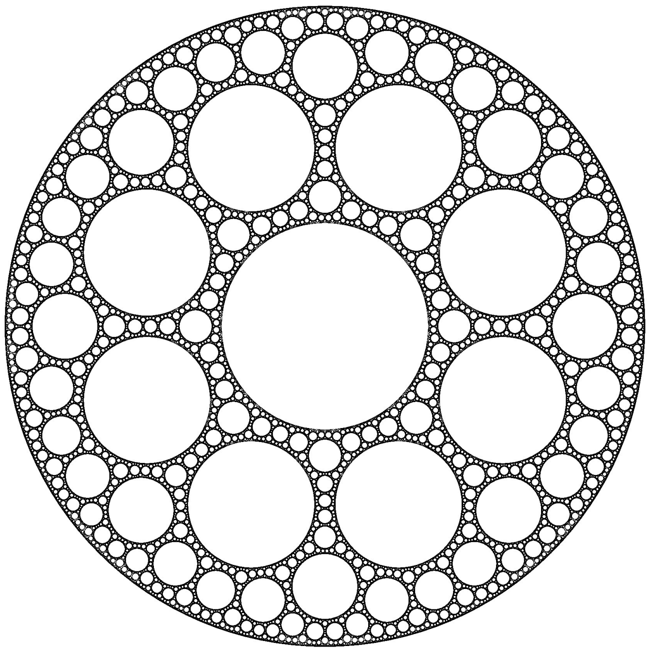
The Laplacian on some self-conformal fractals and Weyl's asymptotics for its eigenvalues

Naotaka Kajino (Kobe University)
梶野 直孝（神戸大学）

12th MSJ-SI: Stoch. Anal., Random Fields & Integrable Probab.

@Kyushu University, Fukuoka, Japan



August 9, 2019

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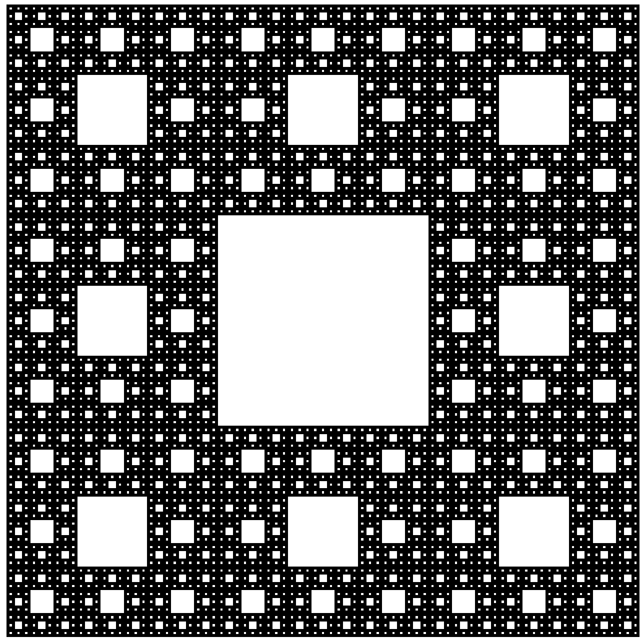
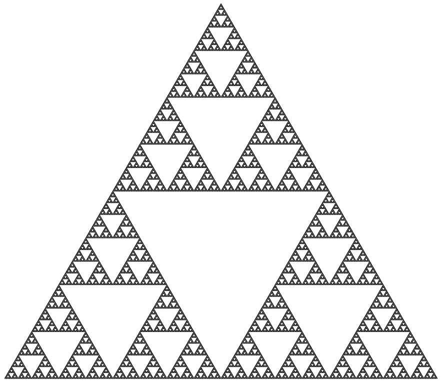


1 Sierpiński gasket & carpet in different geometries

(self-similar) SG



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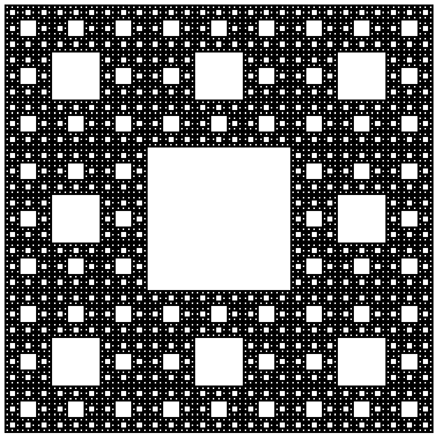

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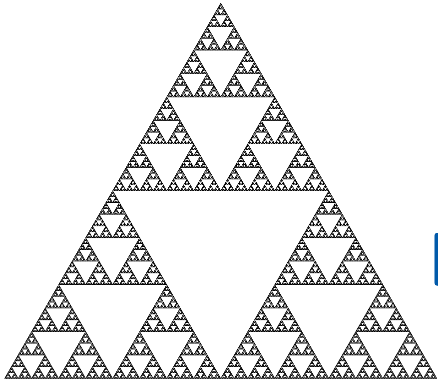

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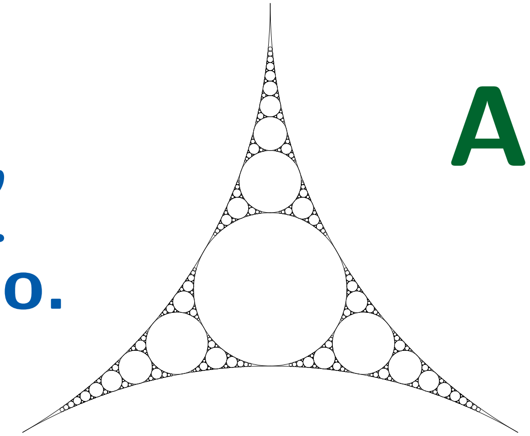

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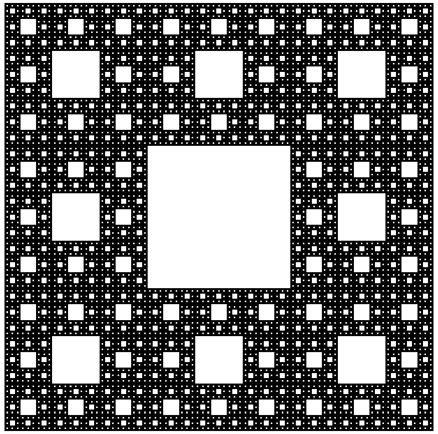

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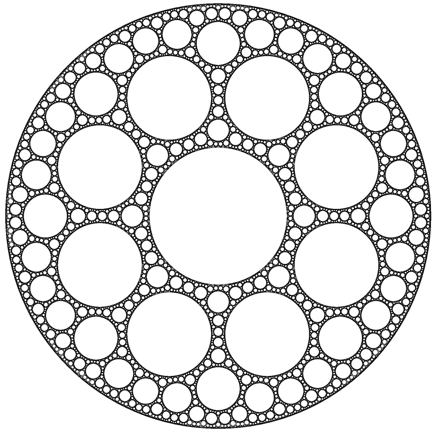

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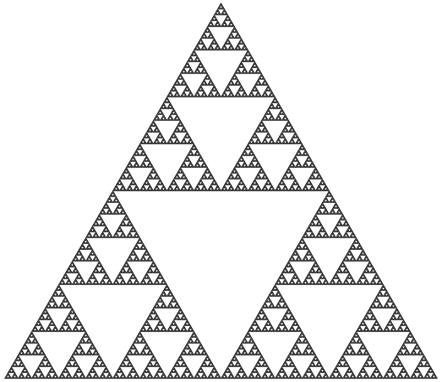

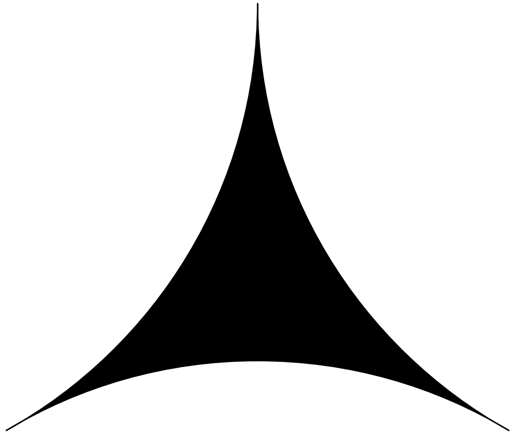

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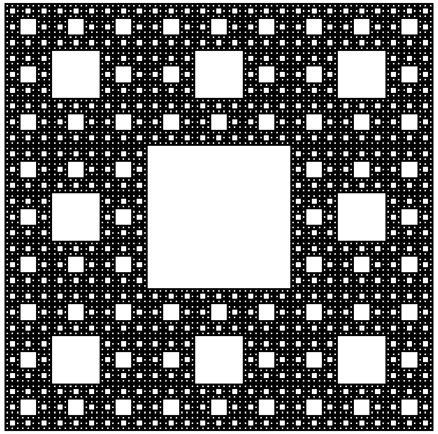

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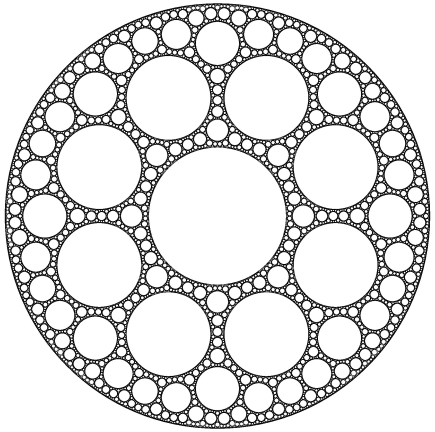

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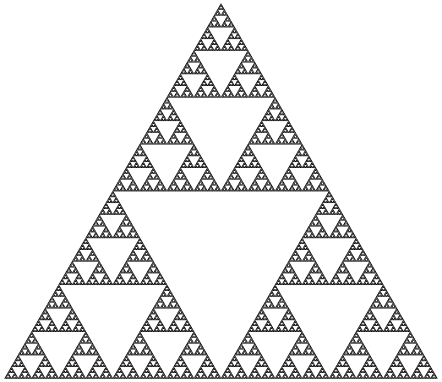

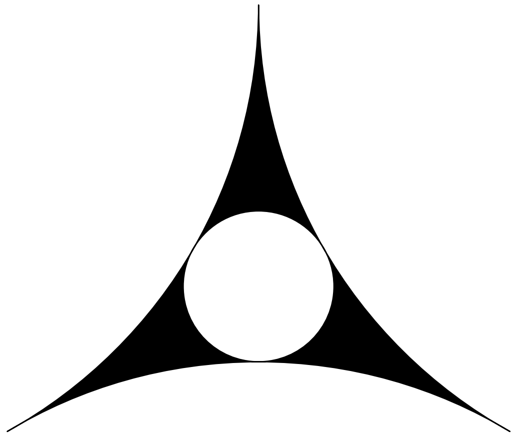

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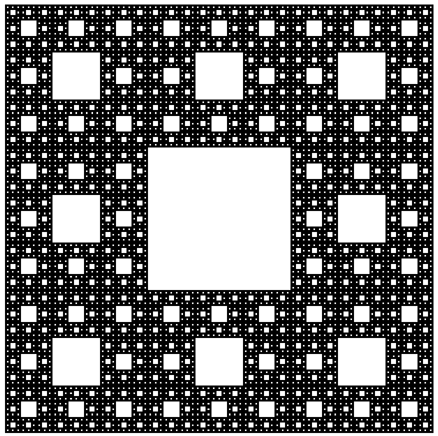

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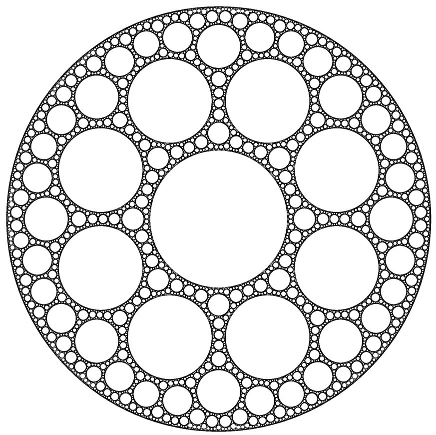

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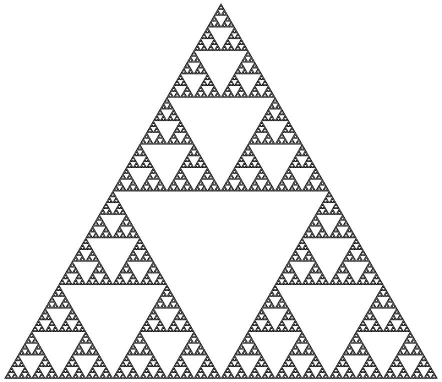

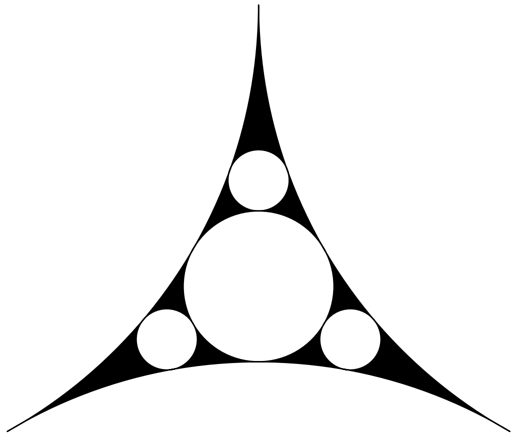

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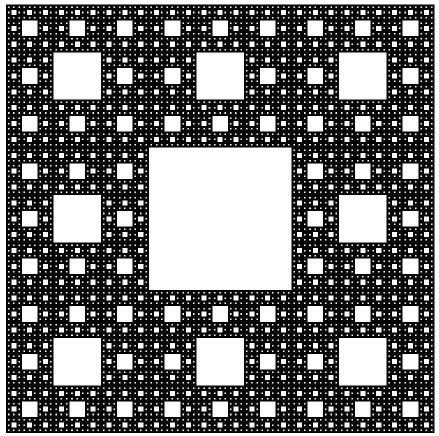

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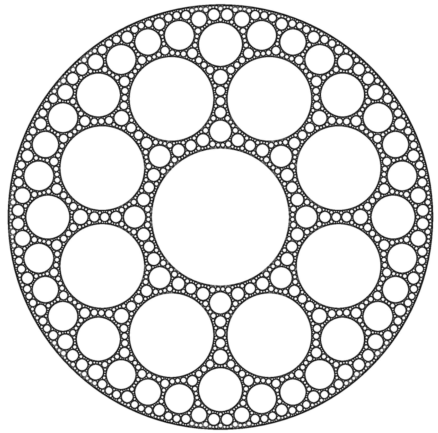

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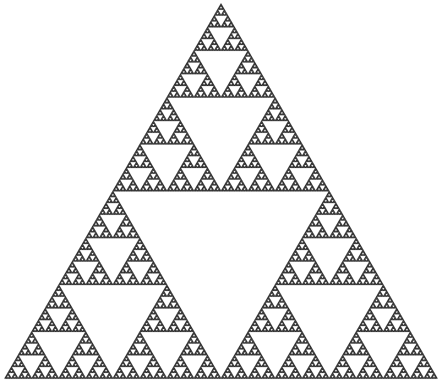

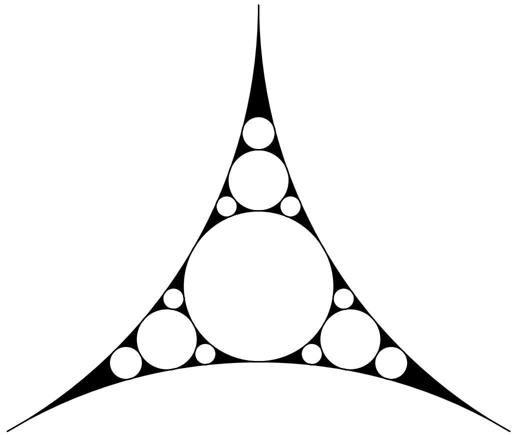

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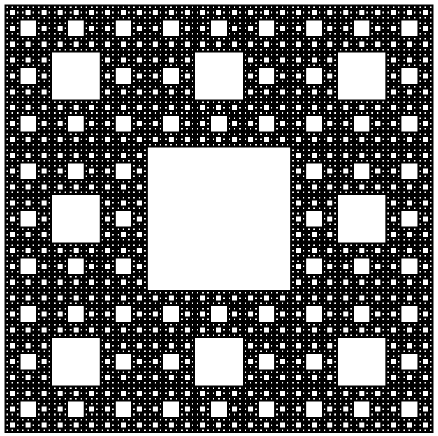

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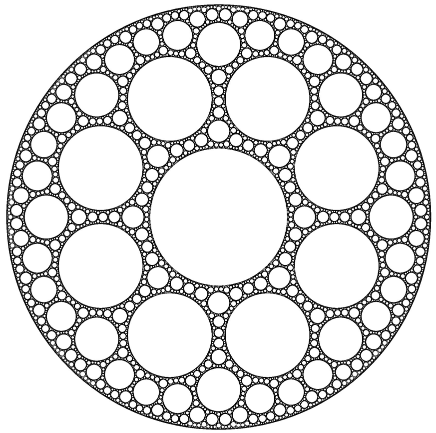

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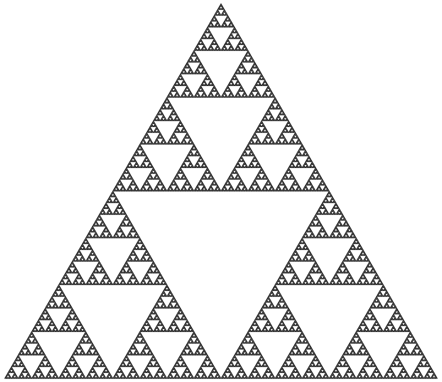

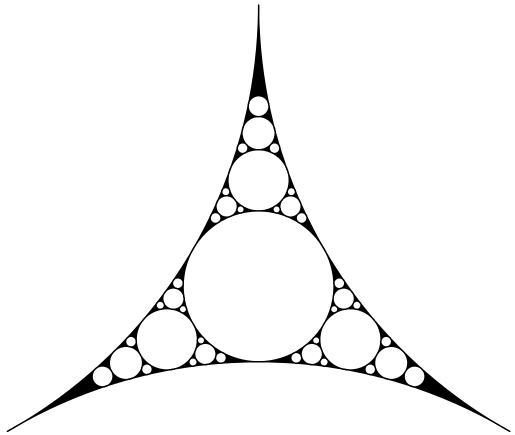

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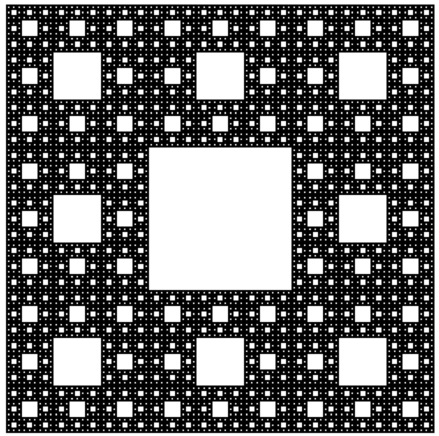

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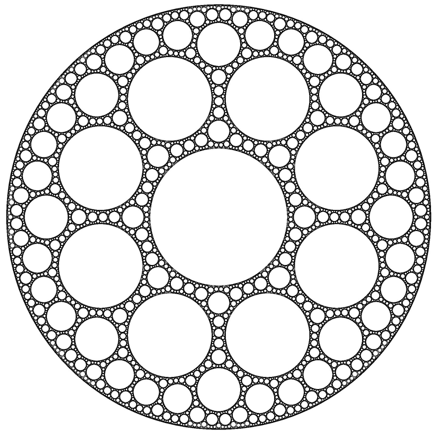

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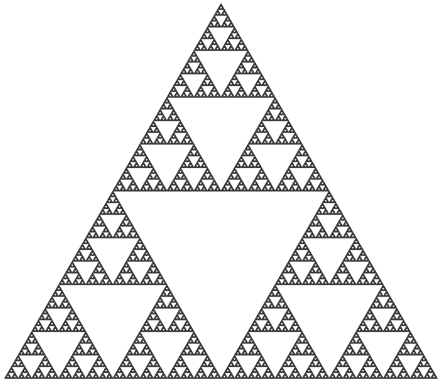

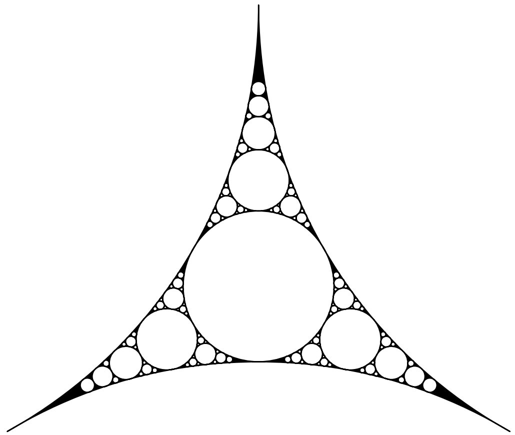

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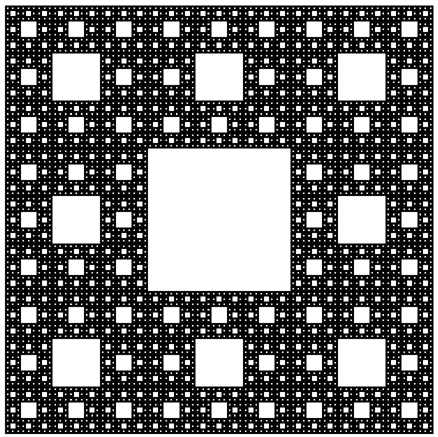

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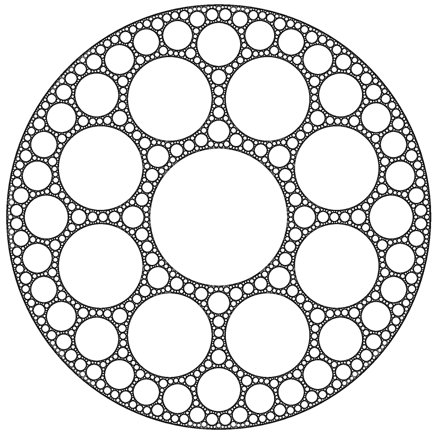

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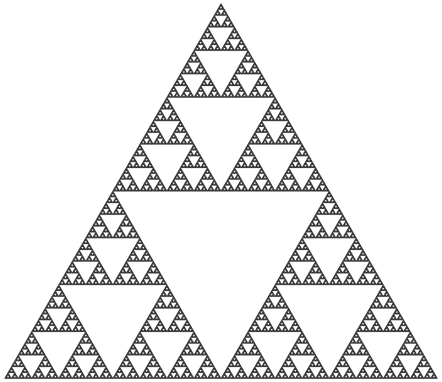
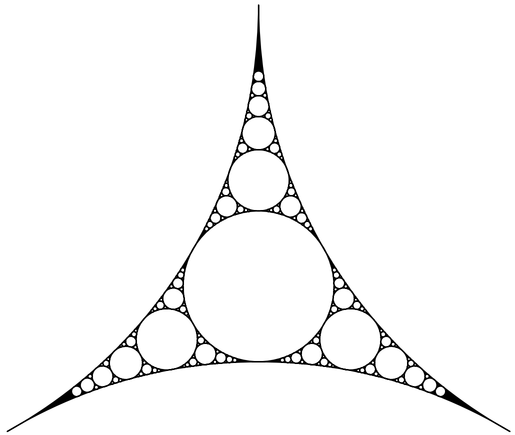
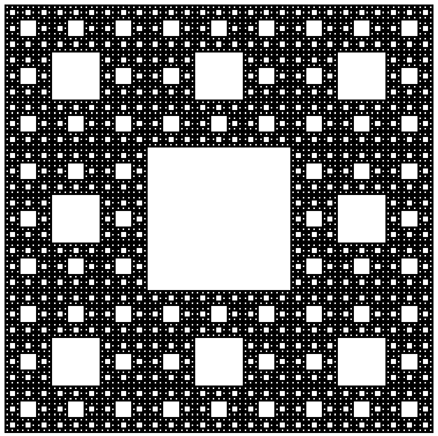
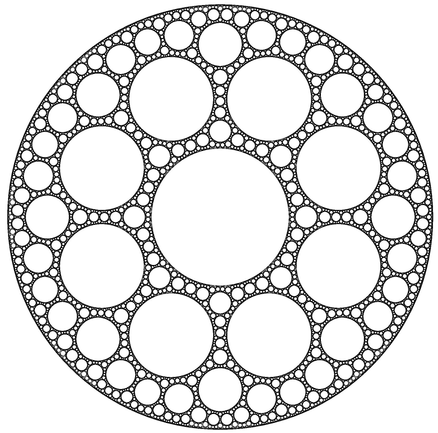
(self-similar) SG  

(self-similar) SC  $\simeq$ homeo.  round SC

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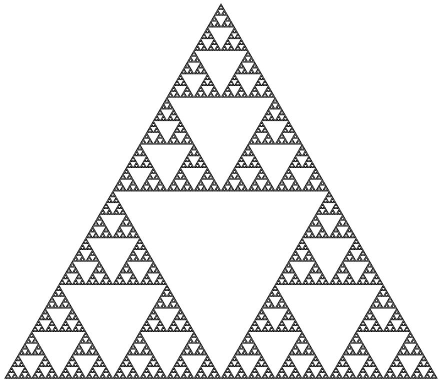

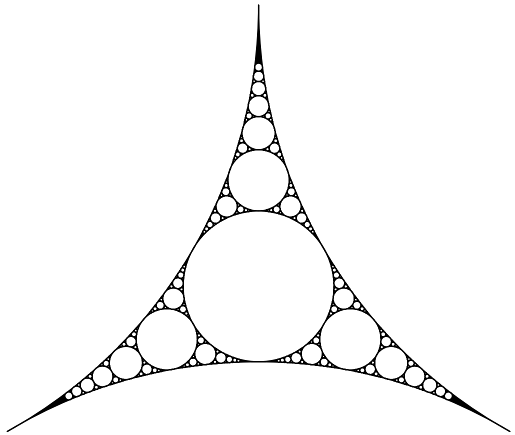

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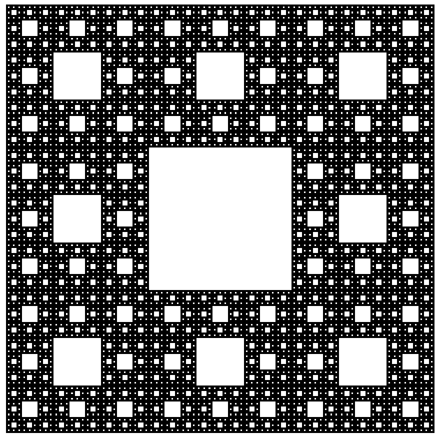

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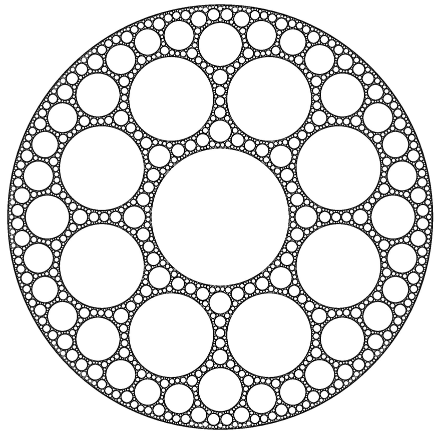

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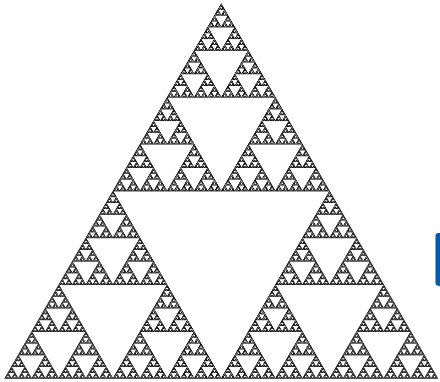

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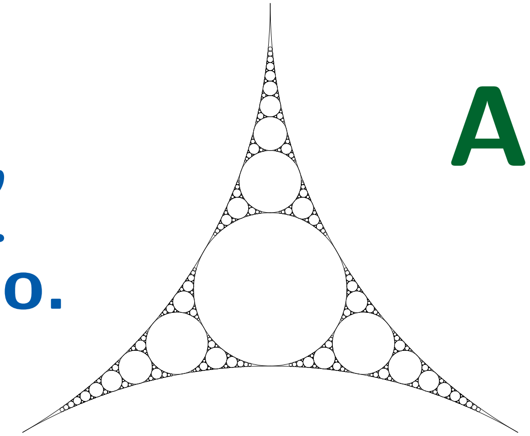

Apollonian gasket

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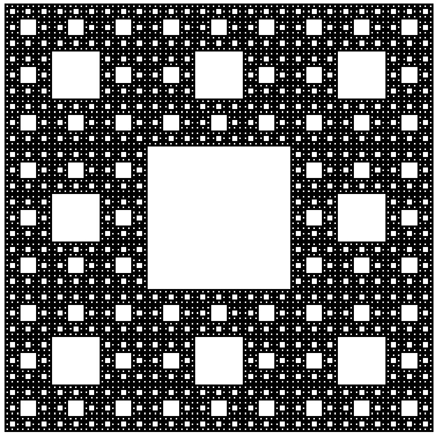

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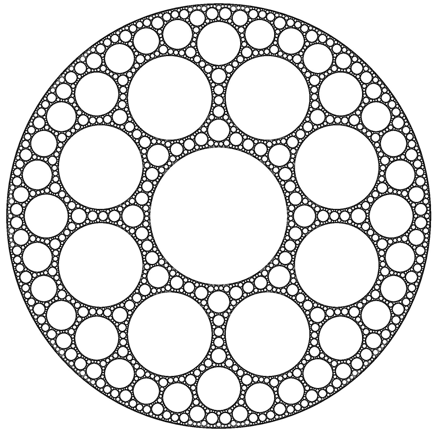

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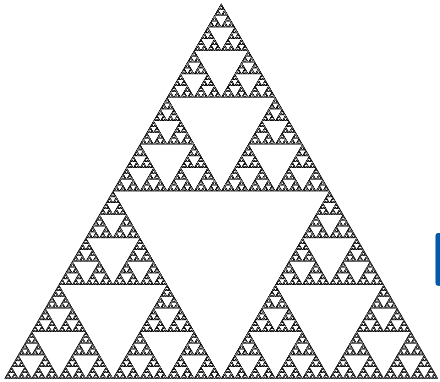

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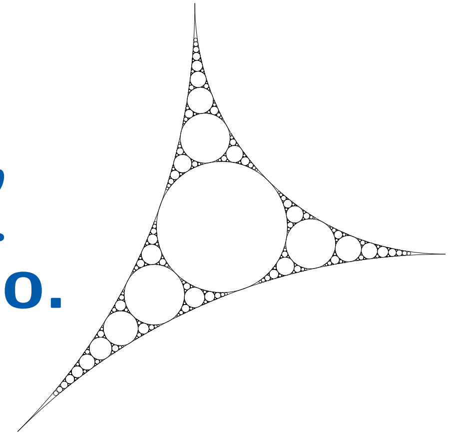

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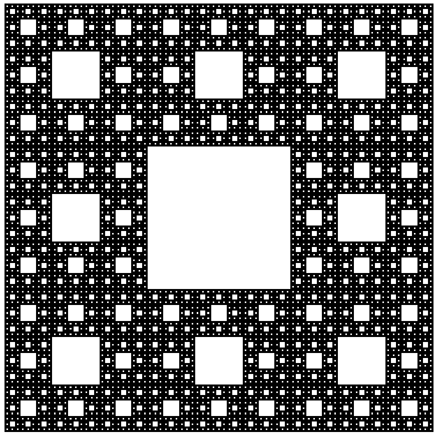

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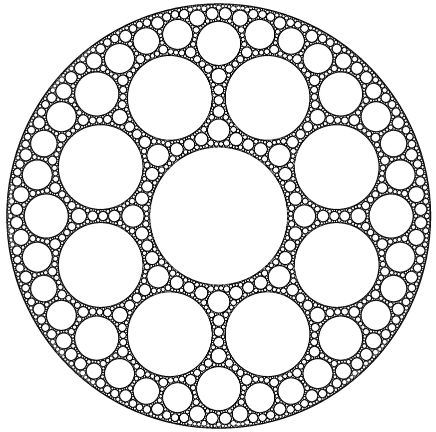

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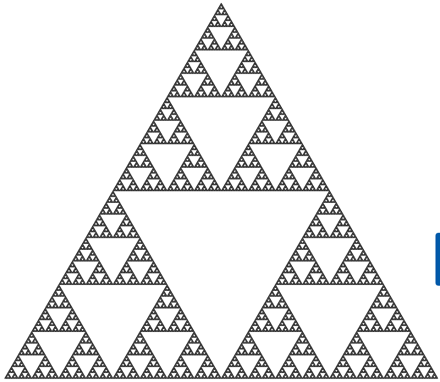

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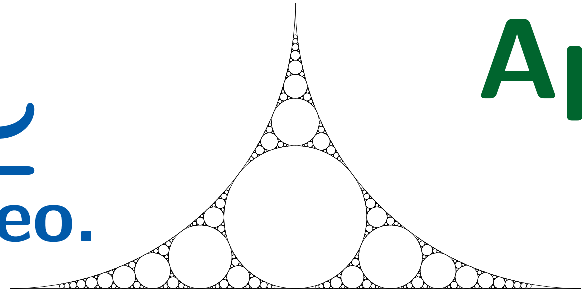

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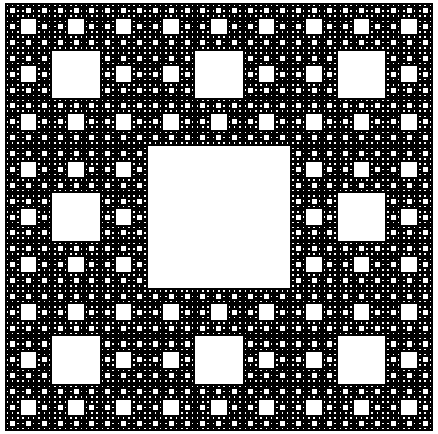

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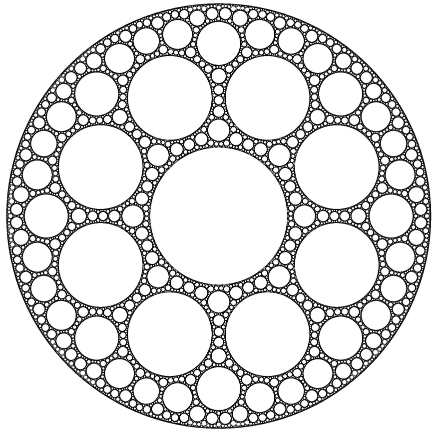

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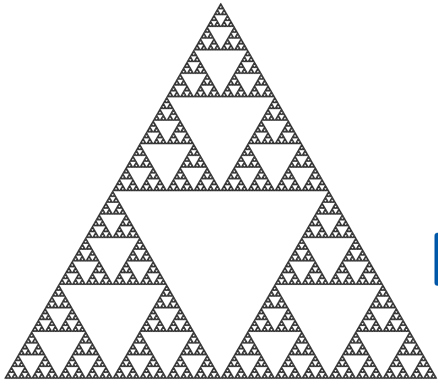
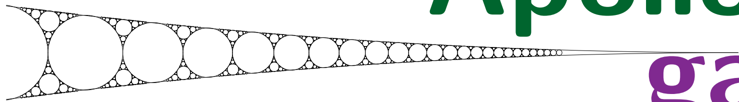
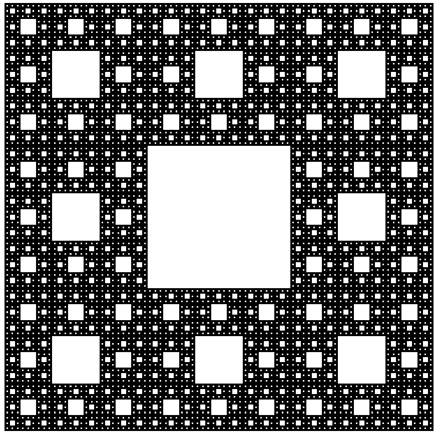
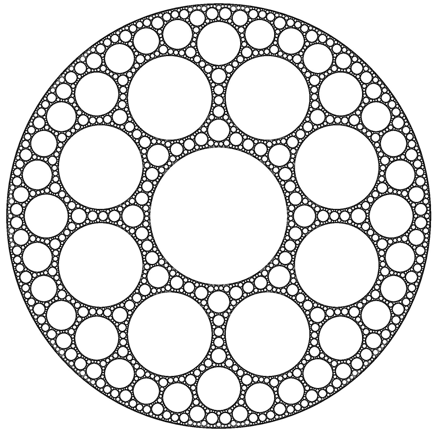
(self-similar) SG  $\underset{\text{homeo.}}{\simeq}$  Apollonian gasket

(self-similar) SC  $\underset{\text{homeo.}}{\simeq}$  round SC

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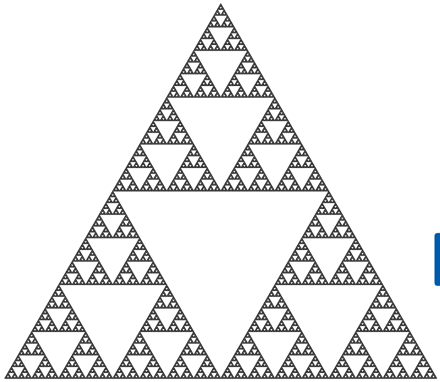

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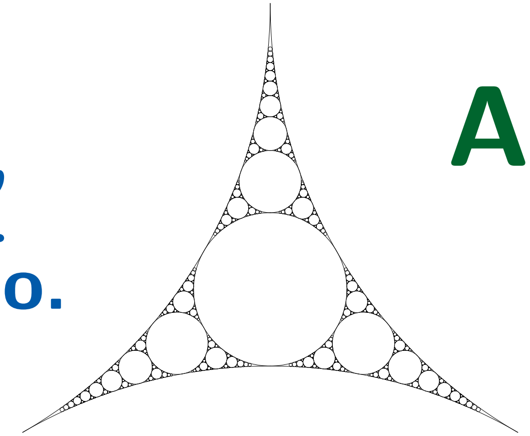

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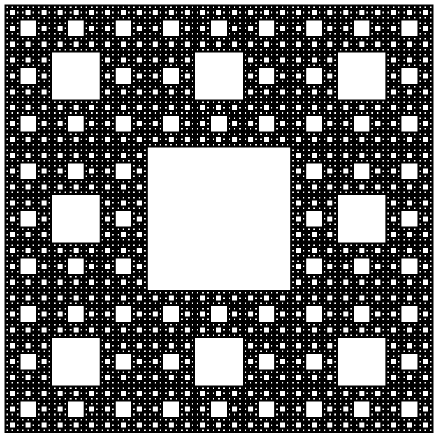

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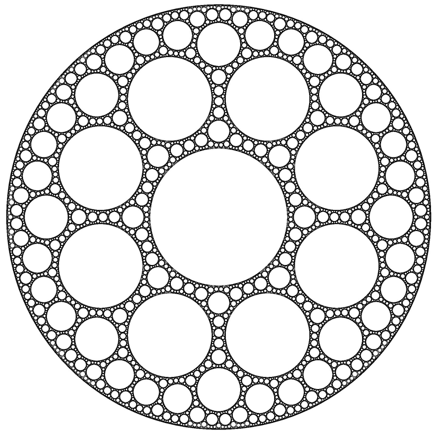

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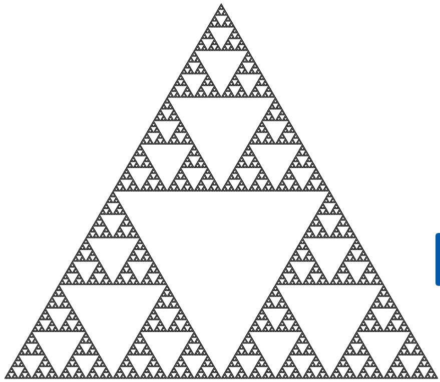

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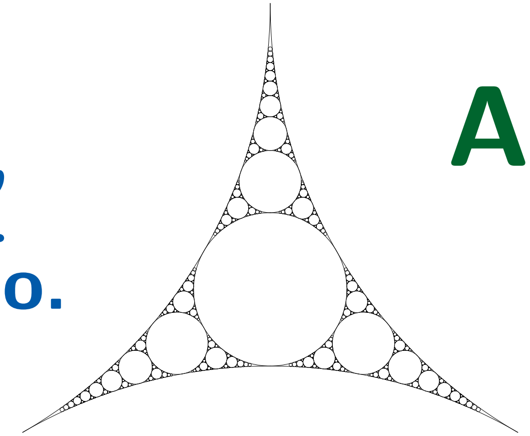

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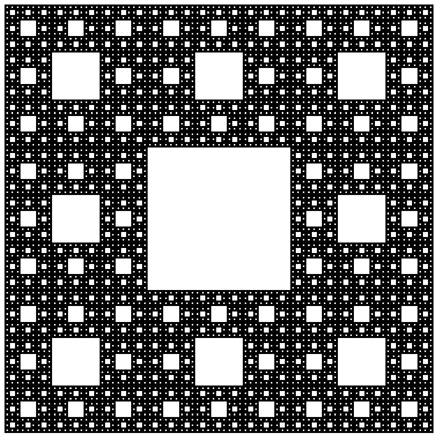
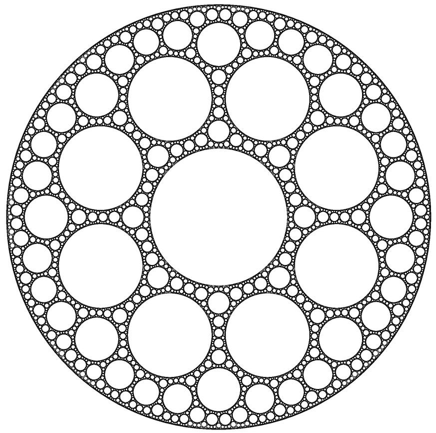


$\simeq$ homeo.



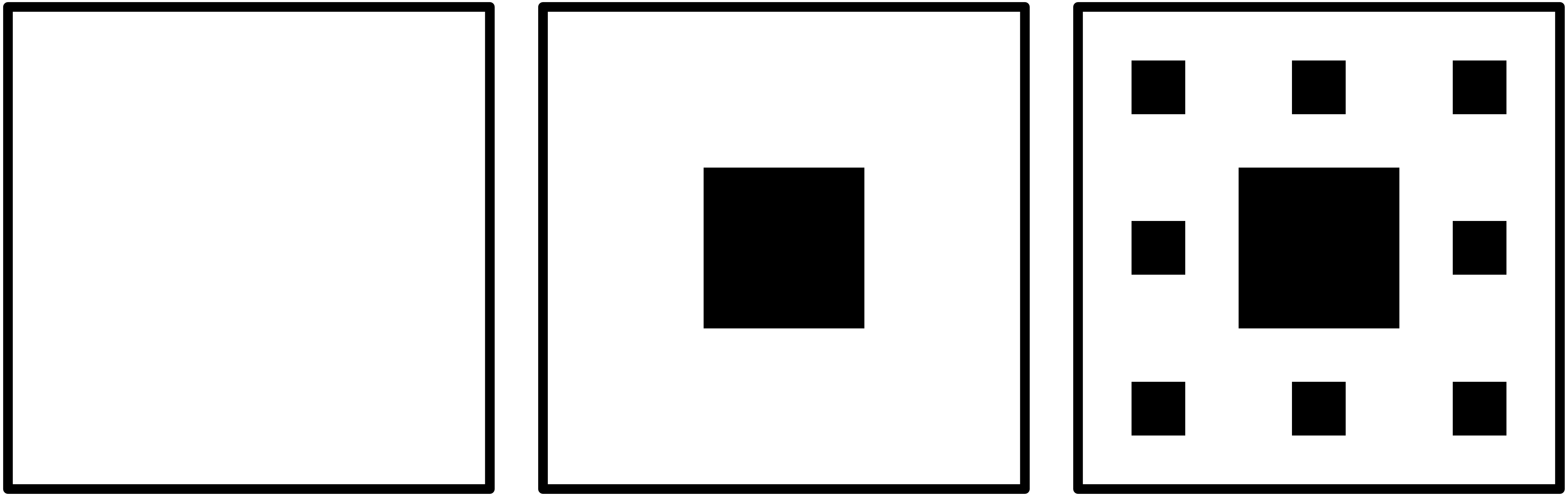
round SC

Constr./Analysis of "B.M."
(SG: Barlow–Perkins '88, Goldstein '87, Kusuoka '87
SC: Barlow–Bass '89, '99)

Problem.
Construction & Analysis of "Laplacian" & "B.M." which respect given geometry?

Dirichlet form & B.M. on self-similar SCs

- A self-similar regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ exists.
  (Barlow–Bass ’89, ’99, Kusuoka–Zhou ’92)



BB ’89: ${}^{\exists 1}\tau > 1$, $\{\mathrm{Law}(\{B^{\mathrm{ref},D_n}_{\tau^n t}\}_{t\geq 0})\}_{n=0}^{\infty}$ is tight.

- Such a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is unique.
  (Barlow–Bass–Kumagai–Teplyaev ’10)

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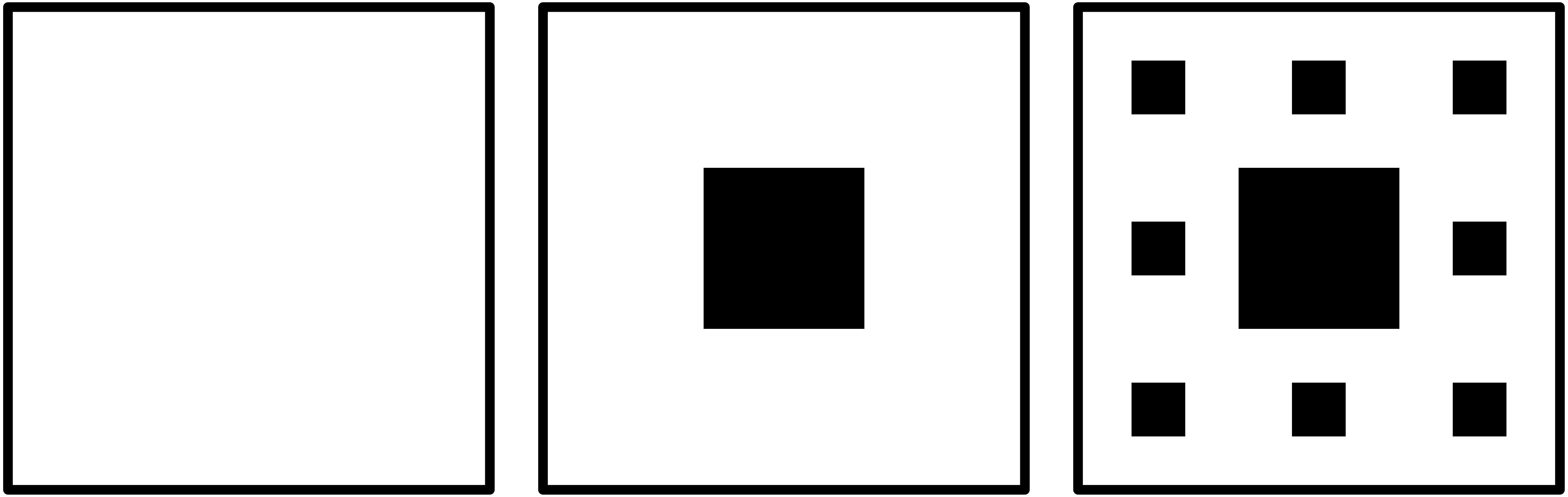

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1 Sierpiński gasket & carpet in different geometries

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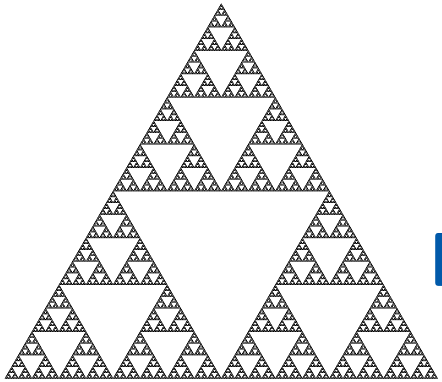

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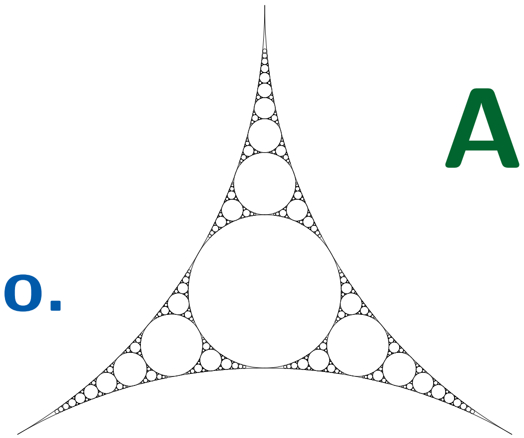

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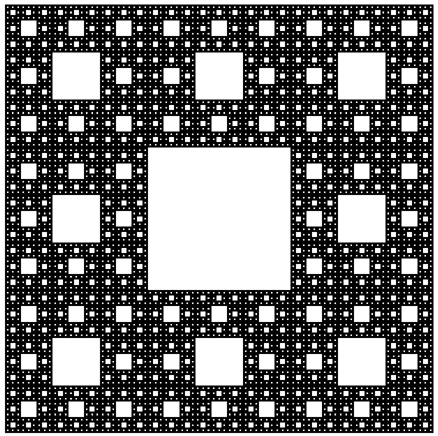

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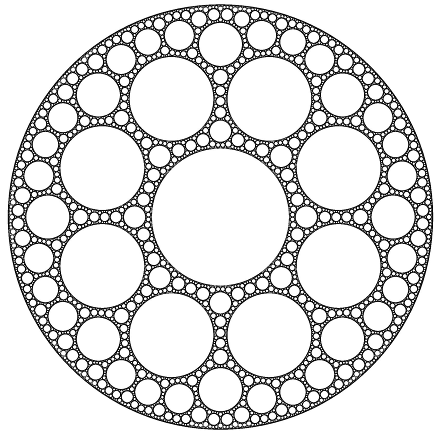

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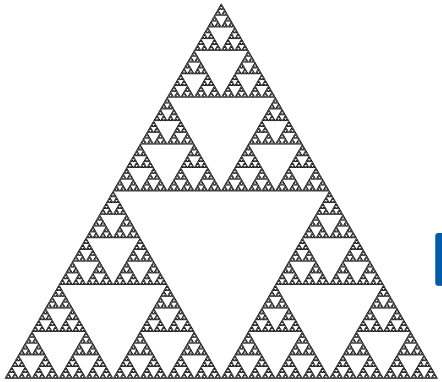

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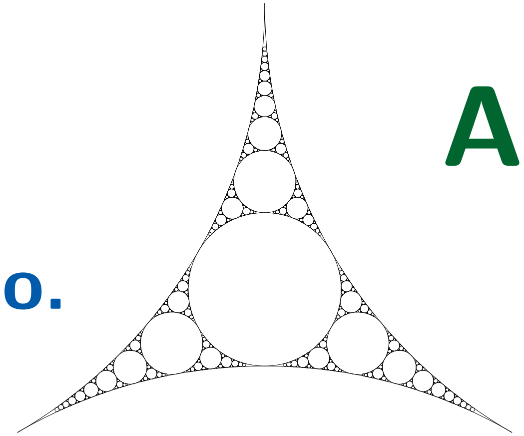

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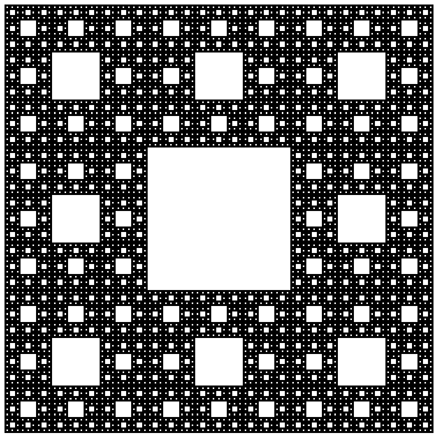

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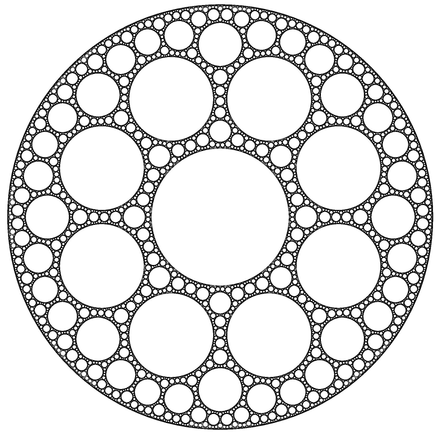

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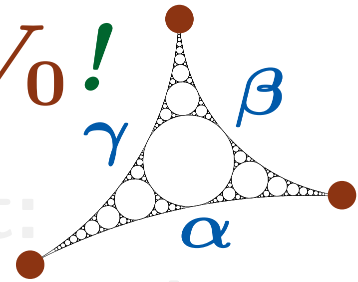
Main Result.

Construction & Analysis of "Laplacian" & "B.M." which respect given geometry!

2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \underset{\text{embedding}}{\overset{\text{harmonic}}{\hookrightarrow}} \mathbb{C}$

Thm (K., *cf.* Teplyaev '04). $^{\exists 1}(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$-*harmonic on* $K_{\alpha,\beta,\gamma} \setminus V_0$*!*



Rmk. Choice of a reference measure is irrelevant:
$\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}}$ are unique.

Thm (K.). $\mathrm{LIP}|_{K_{\alpha,\beta,\gamma}}$ is a core of $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$, and

$$^\forall u \in \mathrm{LIP},\ \mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset_{\mathrm{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 \, d\mathrm{vol}_C.$$

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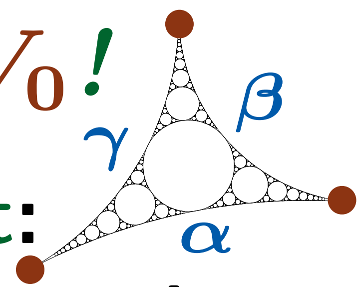

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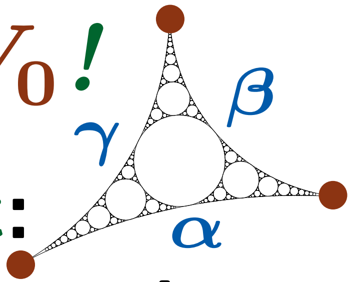

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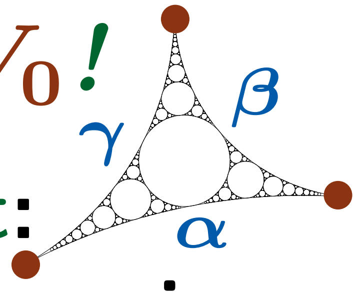

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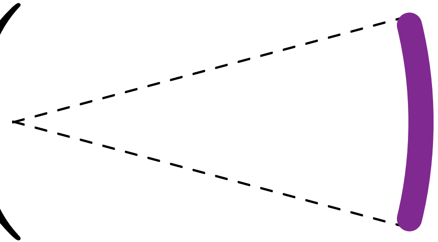
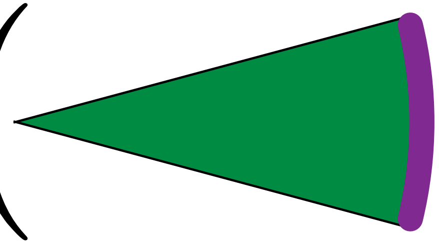
$\mu^{\alpha,\beta,\gamma}\Big($  $\Big) = 2\mathrm{Area}\Big($  $\Big)$!

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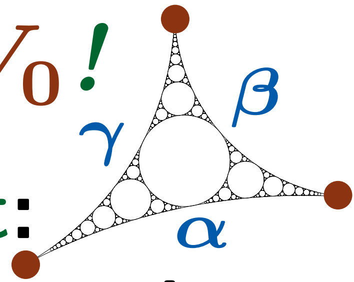

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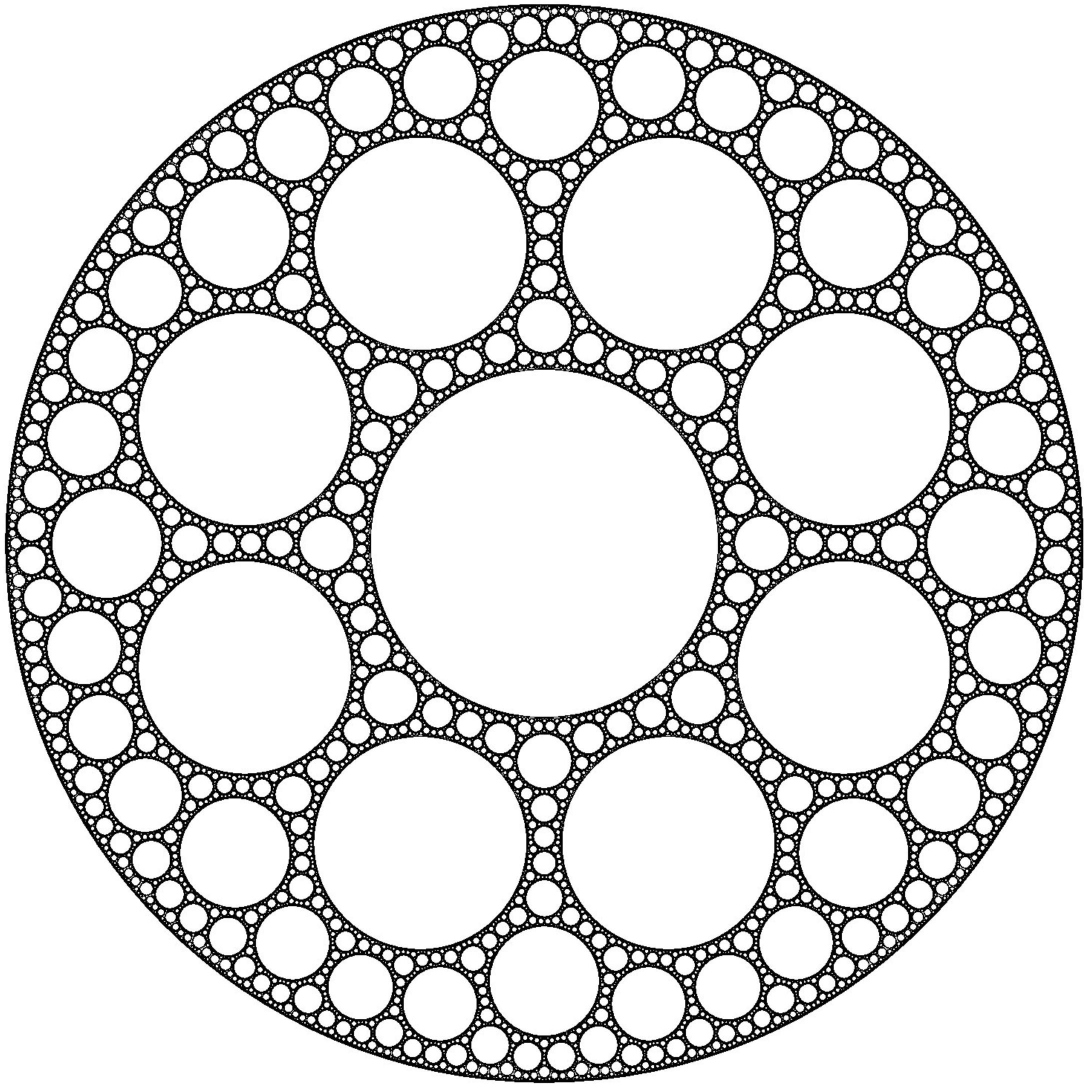
$\lim_{\lambda\to\infty} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda\}/\lambda^{d_{\mathrm{AG}}/2} = c_0 \mathcal{H}^{d_{\mathrm{AG}}}(K_{\alpha,\beta,\gamma})$.

3 Some Kleinian groups G_m with $\partial_\infty G_m$ a RSC



▷ $m>6$ $\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{m}<\pi\right)$

▷ $\{\ell_k\}_{k=1}^{3}$: $\mathbb{B}^2$-geodesics, form $\triangle$, angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$

▷ $\Gamma_m := \left\langle \{\mathrm{Inv}_{\ell_k}\}_{k=1}^{3} \right\rangle$
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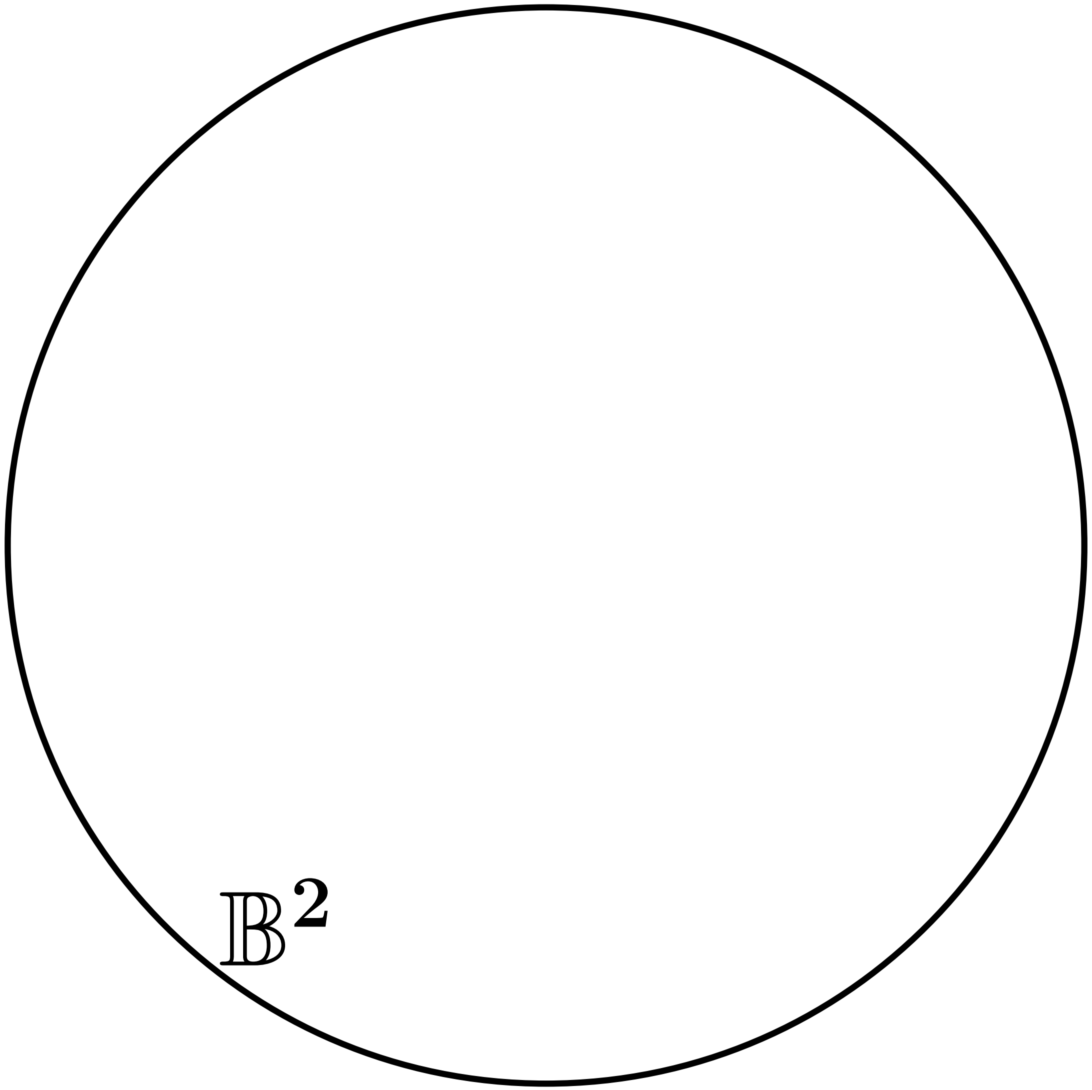

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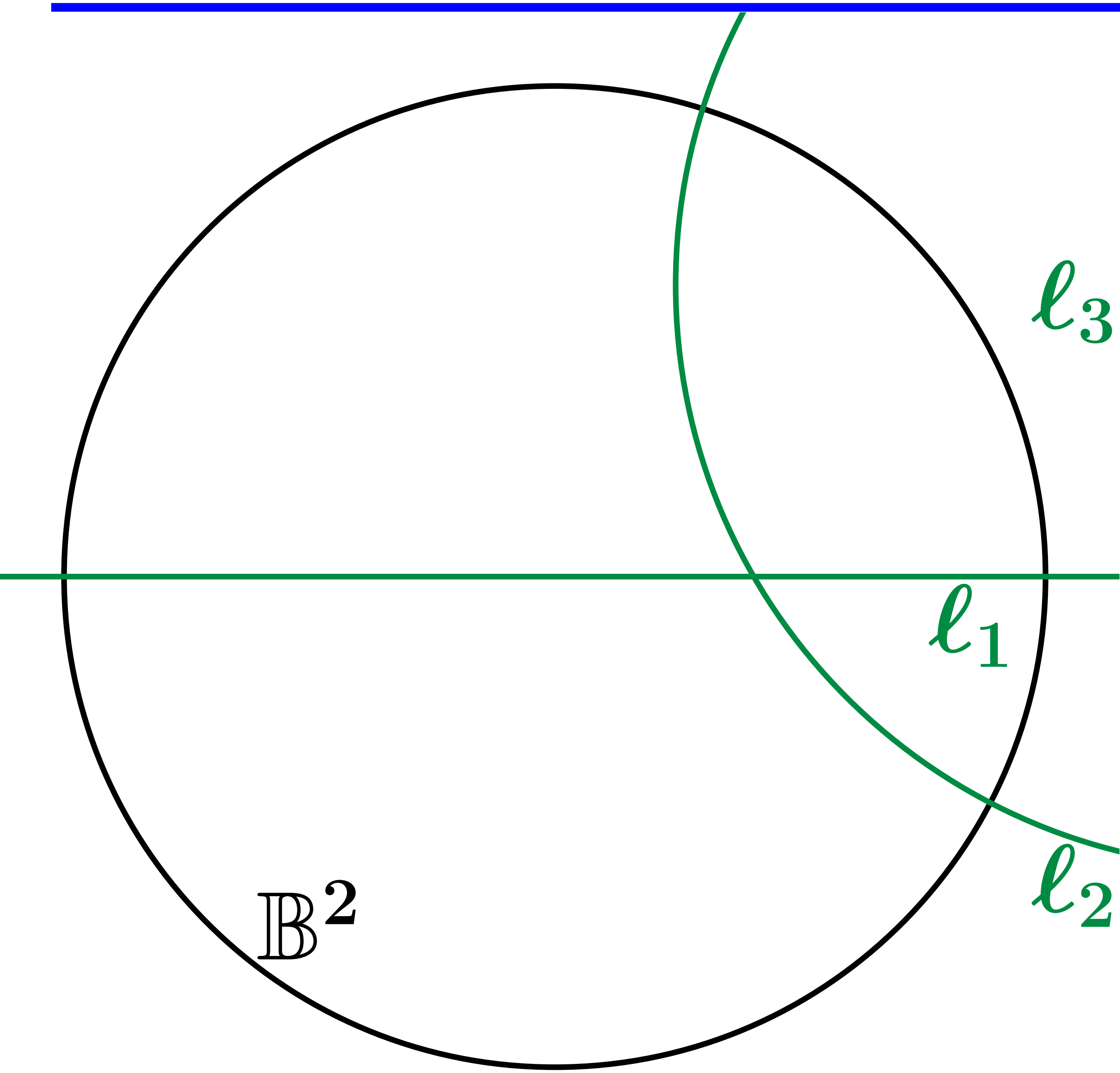

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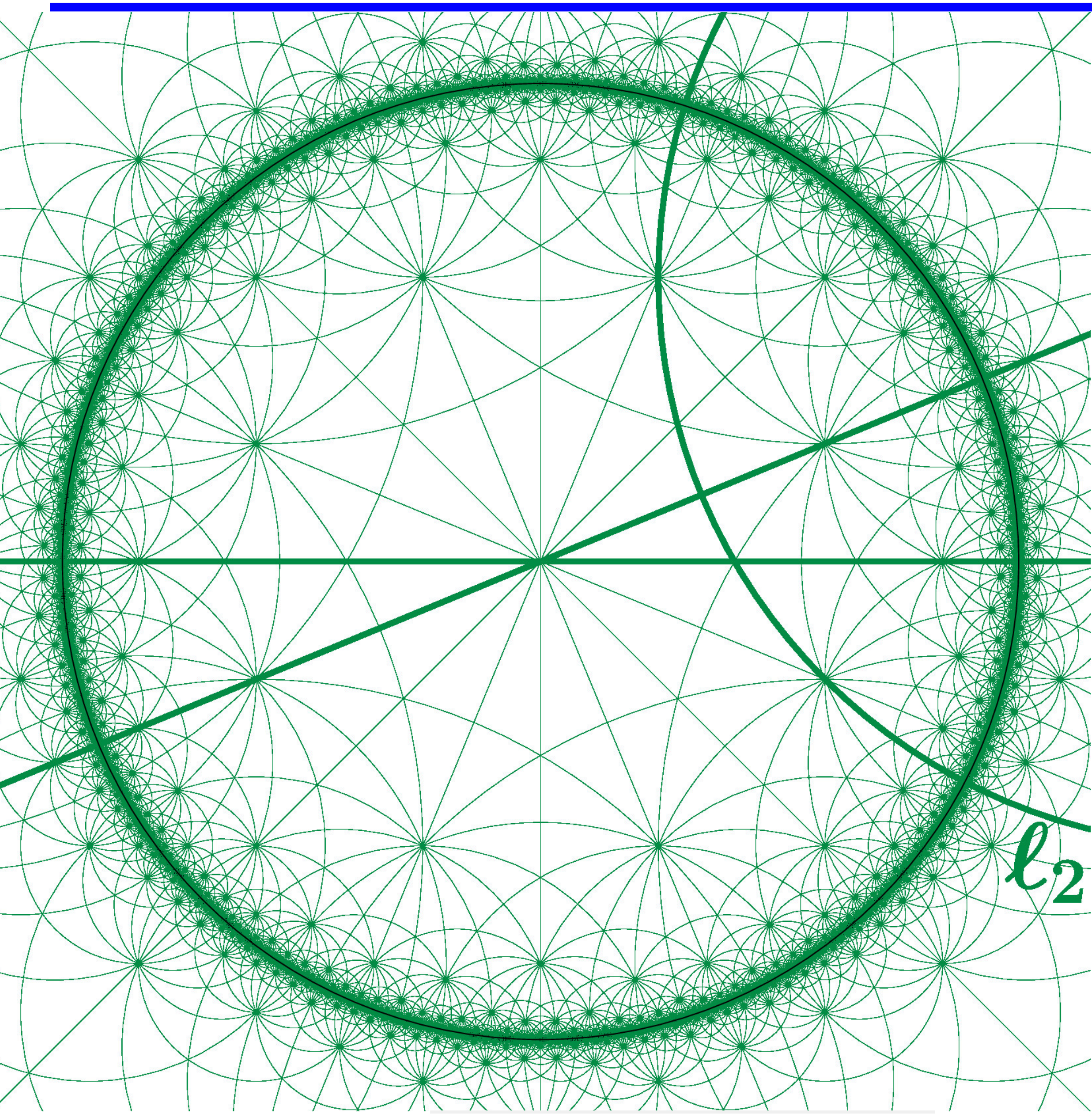

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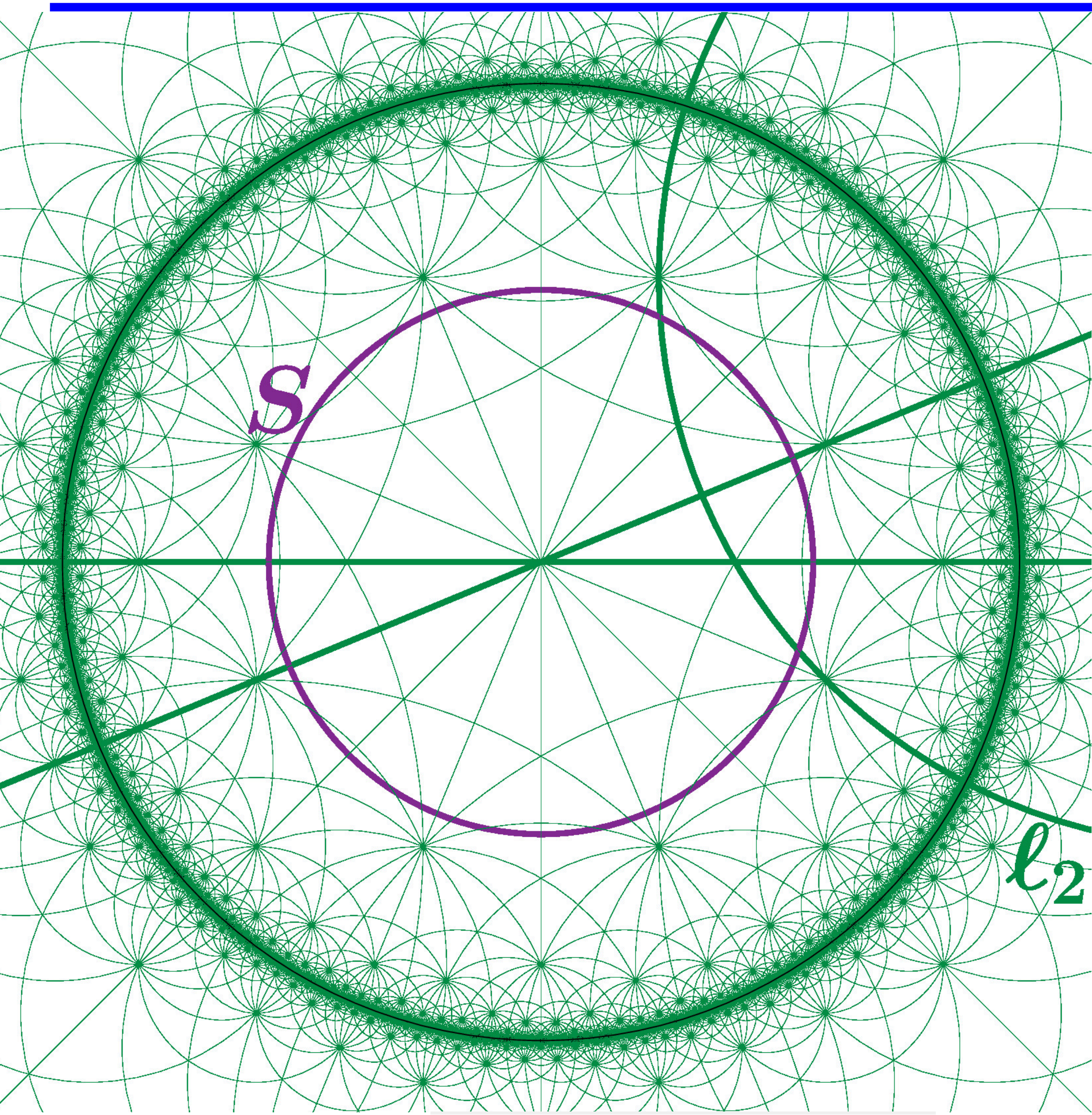

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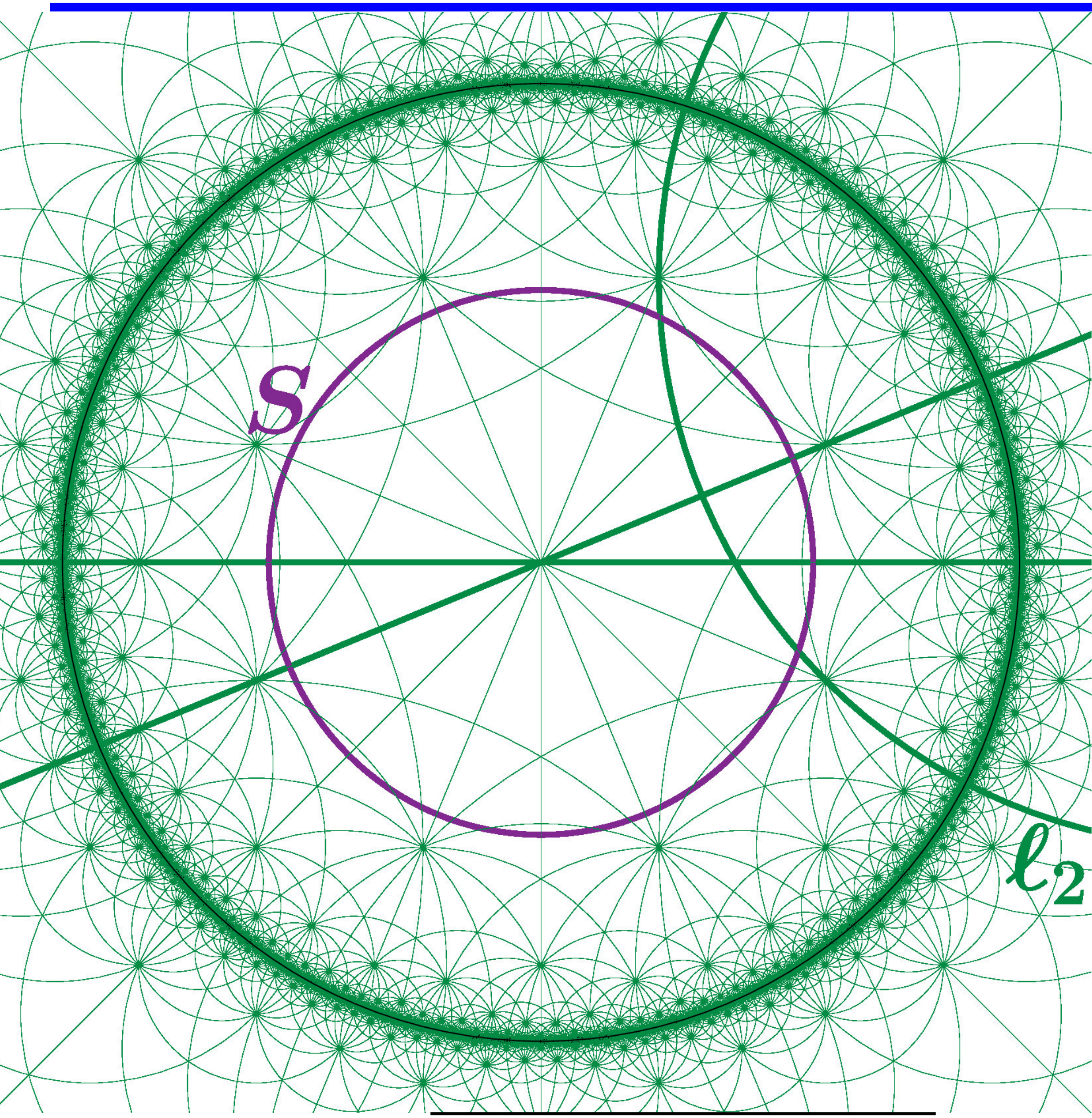

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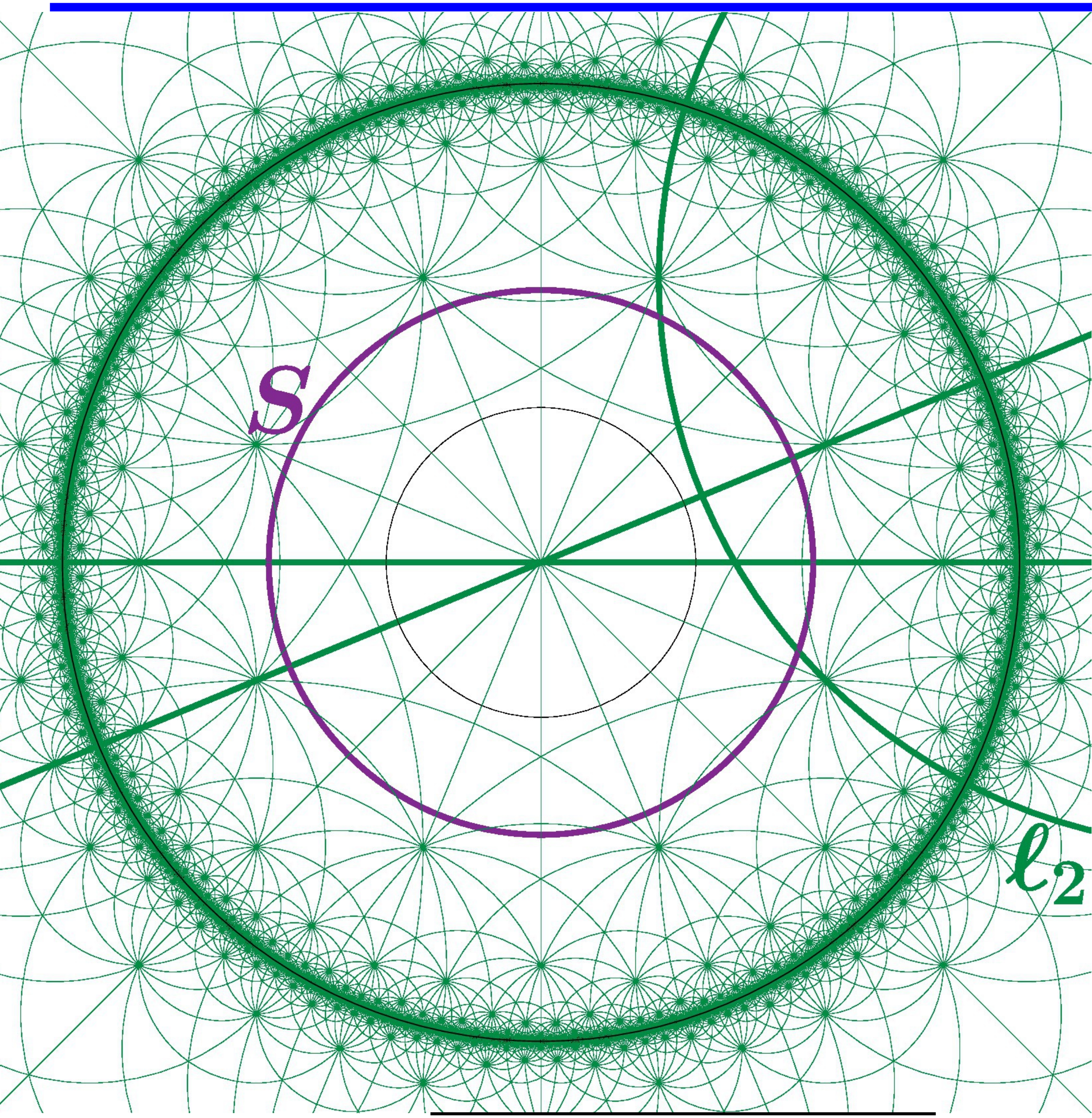

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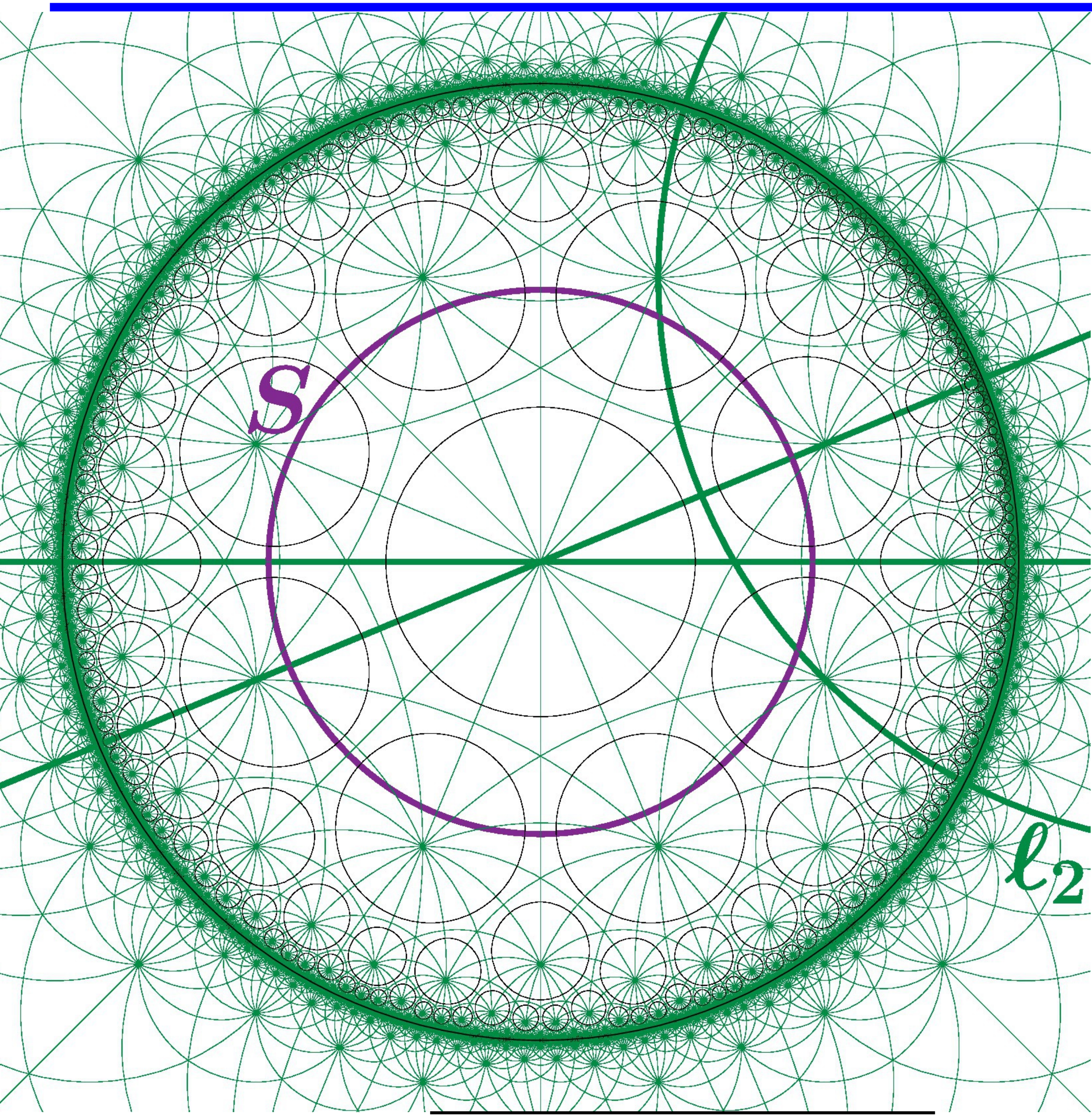

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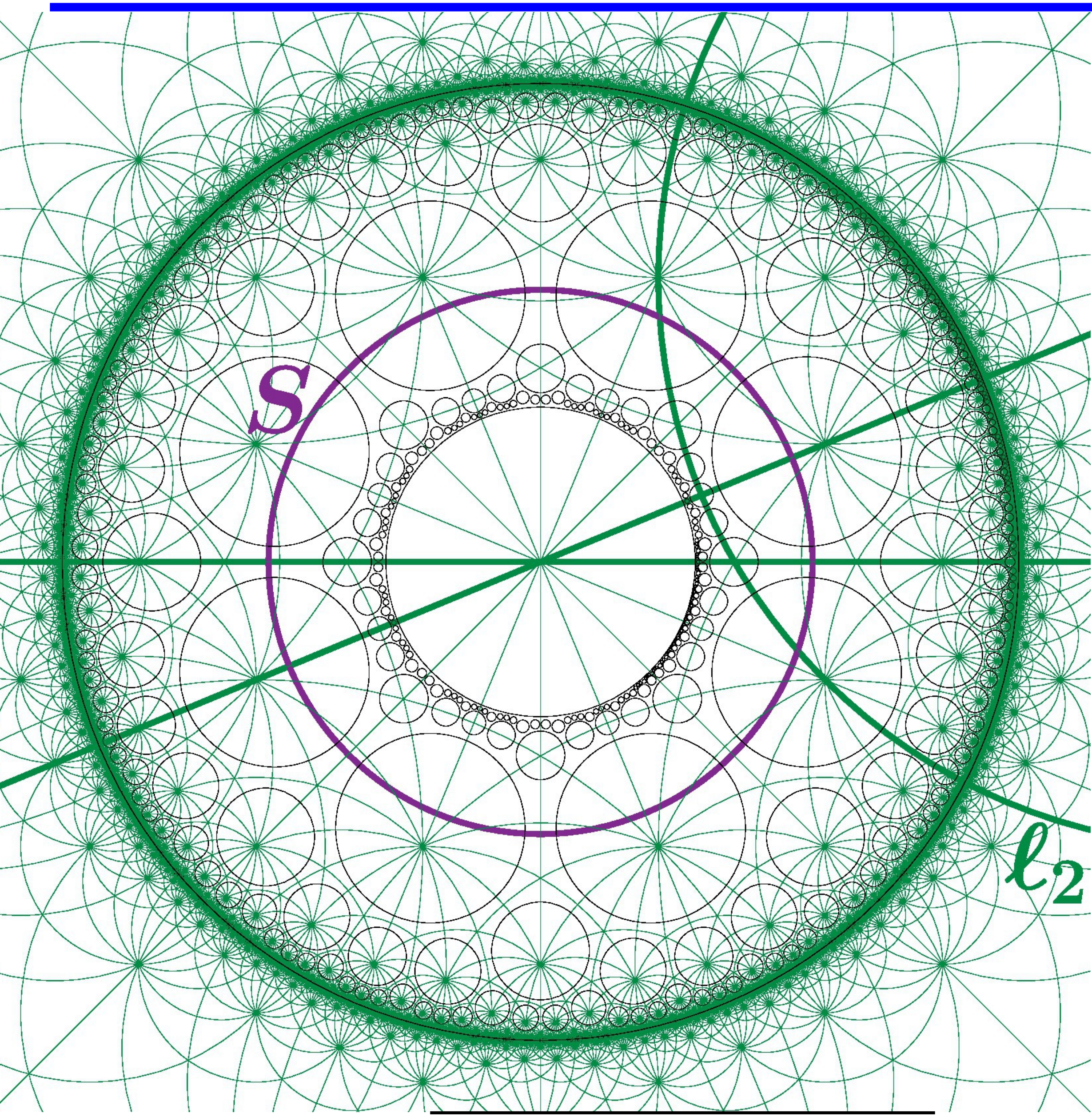

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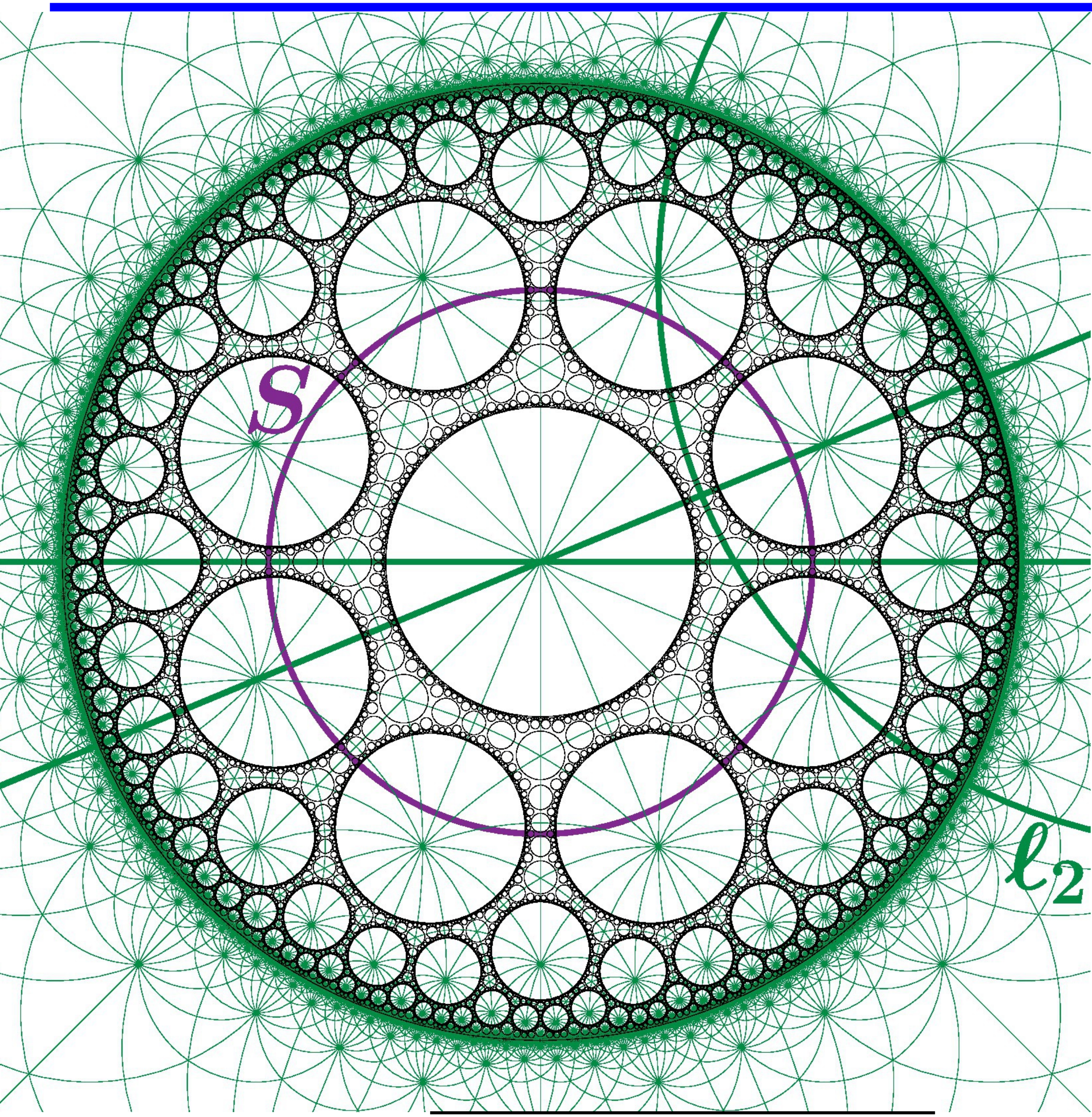

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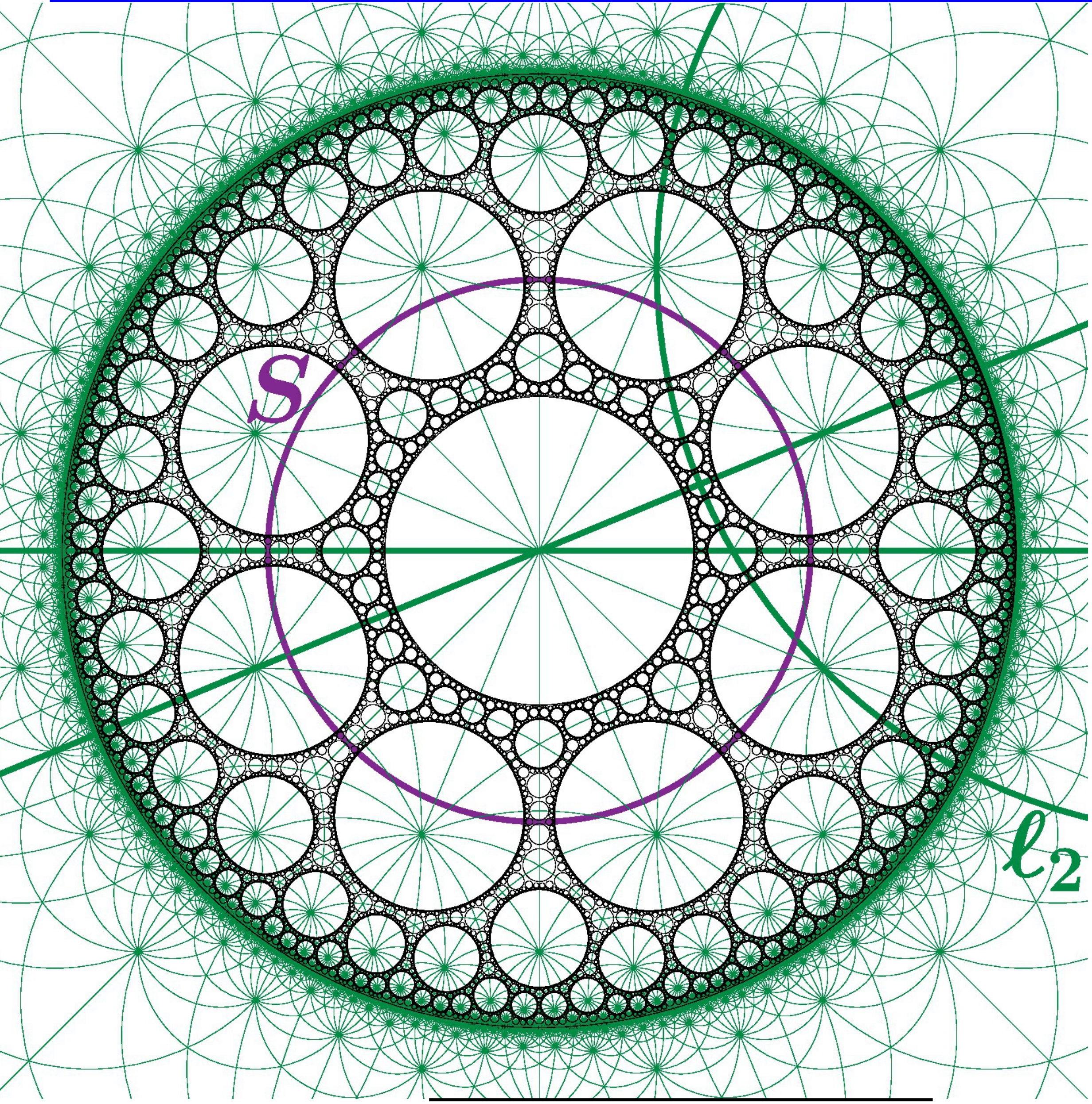

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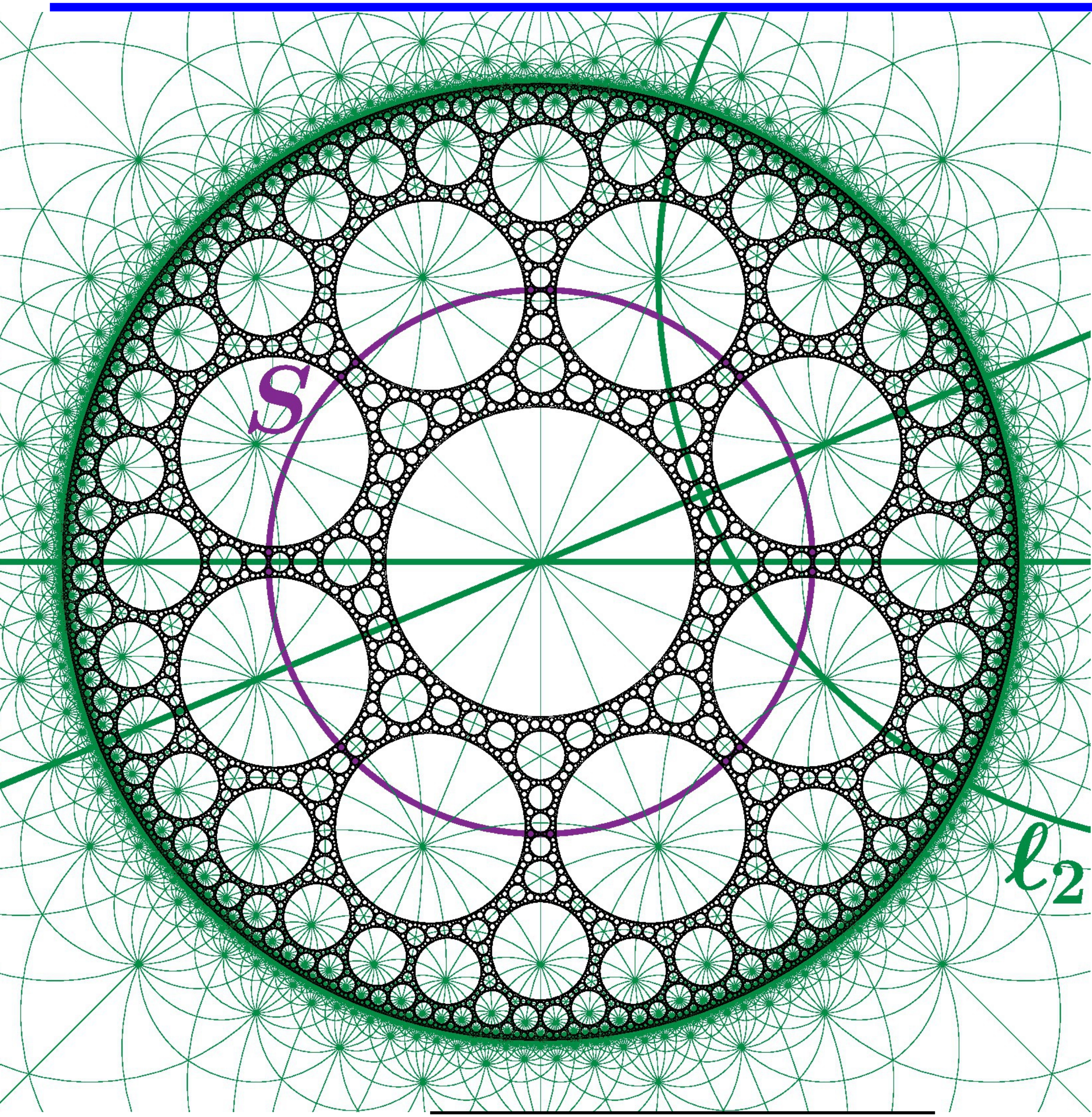

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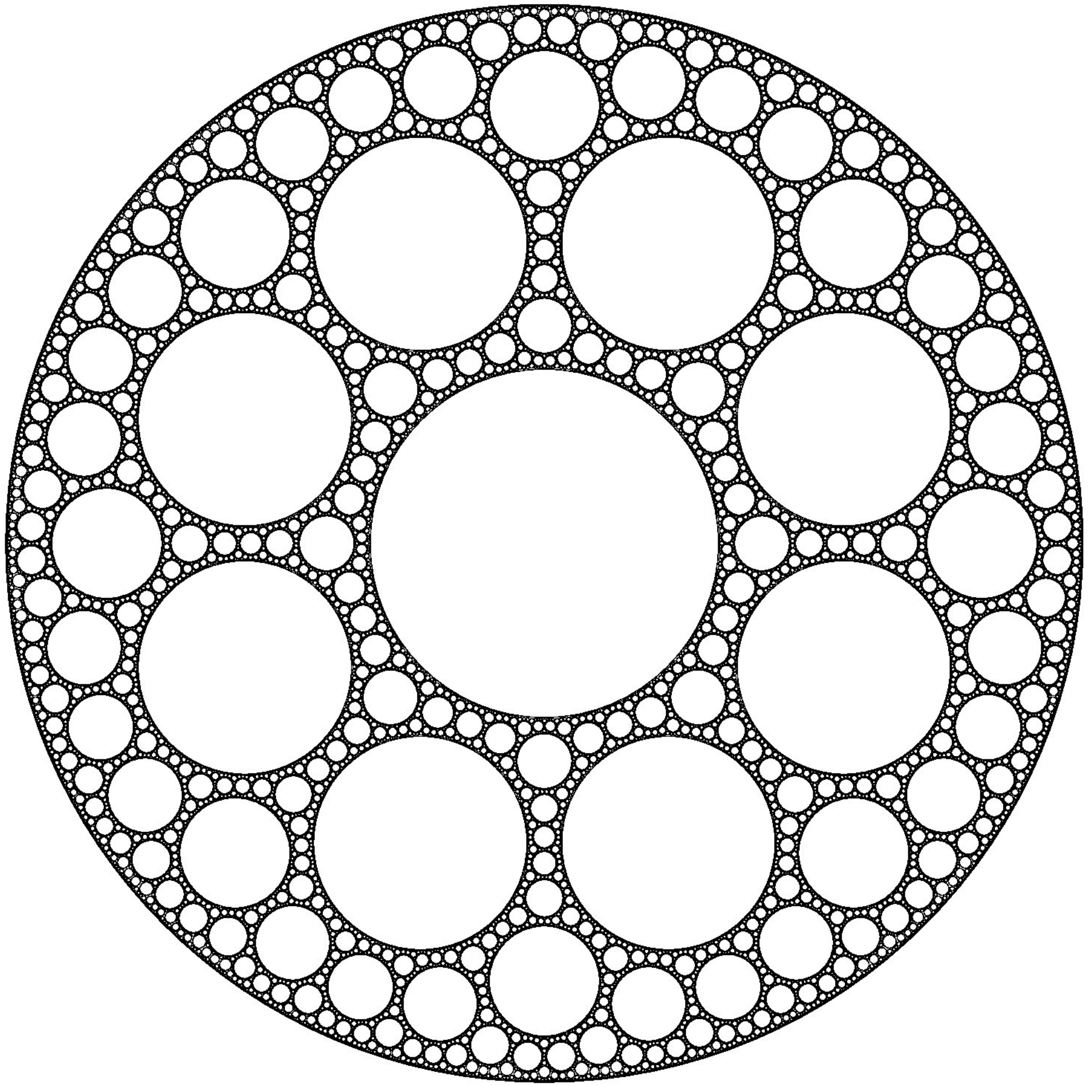

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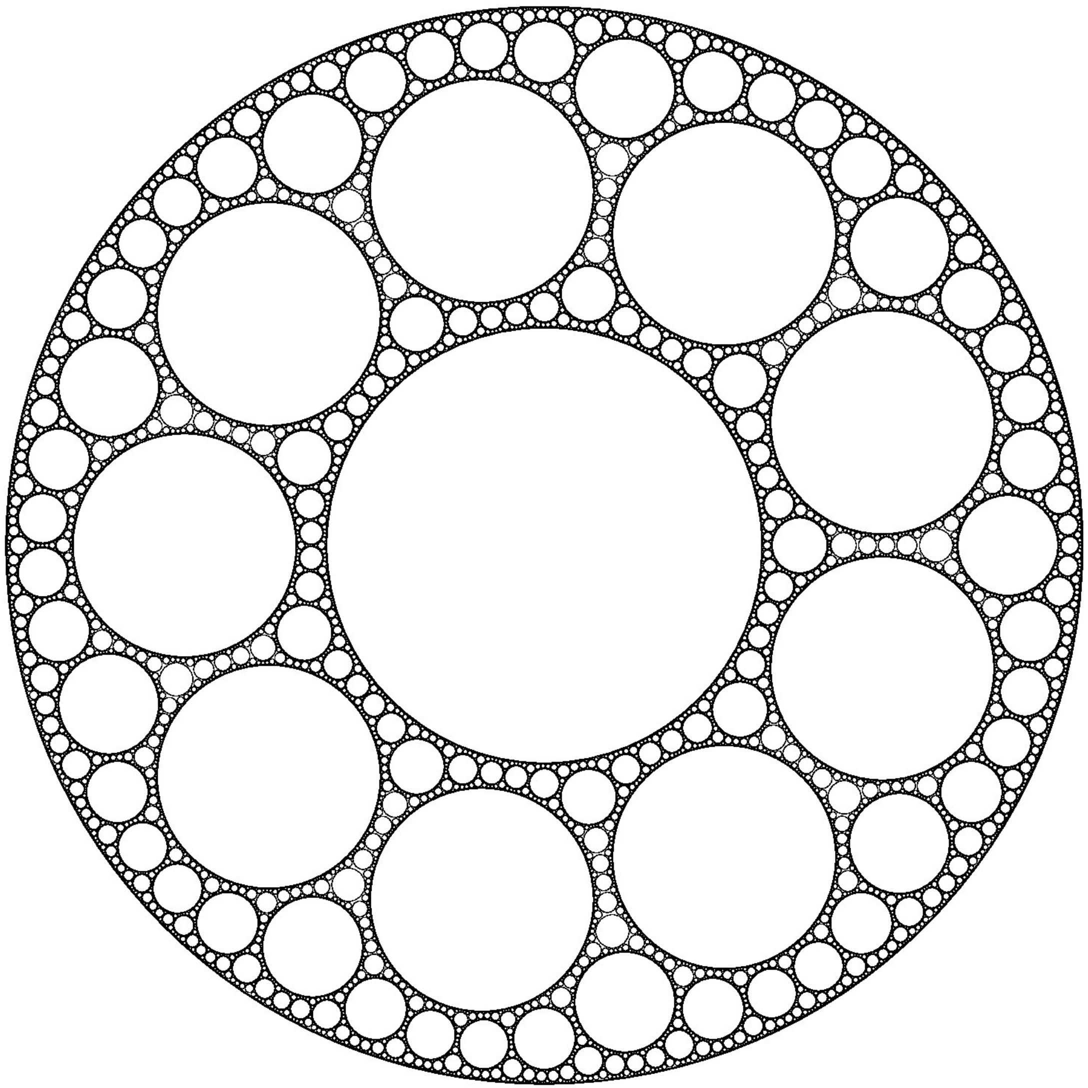

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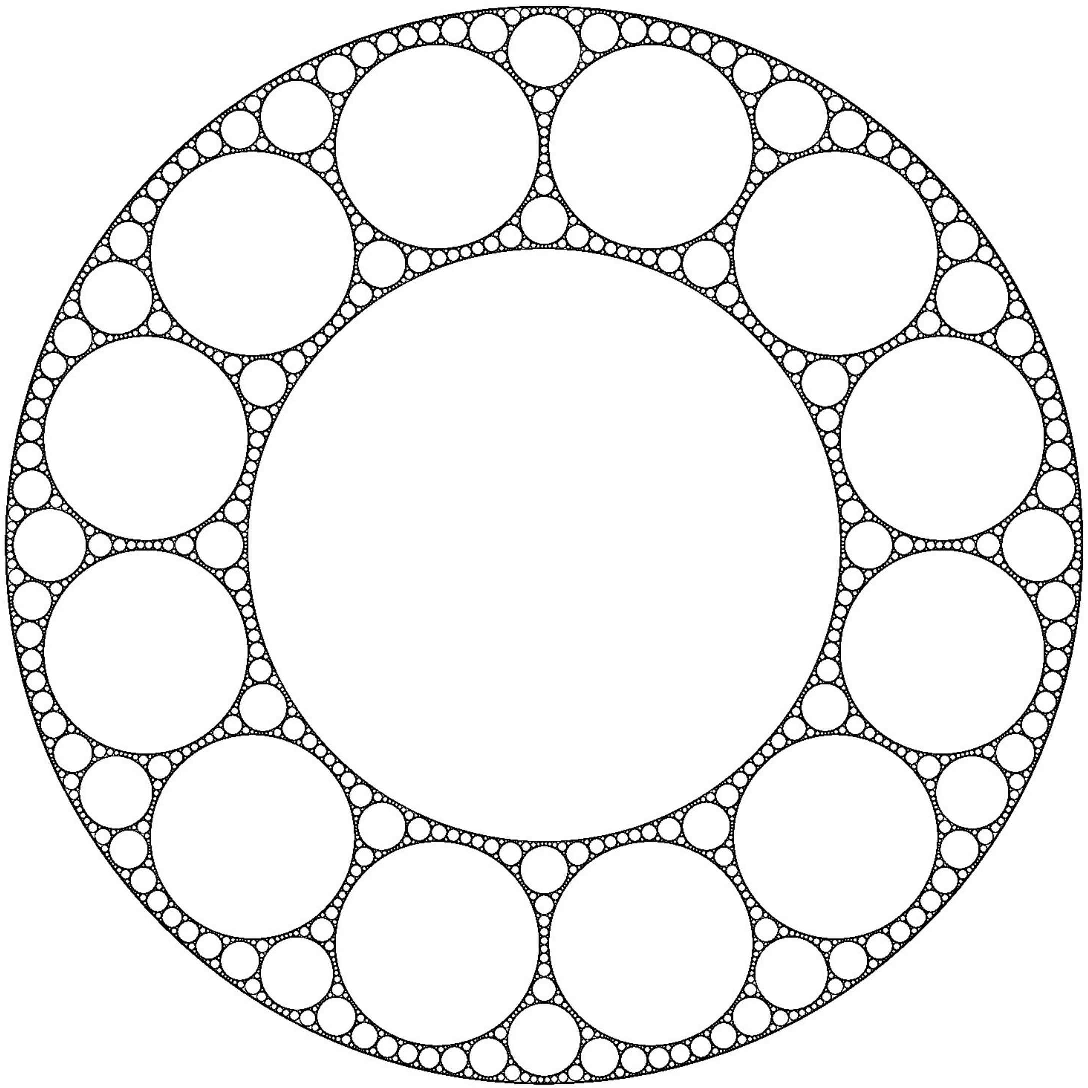

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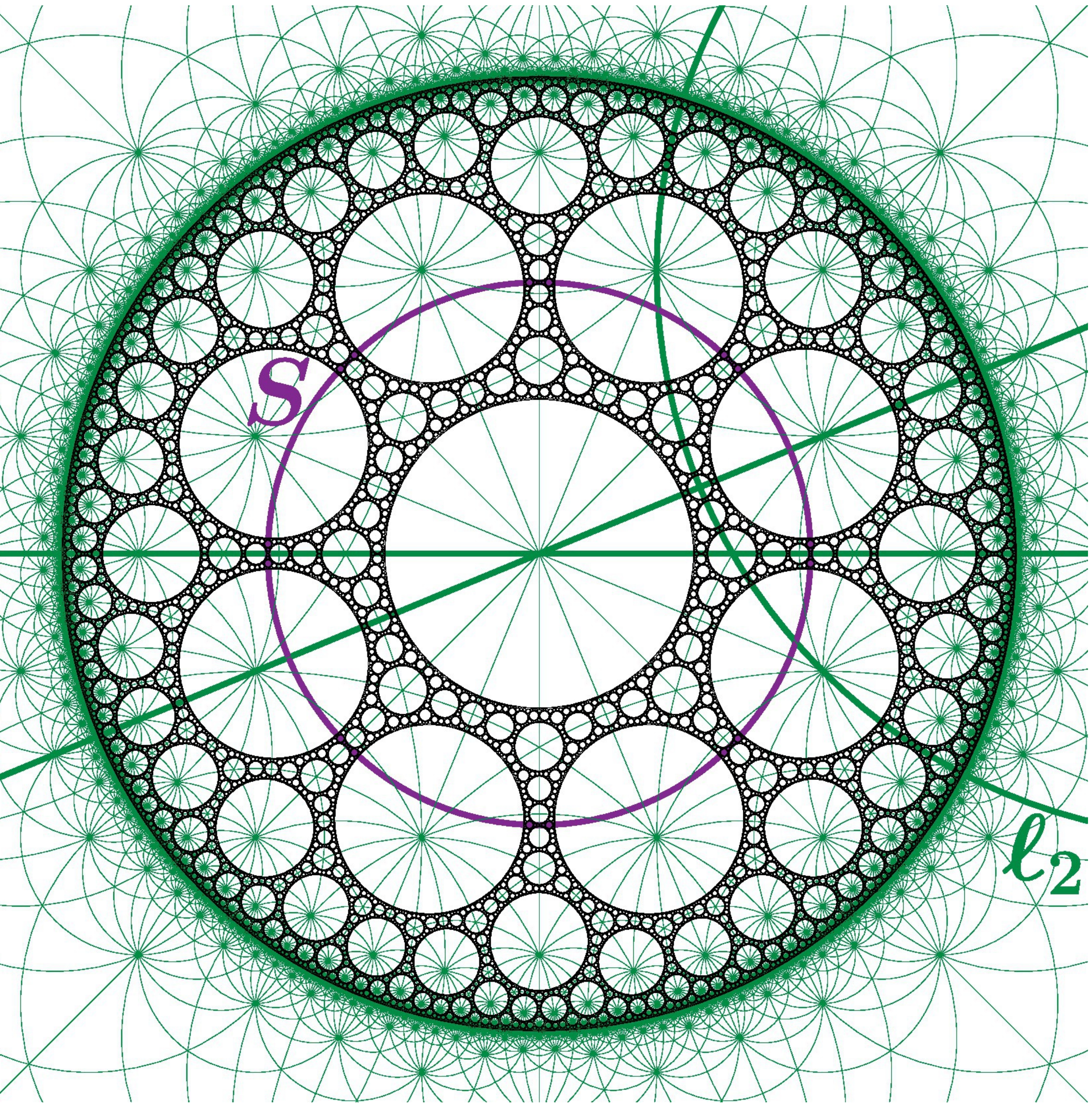

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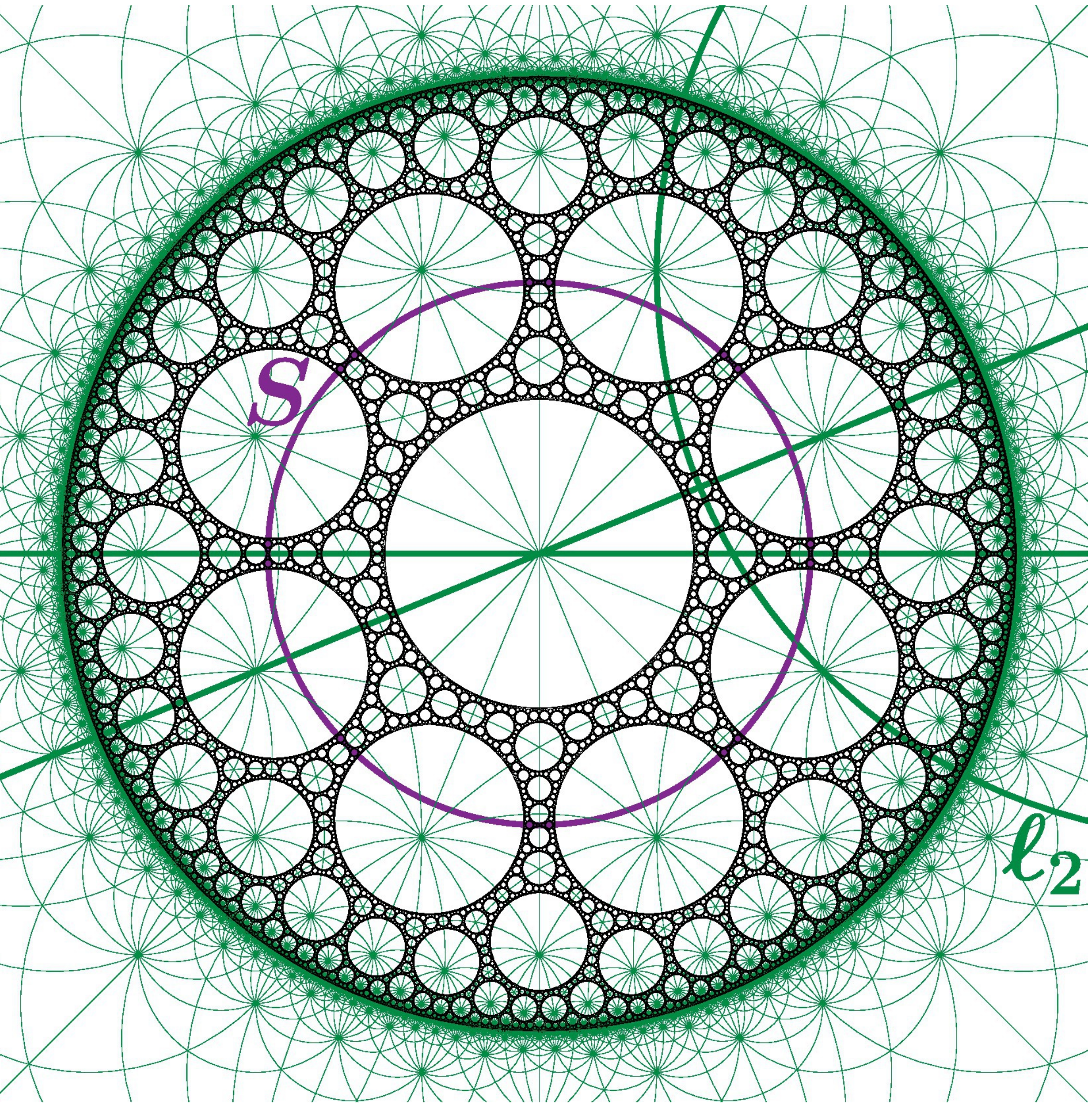

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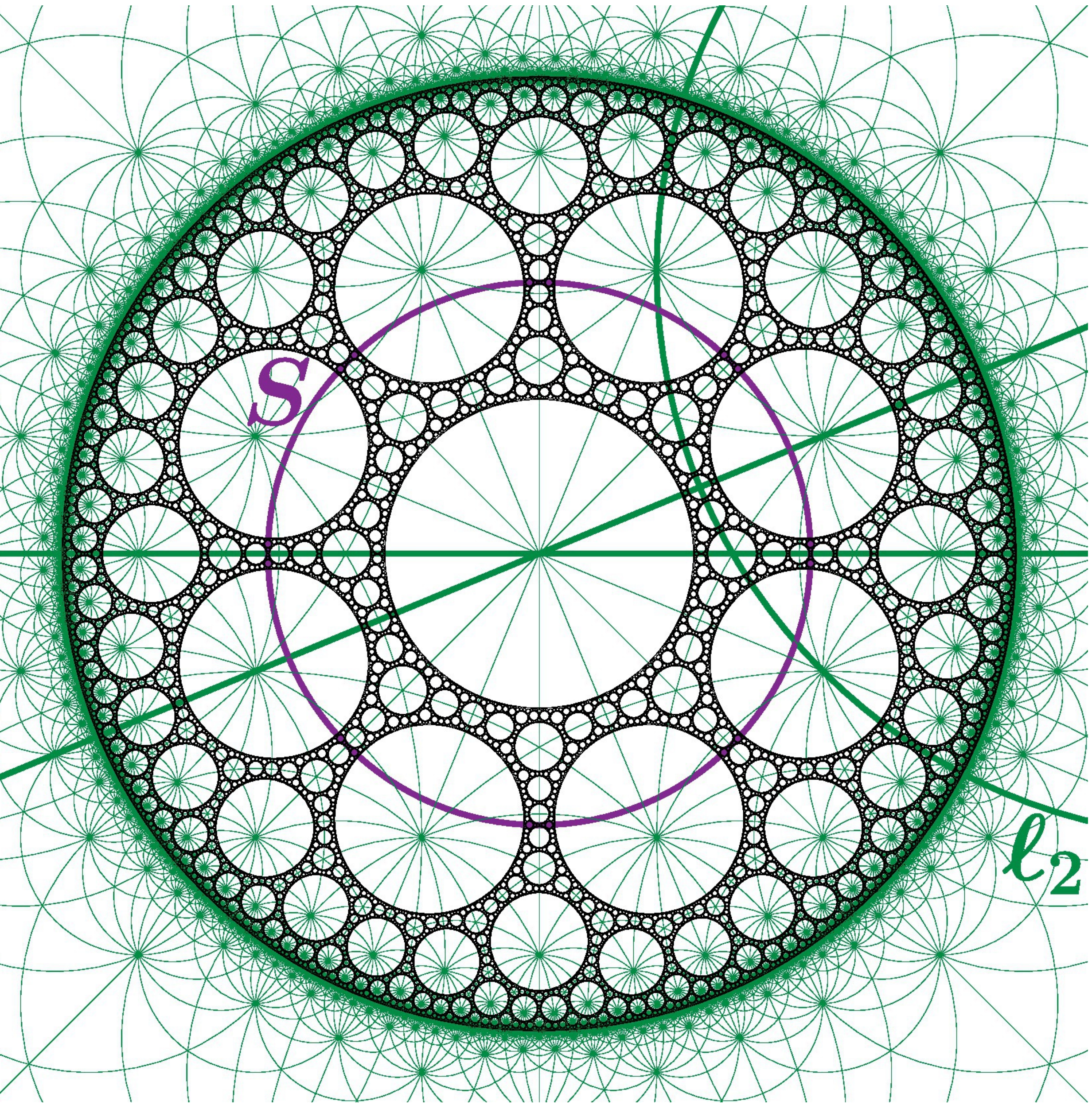

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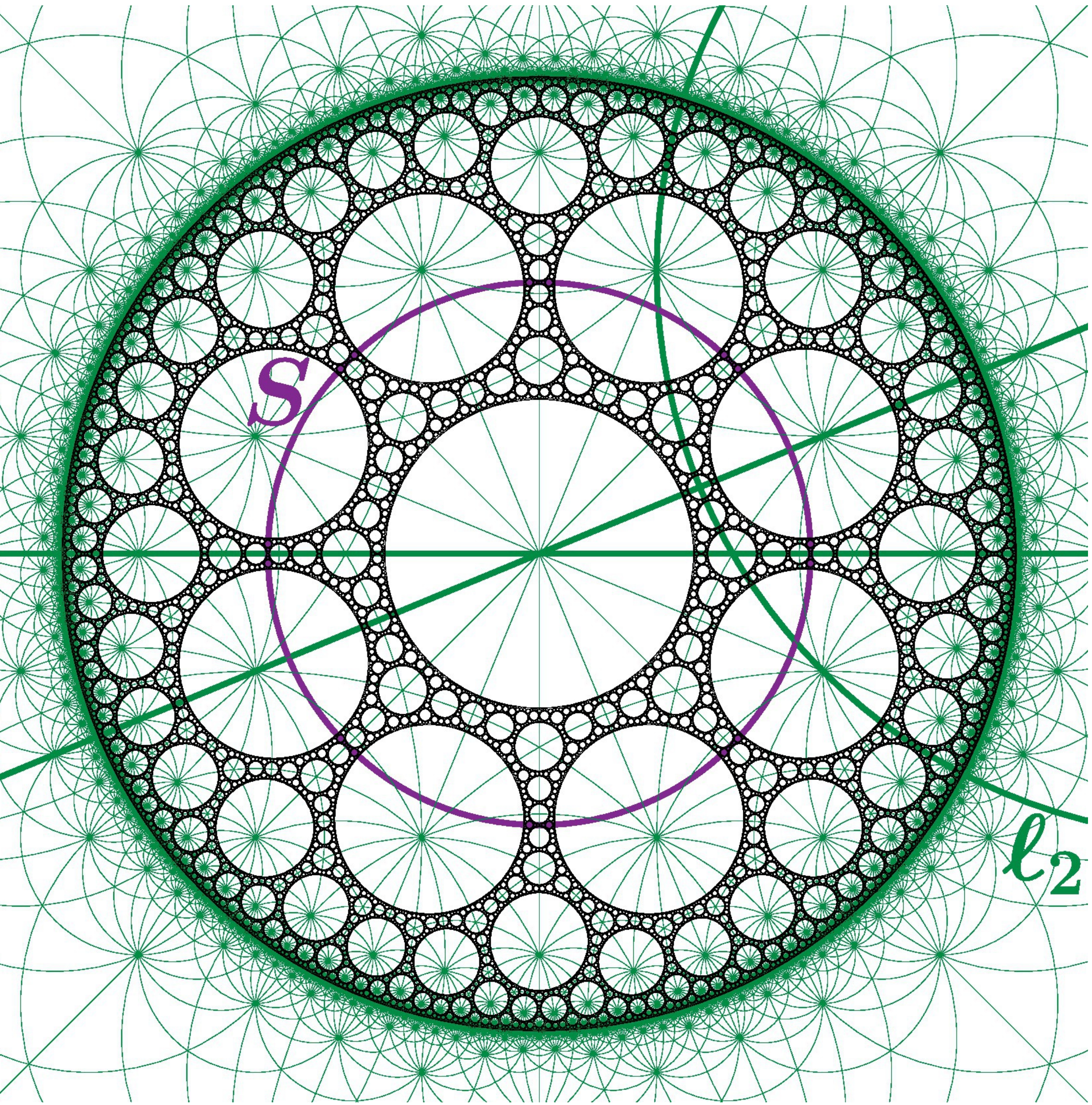

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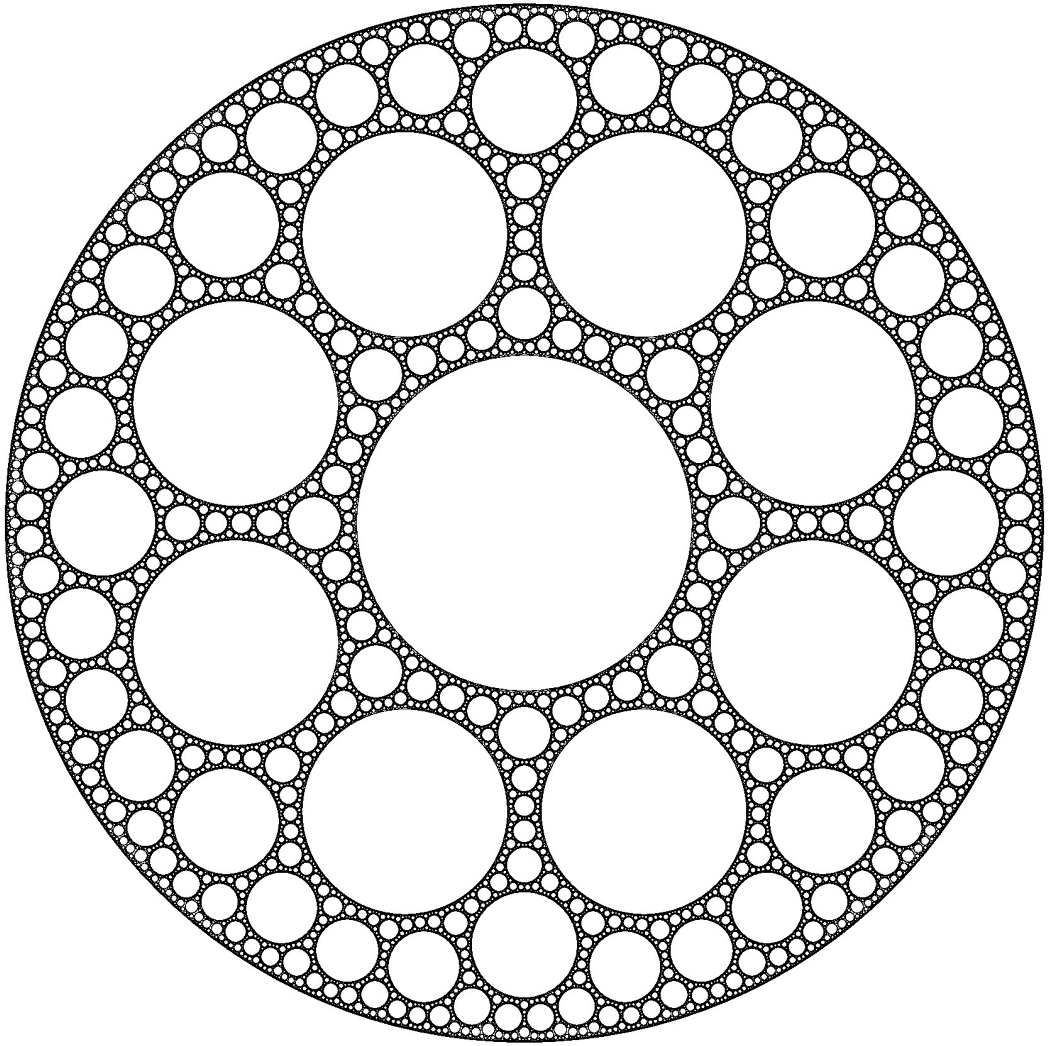
● (Oh–Shah '12) $\#\{\overset{\text{circle}}{C} \subset \partial_\infty G \mid C\cap A \neq \emptyset, \mathrm{ra}(C)^{-1} \leq \lambda\} \sim c\mathcal{H}^d(A)\lambda^d$

4 Laplacian on the limit set $\partial_\infty G$ of $G = G_m$



$$cf.\ \mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset_{\text{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 \, d\mathrm{vol}_C,$$

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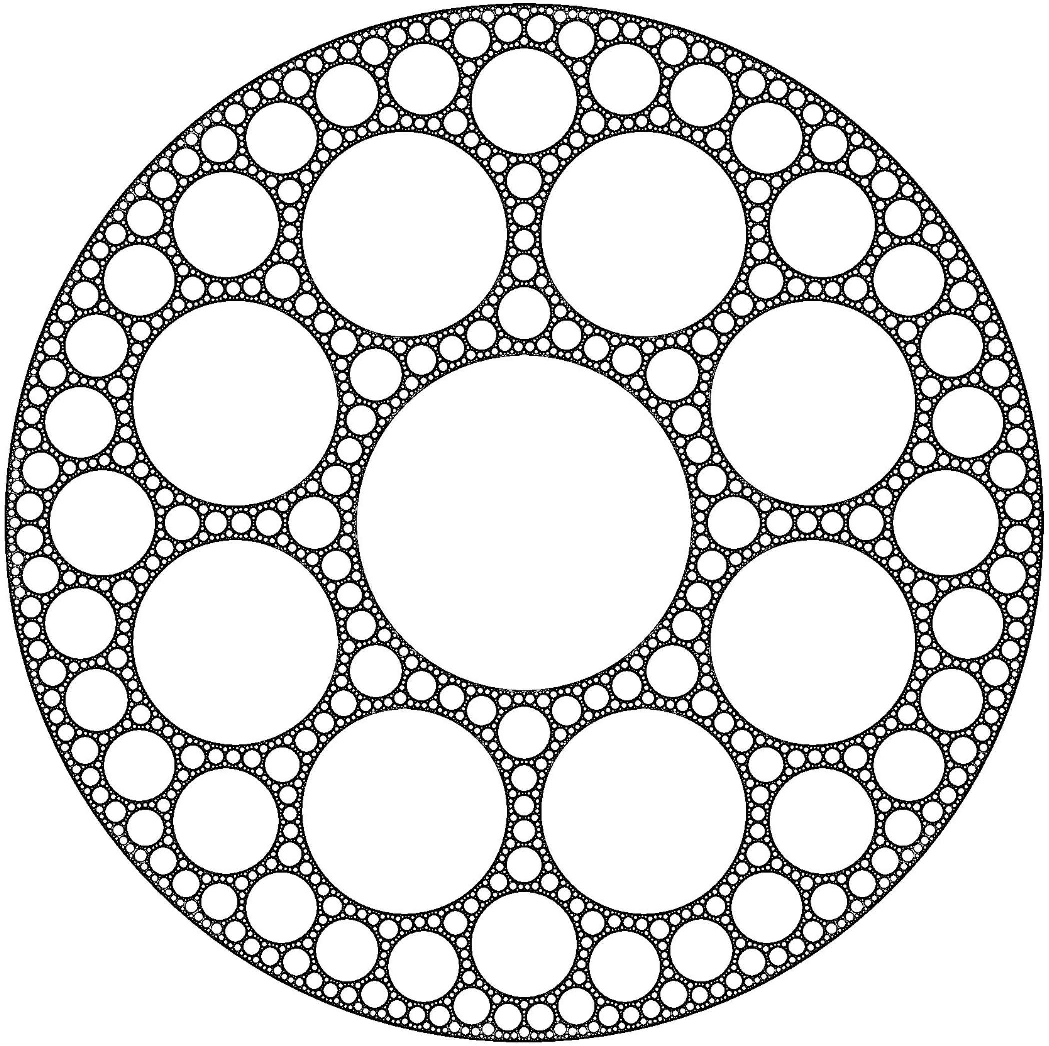

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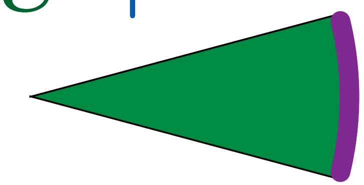

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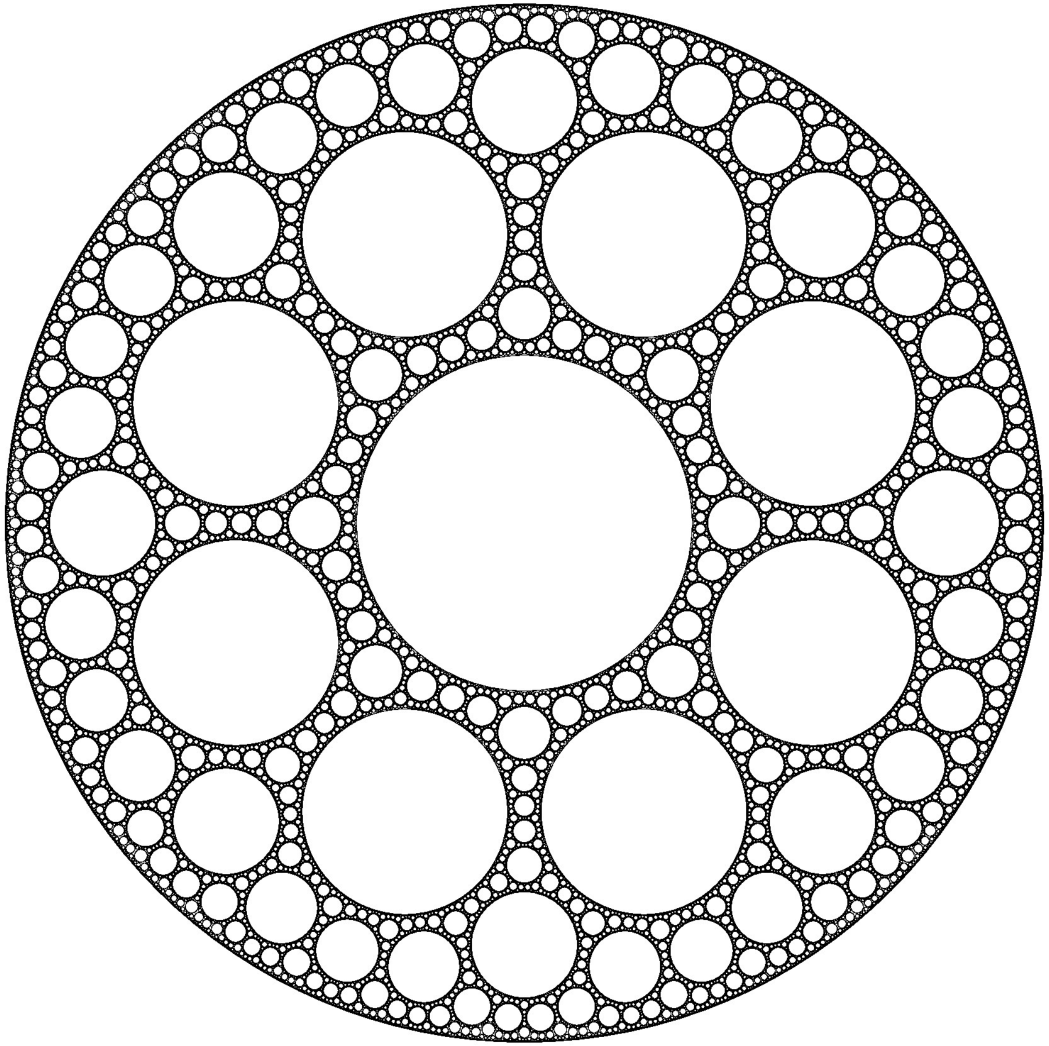

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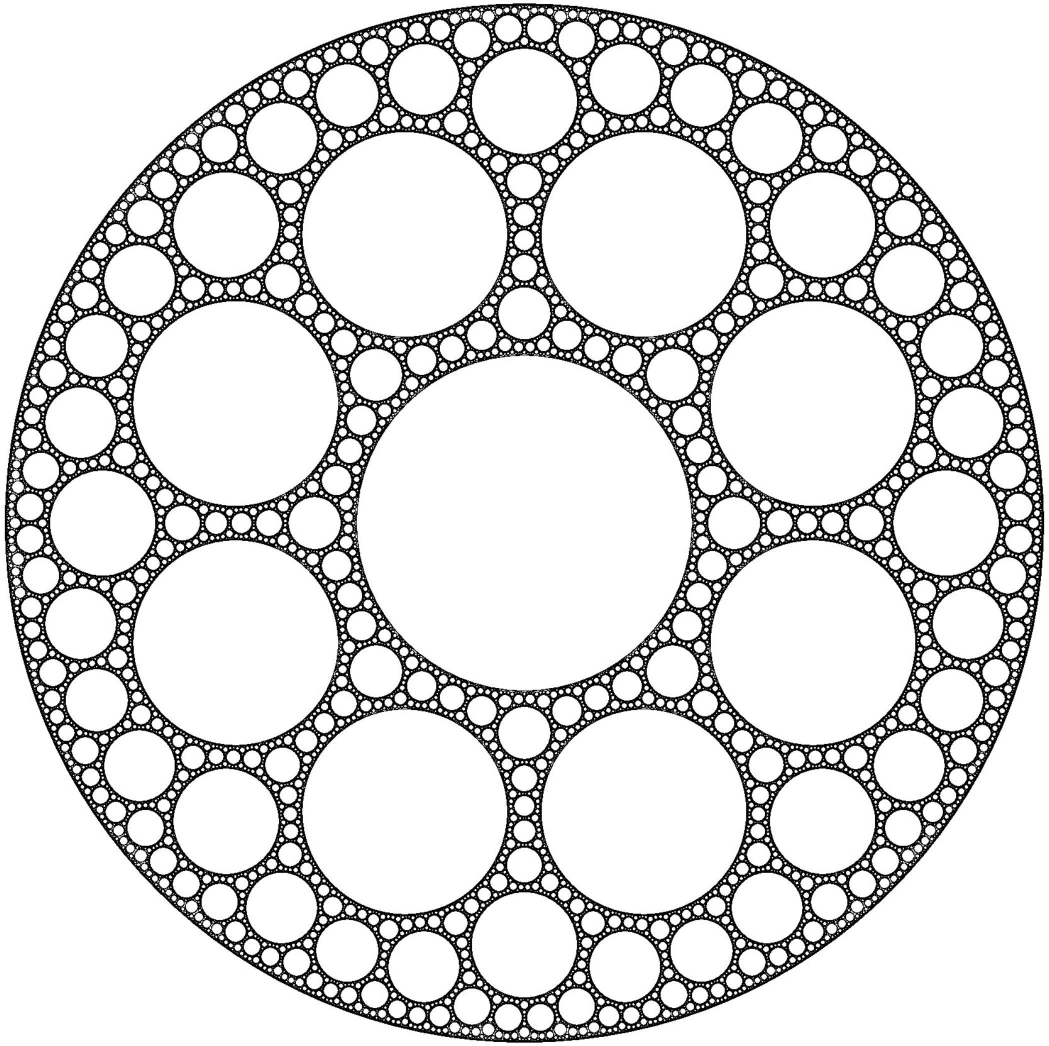

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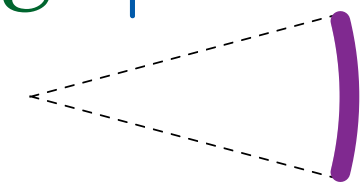

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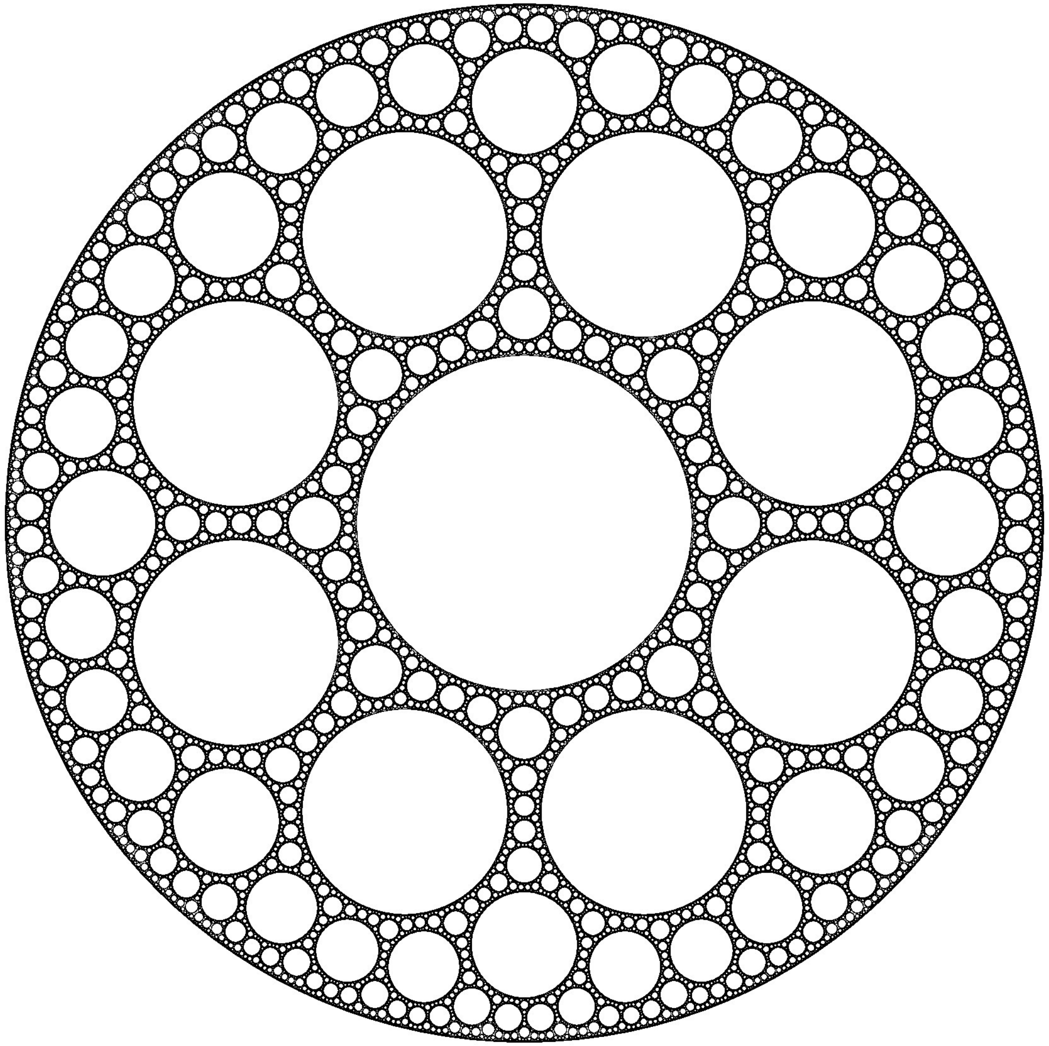
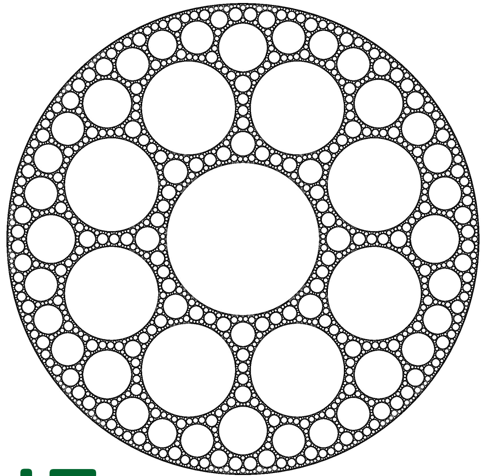


▷ $\mathcal{G} := \{g \in \mathbf{M\ddot{o}b}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\}$

▷ $K_0 := \mathbb{B}^2 \cap \partial_\infty G$, $K_g := g(K_0)$ $\left(\begin{array}{l} g \in \mathcal{G} \text{ represents} \\ \text{choice of initial } \triangle \end{array}\right)$



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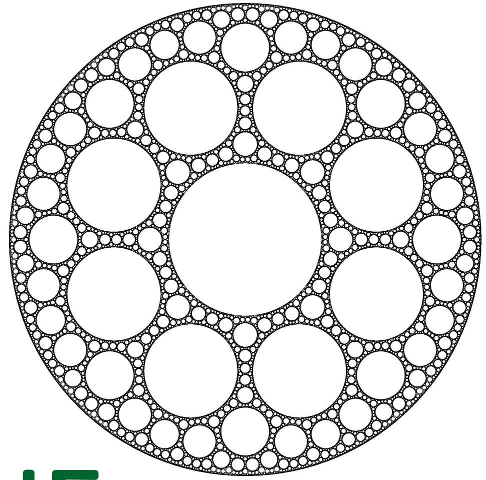
▷ $\nu^g := \sum_{C \subset_{\text{arc}} K_g} \text{rad}(C) \text{vol}_C$ (**NOT doubling!**)

▷ $^\forall u \in \text{LIP}|_{K_g}$, $\mathcal{E}^g(u,u) := \sum_{C \subset_{\text{arc}} K_g} \text{rad}(C) \int_C |\nabla_C u|^2 \, d\text{vol}_C$ ($cf.$ Osada '07)

Prop. On $L^2(K_g, \nu^g)$, $(\mathcal{E}^g, \text{LIP}_c(K_g))$ is closable & its closure $(\mathcal{E}^g, \mathcal{F}_g)$ is a strongly local regular Dirichlet form.

Prop. The inclusion map $\iota : K_g \hookrightarrow \mathbb{R}^2$ is $\mathcal{E}^g$-*harmonic*. (uniqueness NOT known)



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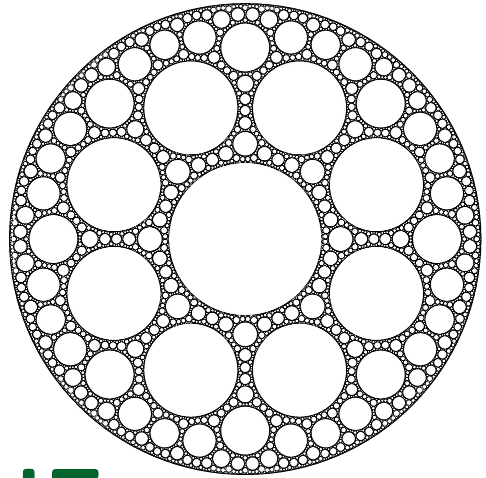

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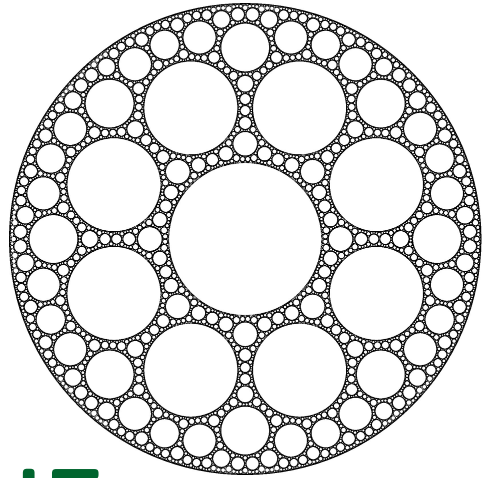

$\triangleright\ \mathcal{G} := \{g \in \text{Möb}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\}$

$\triangleright\ K_0 := \mathbb{B}^2 \cap \partial_\infty G, \quad K_g := g(K_0)$

For each $g \in \mathcal{G}$, on $K_g = g(K_0)$ DEFINE:

$\triangleright\ \nu^g := \sum_{C \subset_{\text{arc}} K_g} \text{rad}(C)\text{vol}_C$ **(NOT doubling!)**

$\triangleright\ {}^\forall u \in \text{LIP}|_{K_g},\ \mathcal{E}^g(u,u) := \sum_{C \subset_{\text{arc}} K_g} \text{rad}(C) \int_C |\nabla_C u|^2 \, d\text{vol}_C$ (*cf.* Osada '07)

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▷ $g \in \mathcal{G}$ (represents choice of the initial $\mathbb{B}^2$-△)

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▷ $\{\lambda_n^{g,U}\}_{n\in\mathbb{N}}$: the eigenvalues of $-\Delta^{\mathrm{Dirichlet}}_{(U,\nu^g,\mathcal{E}^g,\mathcal{F}_g)}$

▷ $\mathcal{N}_{g,U}(\lambda) := \#\{n\in\mathbb{N} \mid \lambda_n^{g,U} \le \lambda\}$ ($\emptyset\neq U\subset K_g$ open)

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$\Leftrightarrow$ **Thm**, BUT

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5 Case of non-circle-packing self-conformal fractals?

- CLE_κ-carpet, $\kappa \in (\frac{8}{3}, 4]$, of Sheffield–Werner '12?
- SLE_κ-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15)
- $f(\partial_\infty G)$ (∂_∞ of quasiconf. deform. fGf^{-1} of G)?
- Self-conformal quasi-circles γ (work in progress):

▷ Resistance on $\gamma :=$ harmonic meas. ω on $\mathrm{INT}(\gamma)$

- ${}^{\exists 1}d_{\mathrm{w}} \geq 2$, with $d_{\mathrm{f}} := \dim_{\mathrm{H}} \gamma$, ${}^{\exists}\mu_\gamma$: meas. on γ,

$\mu_\gamma([x,y]_\gamma) \asymp (|x-y|^{d_{\mathrm{f}}})^{d_{\mathrm{w}}} / \omega([x,y]_\gamma)$. (*cf.* Makarov '90)

$\rightsquigarrow$ For the Laplacian on $L^2(\gamma, \mu_\gamma)$, we should have:

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