Two manifestations of rigidity phenomena in random point sets : forbidden regions and maximal degeneracy

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- Namely, given a domain D, how does the point configuration outside of D impact the distribution of the points inside D ?
- It turns out that such spatial conditioning leads to remarkable singularities in the distribution of the points inside the domain. Roughly speaking, this is what we refer to as rigidity.

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- This has implications in the study of stochastic geometry on these point processes, notably in the use of Burton and Keane type arguments, or the “finite energy” property.

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- Rigidity of particle numbers was also established for the zeros of the planar Gaussian analytic function [G. - Peres]

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}.$$

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- Projection kernel in the above is necessary ! [G.-Krishnapur]

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  for some function φ. Setting $\varphi = \mathbf{1}_D$ gives the number of points in D.
- Natural to ask about rigidity of more general functionals of a point process (other than the particle count), particularly higher moments of the points in D.

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  $\alpha = 1$ recovers the standard planar case.

- For $\alpha \in (\frac{1}{m}, \frac{1}{m-1}]$, the first m moments of the zero process are rigid. [G.-Krishnapur]

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- Rigidity of general observables connected with suppressed fluctuation of other linear statistics.
- Rigidity is also connected with faster decay of hole probabilities and singularity of Palm measures

General picture

- (Moment-matching) [G.] Consider a point process Π having precisely the first m moments rigid, and two collections of points $\underline{\zeta} = (\zeta_1, \cdots, \zeta_k)$ and $\underline{\eta} = (\eta_1, \cdots, \eta_l)$. Then Palm measures $[\Pi]_{\underline{\zeta}}$ and $[\Pi]_{\underline{\eta}}$ are mutually absolutely continuous iff the first m moments of ζ and η match,

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- However, very few rigorous theorems establishing general implications like the above between these concepts.

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- We look at the conditional distribution of points outside D given that D is hole.
- When radius$(D) \to \infty$, how does the outside configuration behave ?
- In other words, what causes a large hole (a rare event) to occur ?

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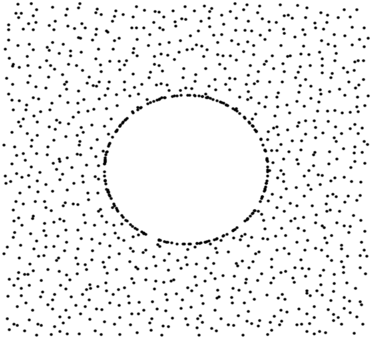

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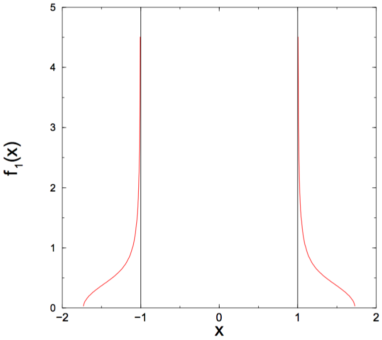

GUE

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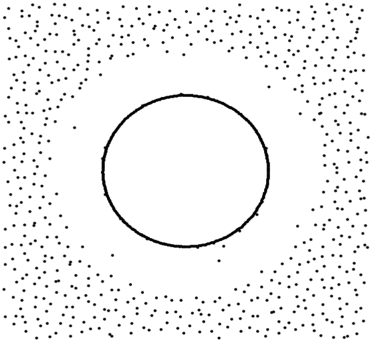

Gaussian Zeros

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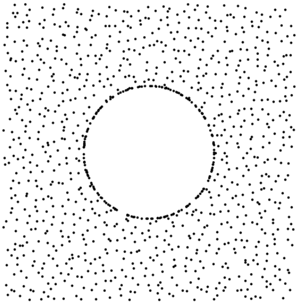

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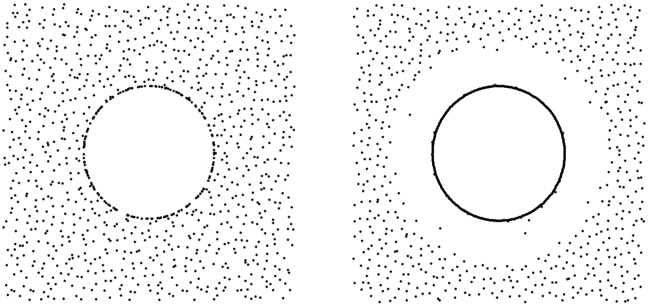

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- No zeros in the hole D is the same as $\underline{Z}(D) = 0$.
- To find the "most likely configuration" given that there is hole is roughly the same as minimizing the rate functional I over the space of probability measures (on $\mathbb{C}$) under the constraint that there is zero mass on D.

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- Heuristics made rigorous by obtaining "effective" versions of large deviation estimates and approximating the analytic function zeros by those of the polynomials.

Stealthy random fields

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- Stealthy particle systems conjectured to have deterministically bounded holes [Zhang-Stillinger-Torquato].

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Special case : Guassian process with a gap (or fast decay) in the spectrum

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- The existence of a gap / fast decay in the spectrum can be exploited to construct linear functionals of the process which have low variance.
- A linear functional with a low variance is approximately constant, so this gives an approximate linear constraint
- Sufficiently rich class of such constraints can be exploited to deduce degenerate behaviour.

Thank you !!