}\det(K_{t,z}(x_i,x_j))_{i,j=1}^n \mathrm{d}^n x$$

$$K_{t,z}(x,x') := \int_{\mathbb{R}_+}(1+\exp(-t^{1/3}\lambda - tz))^{-1}\mathrm{Ai}(x+r)\mathrm{Ai}(x'+r)\mathrm{d}r$$

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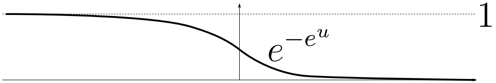

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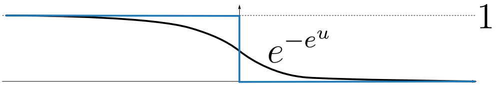

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- Upper tail $z > 0$ as $t \to \infty$, we have $K_{t,z} \to 0$
  - Perturbative: $\det(I - K_{t,z}) = 1 - \mathrm{Tr}(K_{t,z}) + \dots$
  - [Le Doussal Majumdar Schehr 16] predicted $\Phi_+(z) = \frac{4}{3} z^{\frac{3}{2}}$
  - Proof in progress

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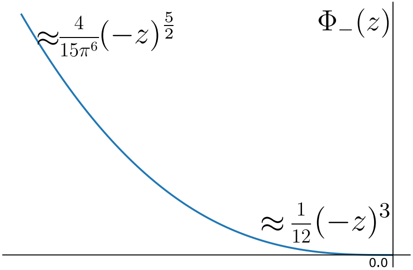
- Lower tail $z < 0$, $K_{t,z} \not\to I$ as $t \to \infty$
  - *Non*-perturbative

The lower-tail of $h(2t, 0)$

Physics results

- [Kolokolov Korshunov 07] and [Meerson Katzav Vilenkin 16] predicted small/large $|z|$ behaviors



Math results

- [Corwin Ghosal 18] obtained bounds ($\forall t \geq t_0$) capturing small/large $|z|$ behaviors

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$$\Phi_-(z) = \frac{4}{15\pi^6}(1 - \pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4} z - \frac{1}{2\pi^2} z^2$$

  by WKB approx of an integral-diff eqn

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Math results

- [Corwin Ghosal 18] obtained bounds $(\forall t \geq t_0)$ capturing small/large $|z|$ behaviors
- [Tsai 18] proof of Φ_- by stochastic Airy operator

Result

Theorem (Tsai 18)

Consider the IC $Z(0,x) = \delta(x)$*. For* $z < 0$*, as* $t \to \infty$*,*

$$\lim_{t\to\infty} \frac{1}{t^2} \log\Big(\mathbf{P}[h(2t,0) + \tfrac{t}{12} < tz]\Big) = -\Phi_-(z)$$

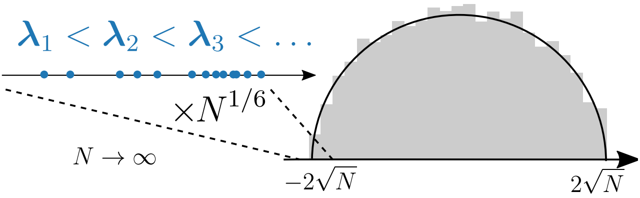
where $\Phi_-(z) := \frac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4} z - \frac{1}{2\pi^2} z^2$*.*

Exponential functional of Airy Point Process

[Borodin Gorin 16]

$$\mathbf{E}\Big[e^{-Z(2t,0)e^{\frac{t}{12}+tz}}\Big] = \mathbf{E}_{\text{Airy}}\Big[\prod_{i=1}^{\infty} \frac{1}{1+e^{-t^{1/3}(\boldsymbol{\lambda}_i+t^{2/3}z)}}\Big]$$

$\boldsymbol{\lambda}_1 < \boldsymbol{\lambda}_2 < \ldots \in \mathbb{R}$ (space-reversed) Airy Point Process



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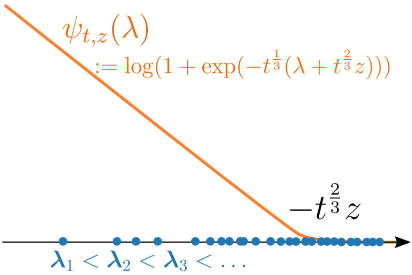
$$\mathbf{E}\left[e^{-Z(2t,0)e^{\frac{t}{12}+tz}}\right] = \mathbf{E}\left[e^{-\sum_{i=1}^{\infty} \psi_{t,z}(\boldsymbol{\lambda}_i)}\right]$$



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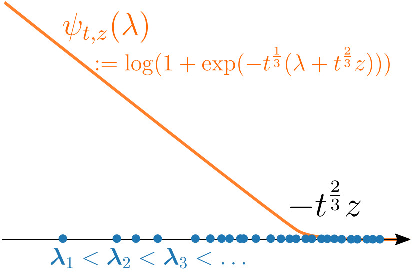

Laplace's method / Varadhan's lemma — a general picture

$$\mathbf{E}\Big[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}\Big] = \int e^{-\psi_{t,z}(\boldsymbol{\rho})}\underbrace{e^{-\mathsf{penalty}(\boldsymbol{\rho})}\mathrm{d}\boldsymbol{\rho}}_{:=\mathrm{d}\mathbf{P}[\boldsymbol{\rho}]}$$

Laplace's method / Varadhan's lemma — a general picture

$$\mathbf{E}\big[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}\big] \approx \exp\big(-\min_{\boldsymbol{\rho}}\big\{\psi_{t,z}(\boldsymbol{\rho}) + \textsf{penalty}(\boldsymbol{\rho})\big\}\big)$$

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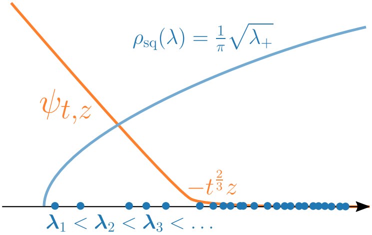
Examples [Corwin Ghosal 18]

$\mathbf{P}[\boldsymbol{\rho} \approx \rho_{\mathsf{sq}}] \approx 1$, but

$e^{-\int_{\mathbb{R}}\psi_{t,z}(\lambda)\rho_{\mathsf{sq}}(\lambda)\mathrm{d}\lambda} \approx e^{-t^2 b_1(z)}$.



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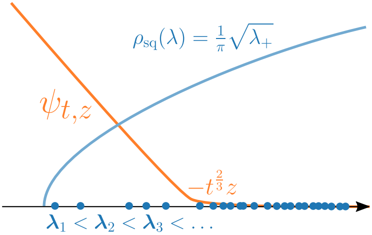
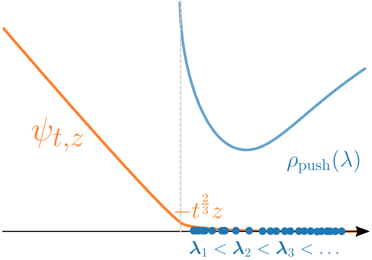


$e^{-\int_{\mathbb{R}}\psi_{t,z}(\lambda)\rho_{\mathsf{push}}(\lambda)\mathrm{d}\lambda} \approx 1$, but

$\mathbf{P}[\boldsymbol{\rho} \approx \rho_{\mathsf{push}}] \approx e^{-t^2 b_2(z)}$.



Stochastic Airy Operator

Theorem (Ramirez Rider Virag 06)

The Stochastic Airy Operator

$$\mathcal{A} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \sqrt{2}W'(x)$$

acting on $\mathrm{Dom}(\mathcal{A}) \subset L^2(\mathbb{R}_+)$ *has spectrum* $\{\boldsymbol{\lambda}_1 < \boldsymbol{\lambda}_2 < \ldots\}$*,*
where $W :=$ *standard BM.*

Large deviations controlled by W'

$$\mathbf{E}[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}] \approx \exp\big(-\min_{\boldsymbol{\rho}}\big\{\psi_{t,z}(\boldsymbol{\rho}) + \mathsf{penalty}(\boldsymbol{\rho})\big\}\big)$$

$\boldsymbol{\rho} =$ eigenvalues distribution of

$$\mathcal{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \sqrt{2}W'(x)$$

Large deviations controlled by W' and then by v

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$\boldsymbol{\rho}$ = eigenvalues distribution of

$$\mathcal{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \sqrt{2}W'(x)$$

Postulate: relevant deviations controlled by $W'(x) \approx t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)$

(eigen prob) $\qquad -f''(x) + xf(x) + \sqrt{2}W'(x)f(x) = \lambda f(x)$

λ of order $t^{\frac{2}{3}}$

Large deviations controlled by W' and then by v

$$\mathbf{E}[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}] \approx \exp\Big(-\min_{v}\big\{\psi_{t,z}(\boldsymbol{\rho}(v)) + \text{penalty}(v)\big\}\Big)$$

$\boldsymbol{\rho}(v)$ = eigenvalues distribution of

$$\mathcal{A}_v = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \sqrt{2}t^{\frac{2}{3}}v(t^{-\frac{2}{3}}x)$$

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penalty(v) =

$\boldsymbol{\rho}(v)$ =

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[LDP of BM] penalty$(v) = \frac{1}{2}\int t^{\frac{4}{3}}v^2(t^{-\frac{2}{3}}x)\mathrm{d}x$

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[WKB approx] $\boldsymbol{\rho}(v) \approx N'_v(\lambda)\mathrm{d}\lambda$

$$N_v(\lambda) = \frac{t}{\pi}\int_0^{\infty}\sqrt{\big(-x + t^{-\frac{2}{3}}\lambda - \sqrt{2}v(x)\big)_+}\mathrm{d}x$$

Large deviations controlled by W' and then by v

$$\mathbf{E}[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}] \approx \exp\Big(-\min_{v}\big\{\psi_{t,z}(\boldsymbol{\rho}(v)) + \text{penalty}(v)\big\}\Big)$$

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Putting things together gives $(\psi_{t,z}(\boldsymbol{\rho}(v)) + \text{penalty}(v)) \approx t^2 J(v)$

$$J(v) = \int_0^{\infty}(\tfrac{1}{2}v^2(x) + ((-x + z - \sqrt{2}v(x))_+)^{\frac{3}{2}})\mathrm{d}x$$

Large deviations controlled by W' and then by v

$$\mathbf{E}[e^{-\sum_{i=1}^{\infty}\psi_{t,z}(\boldsymbol{\lambda}_i)}] \approx \exp\Big(-\min_{v}\big\{\psi_{t,z}(\boldsymbol{\rho}(v)) + \mathsf{penalty}(v)\big\}\Big)$$

Postulate: relevant deviations controlled by $W'(x) \approx t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)$

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$$J(v) = \int_0^{\infty}\big(\tfrac{1}{2}v^2(x) + ((-x + z - \sqrt{2}v(x))_+)^{\frac{3}{2}}\big)\mathrm{d}x$$

which optimized to be

$$\min_{v} J(v) = \frac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2 = \Phi_-(z)$$

Further discussion

- More general cost functions, when does *Postulate* hold?
- What happens when *Postulate* fails?
- Full LDP of $\{\boldsymbol{\lambda}_i\}_{i=1}^{\infty}$. Rate function conjectured in [Corwin Ghosal Krajenbrink Le Doussal Tsai 18]