Mini-course part 3
Branching Brownian motion and the non local Fisher KPP equation

12'th MSJ-SI
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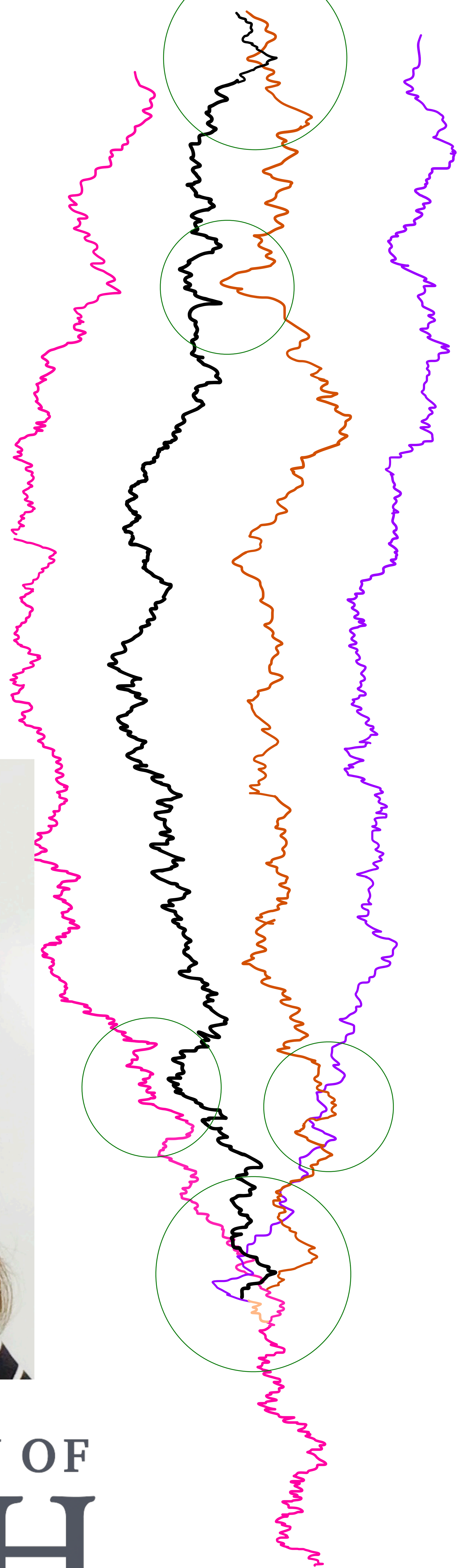


McGill

University of Oxford

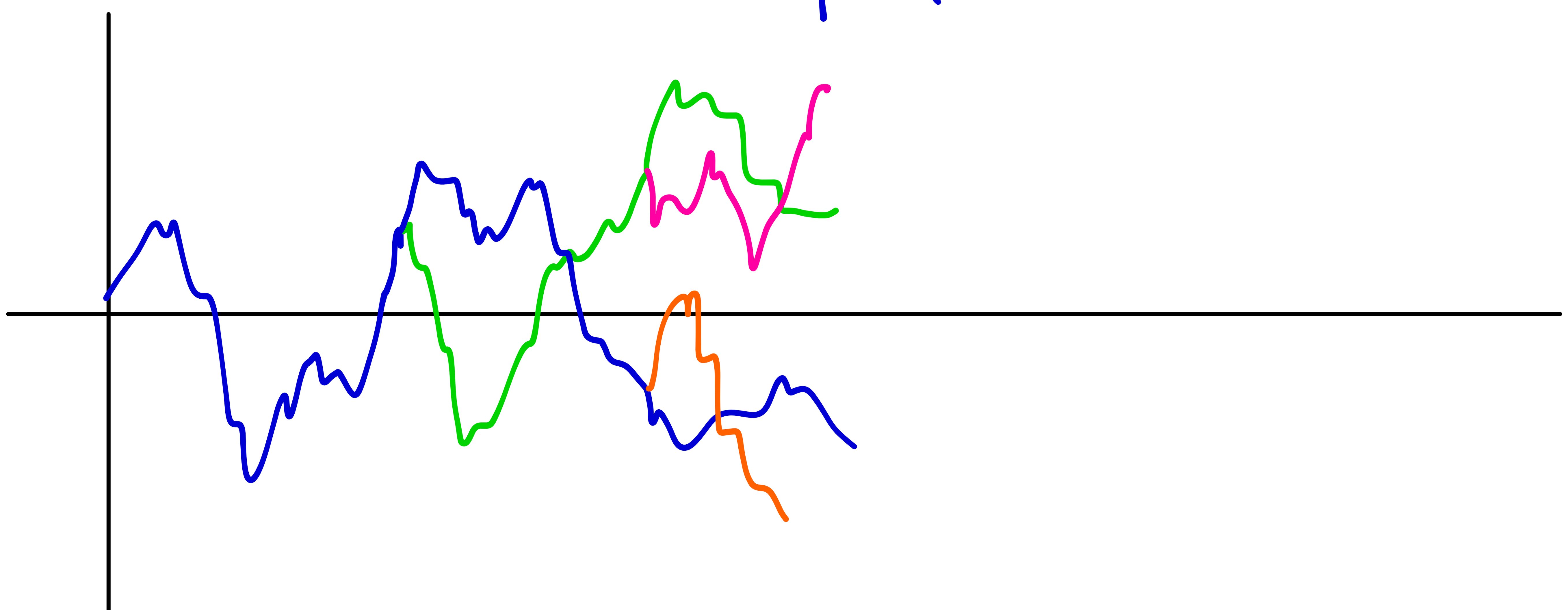
University of Bath

Branching Brownian Motion

- Time 0: Particle at $0 \in \mathbb{R}$
- Time t: Particles at positions $(X_1(t), \ldots, X_{N_t}(t))$

  N_t = # particles at time t.
- Particles branch at rate 1 (i.e. Time for a particle to branch is Exp(1)-distributed; or,

  $$\mathbb{P}(\text{fixed particle branches} \in \underbrace{[t, t+dt)}_{dt}) \sim dt$$)
- Particles move as Brownian Motion
- Particles move and branch indep. from one-another.



Competition

- Particles $(X_i(t), 1 \leq i \leq N_t)$ have masses $(M_i(t), 1 \leq i \leq N_t)$
- A particle at x with mass m causes a particle at y (with $|y-x|<1$) to lose mass at rate m.
- Write $y \sim x$ if $|y-x| \in (0,1)$

Let $\boxed{\zeta(t,x) = \sum_{\{i : X_i(t) \sim x\}} M_i(t)}$ = total mass near x

- Let $(X_{i,t}(s), 0 \leq s \leq t)$ be the ancestral trajectory of $X_i(t)$.
- Set $\boxed{M_i(t) = \exp\left(-\int_0^t \zeta(s, X_{i,t}(s))\,ds\right).}$

Competition

- Picture: For a single B.M., environmental resources (food) exactly balance energetic cost of motion.
- Particles compete for resources if
  dist < 1 from each other.
- Competition $\Rightarrow$ insufficient resources $\Rightarrow$ Loss of mass.

N.B. Branching $\Rightarrow$ Mass doubles (children each inherit mass of parent).

- Parts of analysis work for:
  - $M_i(t+dt) = M_i(t)\left(2 - dt \cdot \sum_{\{j : |X_j(t) - x| \in [0,1)\}} M_j(t)\right)$
  - Branching $\longrightarrow$ Mass splits

Basic facts

- Obs: $\mathbb{E} N_t = e^t$.

  Proof: $\mathbb{E}[N_{t+dt} \mid N_t = n] \sim n + n\,dt$

  so $\mathbb{E}[N_{t+dt}] \simeq \mathbb{E} N_t \cdot (1 + dt)$

  so $\frac{d}{dt} \mathbb{E} N_t = \mathbb{E} N_t$. $\blacksquare$

- Particle density

$$\mu(t,x) := \mathbb{E}[\#\{i : X_i(t) \in dx\}] = \mathbb{E} N_t \cdot \mathbb{P}(X_1(t) \in dx)$$

$$= e^t \, \mathbb{P}(N(0,t) \in dx)$$

$$= e^t \cdot \frac{1}{\sqrt{2\pi t}} \cdot \exp(-x^2/2t)$$

Results:
Front location

$$\boxed{M_i(t) = \exp\left(-\int_0^t \zeta(s, X_{i,t}(s))\,\mathrm{d}s\right)}$$

$$\boxed{\zeta(t,x) = \sum_{\{i:\, X_i(t) \sim x\}} M_i(t)}$$

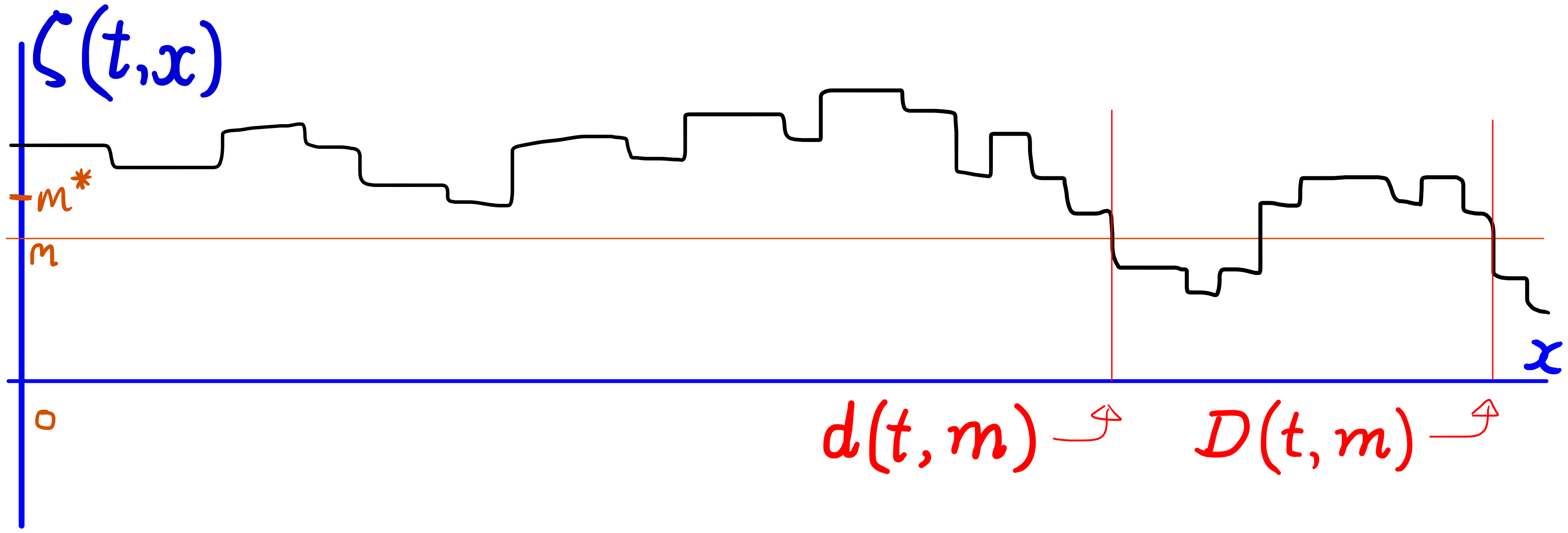
Results: Front location

$$M_i(t) = \exp\left(-\int_0^t \zeta(s, X_{i,t}(s))\,ds\right)$$

$$\zeta(t,x) = \sum_{\{i:\, X_i(t)\sim x\}} M_i(t)$$

$$D(t,m) := \max\{x>0 : \zeta(t,x) > m\}$$

$$d(t,m) := \min\{x>0 : \zeta(t,x) < m\}$$



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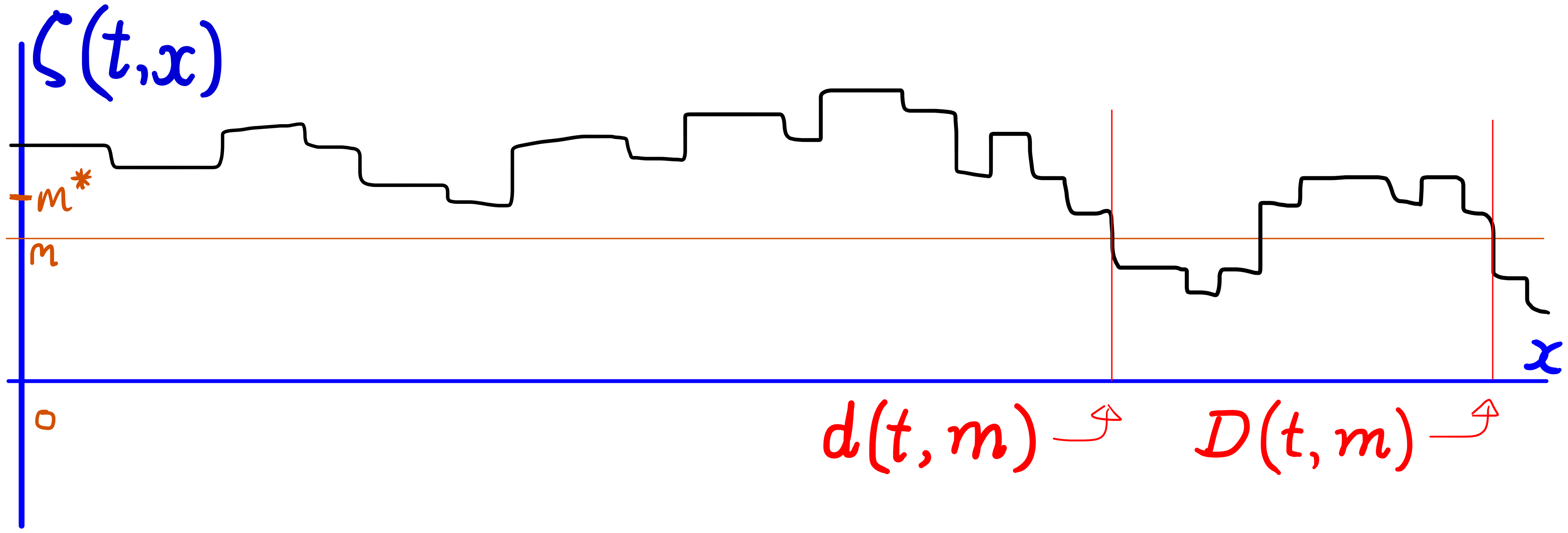

Theorem There exists $m^* > 0$ such that $\forall\, m \in [0, m^*)$, $\exists\, R^* > 0$ such that a.s., for all t sufficiently large, $\inf_{s\geq 0} d(t + R\log t + s, m) \geq D(t,m)$

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Front location

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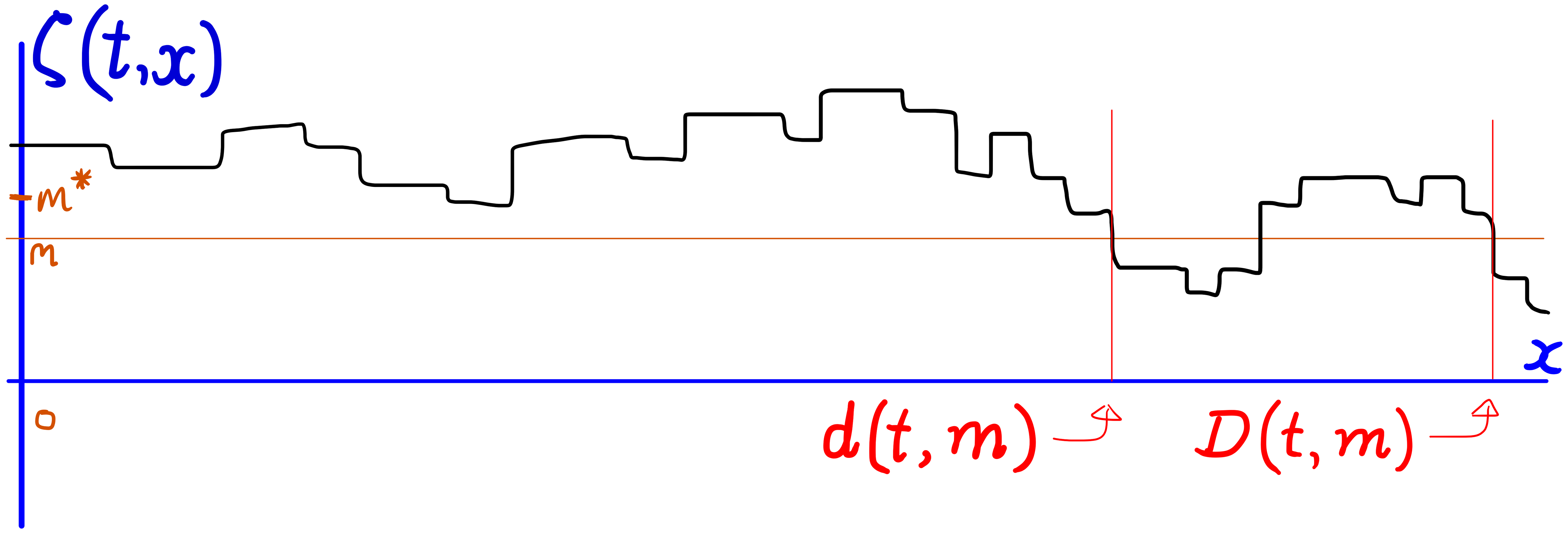

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Theorem *There exists* $m^*>0$ *such that* $\forall\, m\in[0,m^*)$, $\lim_{t\to\infty} \frac{D(t,m)}{t} \overset{a.s.}{=} \sqrt{2}$

Results: Front location

$$\boxed{M_i(t) = \exp\left(-\int_0^t \zeta(s, X_{i,t}(s))ds\right)}$$

$$\boxed{\zeta(t,x) = \sum_{\{i:\, X_i(t)\sim x\}} M_i(t)}$$

$$D(t,m) := \max\{x>0 : \zeta(t,x) > m\}$$

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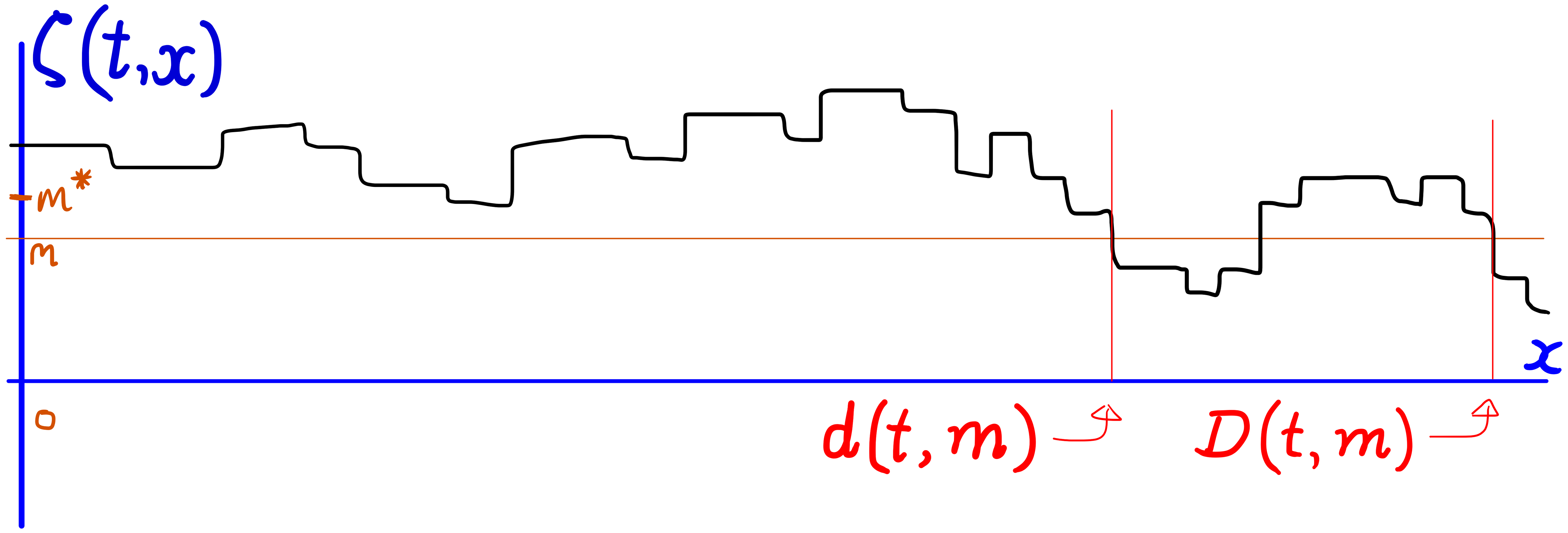

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Theorem There exists $m^*>0$ such that $\forall\, m\in[0,m^*)$, with $c = \frac{3^{4/3}\pi^{2/3}}{2^{7/6}}$, a.s.

$$\limsup_{t\to\infty}\frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \geq c \qquad \liminf_{t\to\infty}\frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c$$

Results:
Particle masses

$$M_i(t) = \exp\left(-\int_0^t \zeta(s, X_{i,t}(s))\,ds\right)$$

$$\zeta(t,x) = \sum_{\{i:\, X_i(t)\sim x\}} M_i(t)$$

Theorem:

There is $m^* > 0$ s.t. for all $c \in (0, \sqrt{2})$ and $n \in \mathbb{N}$, for t large,

$$\mathbb{P}\left(\inf_{s>t} \inf_{|x| \le cs} \zeta(s,x) < m^*\right) \le t^{-n},$$

and

$$\mathbb{P}\left(\sup_{s>t} \sup_{x\in\mathbb{R}} \zeta(s,x) > \frac{1}{m^*}\right) \le t^{-n}.$$

Theorem

For any $\alpha < 1$ and $n \in \mathbb{N}$, for t sufficiently large,

$$\mathbb{P}\left(\max_{i \le N(t)} M_i(t) > \frac{1}{t^\alpha}\right) \le t^{-n}.$$

Results:
Hydrodynamic limit

$$\boxed{\zeta(t,x) = \sum_{\{i:\, X_i(t)\sim x\}} M_i(t)} \qquad \boxed{\zeta_\delta(t,x) = \frac{1}{\delta} \sum_{\{i:\, |X_i(t)-x|\in(0,\delta)\}} M_i(t)}$$

Theorem:

For all $T>0$, and $n\in\mathbb{N}$ there is $C>0$ s.t. for $\delta\in(0,1)$, for times $t>C/\delta^5$,

$$\mathbb{P}\left\{ \sup_{s\leq T,\, x\in\mathbb{R}} \left| \zeta_\delta(t+s,x) - u^t(s,x) \right| \geq \delta \right\} \leq \frac{1}{t^n}$$

where $u^t(s,x)$ is the solution of the non-local Fisher-KPP equation

$$\frac{\partial u^t}{\partial s} = \frac{1}{2}\Delta u^t + u^t - u^t \int_{-1}^{1} u(t,x-y)\,dy$$

with initial condition

$$u^t(0,x) = \zeta_\delta(t,x).$$

(In a solo paper, Penington has derived precise behaviour about the front location for this PDE with compact initial condition.)

Basic facts

$$\boxed{\mu(t,x) = (2\pi t)^{-1/2} \exp\left(+t - x^2/2t\right)}$$

Fact: $\mu(t,x) = 1$ when $t = \underbrace{\sqrt{2}\,t - \frac{1}{2\sqrt{2}}\log t}_{V_t} + O(1)$

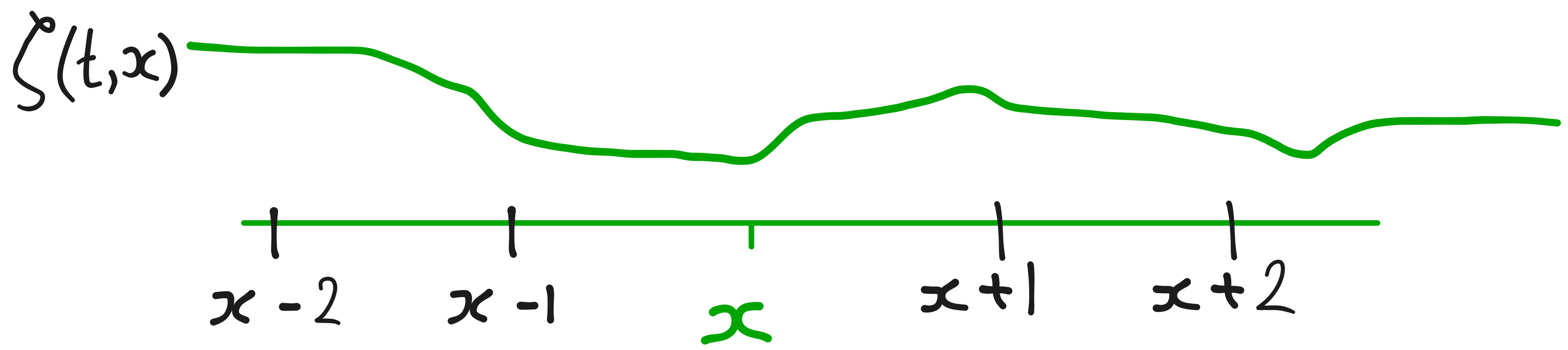
Fact: $\mu(t, V_t - x) = \exp\left((1+o(1))\cdot\sqrt{2}\,x\right)$ for $x = o(t)$.

- I.e. expected particle density decays exponentially beyond V_t, grows exponentially before V_t.

$$\boxed{\text{Intuition (For some purposes): } e^t \text{ indep. Brownian Motions.}}$$

Proof idea I. Density stabilizes quickly a)

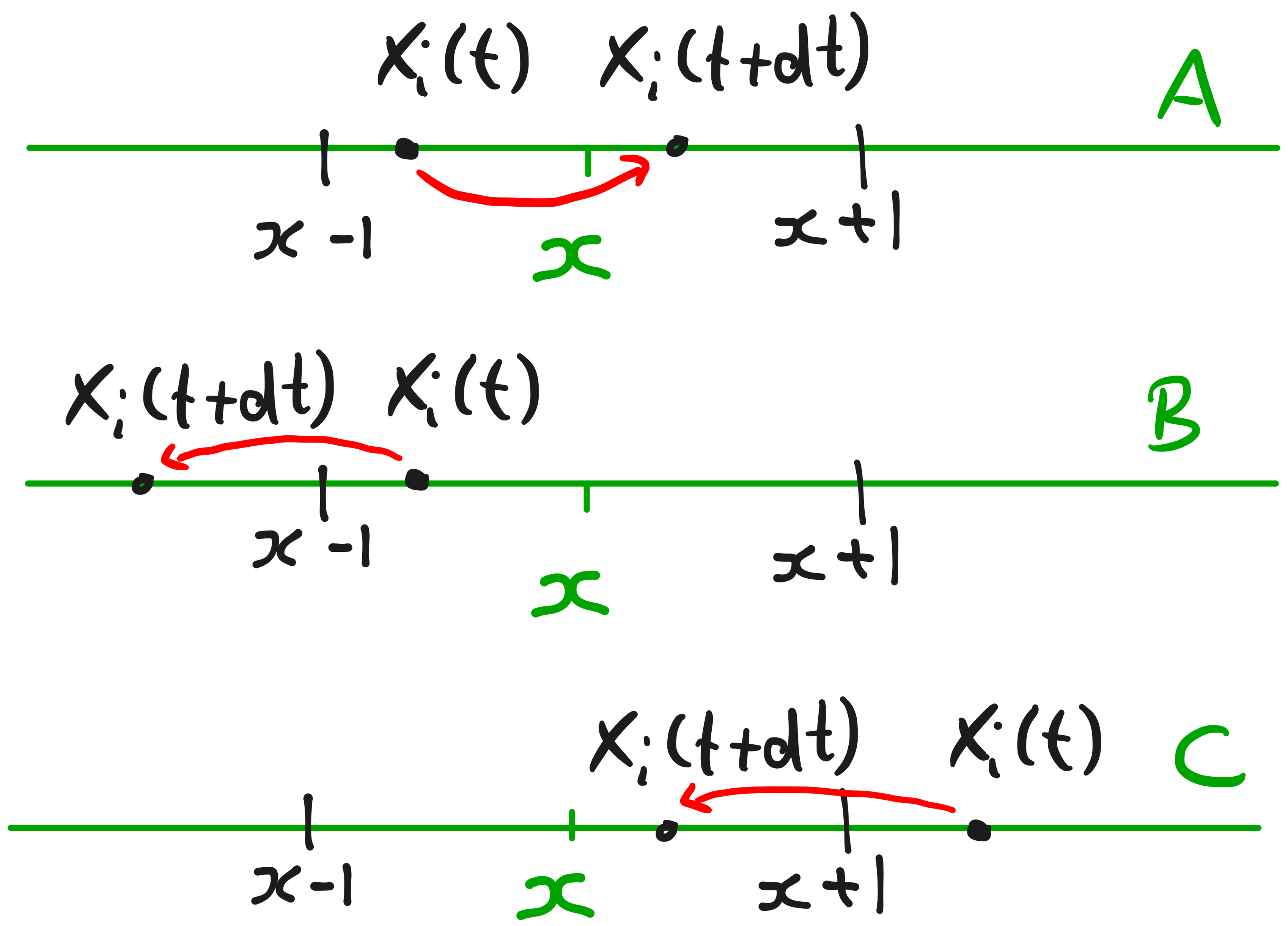
$$\rho(t+dt,x)-\rho(t,x) =$$



$$\sum_{\substack{i\leq N(t)\\ i\in A}} \left(M_i(t+dt)-M_i(t)\right)$$

$$-\sum_{\substack{i\leq N(t)\\ i\in B}} M_i(t) + \sum_{\substack{i\leq N(t)\\ i\in C}} M_i(t) + \sum_{i>N(t)} M_i(t)\cdot \mathbb{1}[X_i(t)\sim x]$$

Branching



Motion

$$\mathbb{P}(X_i(t+dt)\sim x \mid X_i(t)) = \begin{cases} 1-o(dt) & \text{if } X_i(t)\sim x \\ o(dt) & \text{if } X_i(t)\not\sim x,\ X_i(t)\neq x \end{cases}$$

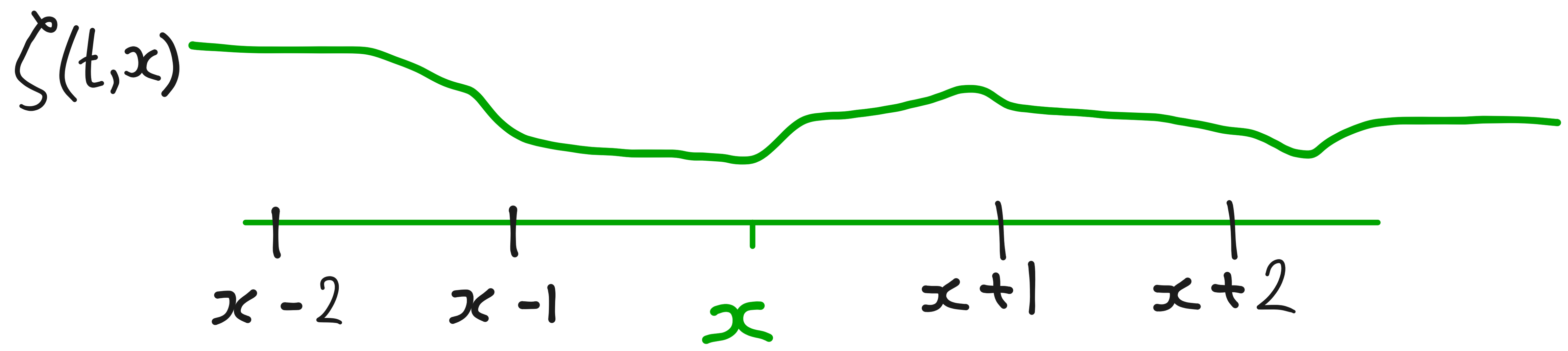
So B,C typically negligible.

$$\sum_{\substack{i\leq N(t)\\ i\in A}} \left(M_i(t+dt)-M_i(t)\right) \cong \sum_{i:X_i(t)\sim x} M_i(t)\left(1-dt\cdot\rho(t,X_i(t))\right)-M_i(t)$$

$$= -dt\cdot \sum_{i:X_i(t)\sim x} \rho(t,X_i(t))$$

$$\sum_{i>N(t)} M_i(t)\cdot\mathbb{1}[X_i(t)\sim x] \cong dt \sum_{i:X_i(t)\sim x} M_i(t) = dt\,\rho(t,x)$$

Proof idea I. Density stabilizes quickly b)



$$\frac{d}{dt}\zeta(t,x) \cong \zeta(t,x) - \sum_{i:X_i(t)\sim x} \rho(t,X_i(t))$$

$$< \zeta(t,x)\left(1-\min\{\zeta(t,y): |y-x|<1\}\right)$$

$$> \zeta(t,x)\left(1-\max\{\zeta(t,y): |y-x|<1\}\right)$$

- If $\zeta(t,y)<1-\varepsilon$ $\forall$ y s.t. $|y-x|<1$ then $\frac{d}{dt}\zeta(t,x) > \varepsilon\cdot\zeta(t,x)$
- If $\zeta(t,y)>M+1$ $\forall$ y s.t. $|y-x|<1$ then $\frac{d}{dt}\zeta(t,x) < -M\,\zeta(t,x)$

Density $\delta<1$ $\rightarrow$ Density ~ 1 in time $O(\log\frac{1}{\delta})$

Density $\Delta>1$ $\rightarrow$ Density ~ 1 in time $O(1)$

Proof idea II. a) Proof by contradiction.

Write $d(t) = d(t, \tfrac{1}{2})$, so $\zeta(t,x) \geq \frac{1}{2}$ $\forall\, 0 < x \leq d(t)$

$D(t) = D(t, \tfrac{1}{2})$ so $\zeta(t,x) < \frac{1}{2}$ $\forall\, x > D(t)$

Idea

$(d(s), s \leq t)$ too big $\Rightarrow$

every trajectory $(X_{i,t}(s), s \leq t)$ spends much time behind $(d(s), s \leq t)$ $\Rightarrow$

particles all have small mass $\Rightarrow$

$d(t)$ must be smaller

A particle $X_i(t)$ whose ancestral path $(X_{i,t}(s), s \leq t)$ has $\mathrm{Leb}(\{s : |X_{i,t}(s)| \leq d(s)\}) \geq t/2$ (a slowpoke) then has $M_t(i) \leq e^{-t/4}$.

Proof idea II.b). Slowpokes miss supper

Say $(g(t), t \geq 0)$ is a slowpoke threshold if almost surely,

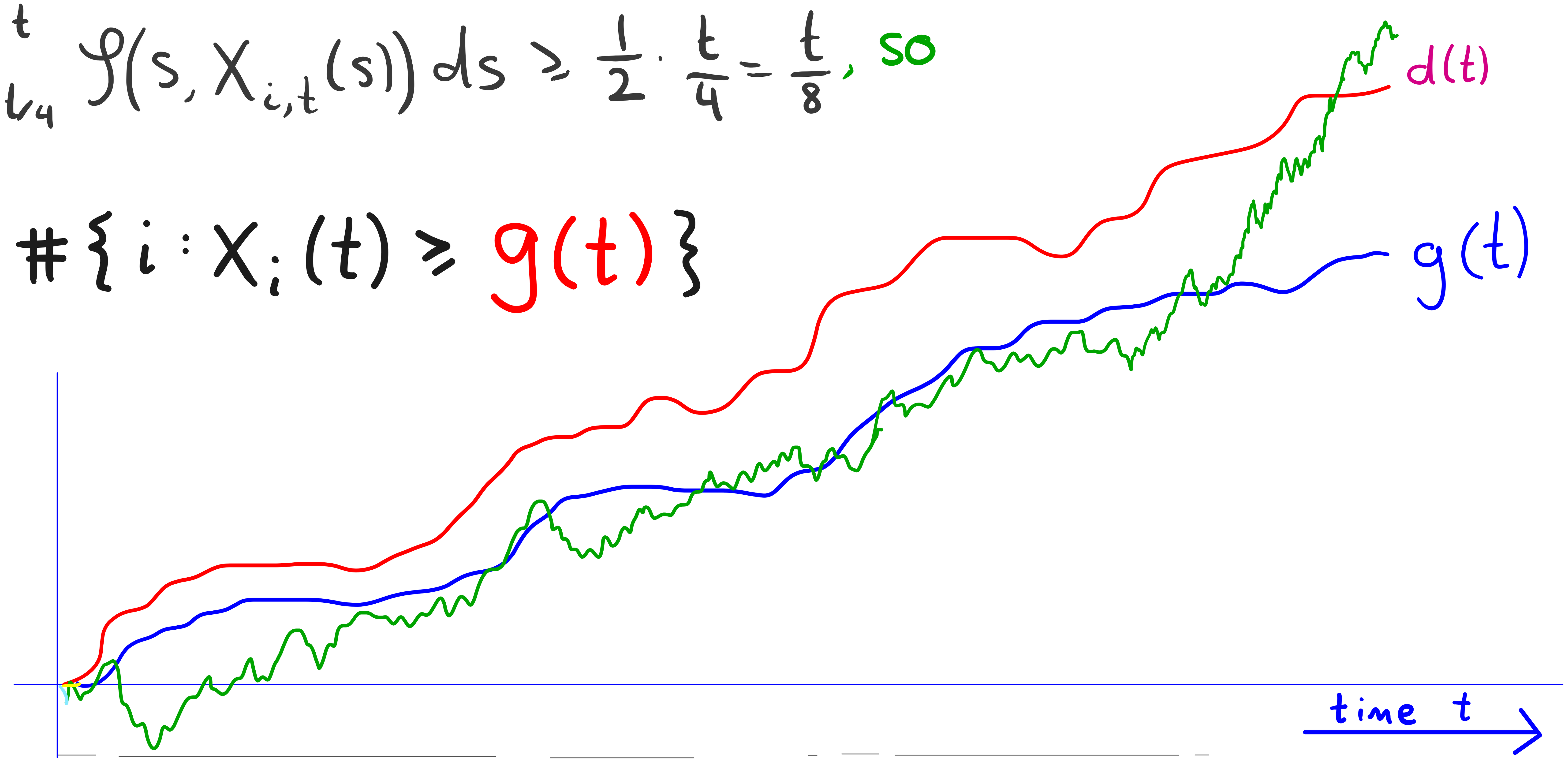
$\exists t_0$ s.t. $\forall t > t_0$, $\forall i \leq N(t)$ s.t. $X_i(t) > g(t)$,

$$\mathrm{Leb}(\{s : |X_{i,t}(s)| \leq g(s)\}) \geq t/2$$

If $d(s) \geq g(s)$ $\forall s \in (t/4, t)$ then whp

$\forall i$ s.t. $X_i(t) \geq g(t)$, $\int_{t/4}^{t} \mathcal{g}(s, X_{i,t}(s))\, ds \geq \frac{1}{2} \cdot \frac{t}{4} = \frac{t}{8}$, so

$$\sum_{\{i : X_i(t) \geq g(t)\}} M_i(t) \leq e^{-t/8} \cdot \#\{i : X_i(t) \geq g(t)\}$$



In this case at time t

NEED $\exp(t/8)/2$ slowpokes to make $\sum_{\{i : X_i(t) \geq g(t)\}} M_i(t) > 1/2$.

Proof idea II c). The slowpoke threshold

Lemma: There is $c>0$ s.t. $g(t)=\sqrt{2}\,t - c\,t^{1/3}$ is a slowpoke threshold.

"Proof": Let $B=(B(t), t\geq 0)$ be Brownian motion.

i) Easy: $\mathbb{P}\left(\mathrm{Leb}\{s\in[0,t] : |B_s|<1\} > t/2\right) = e^{-\Theta(t)}$.

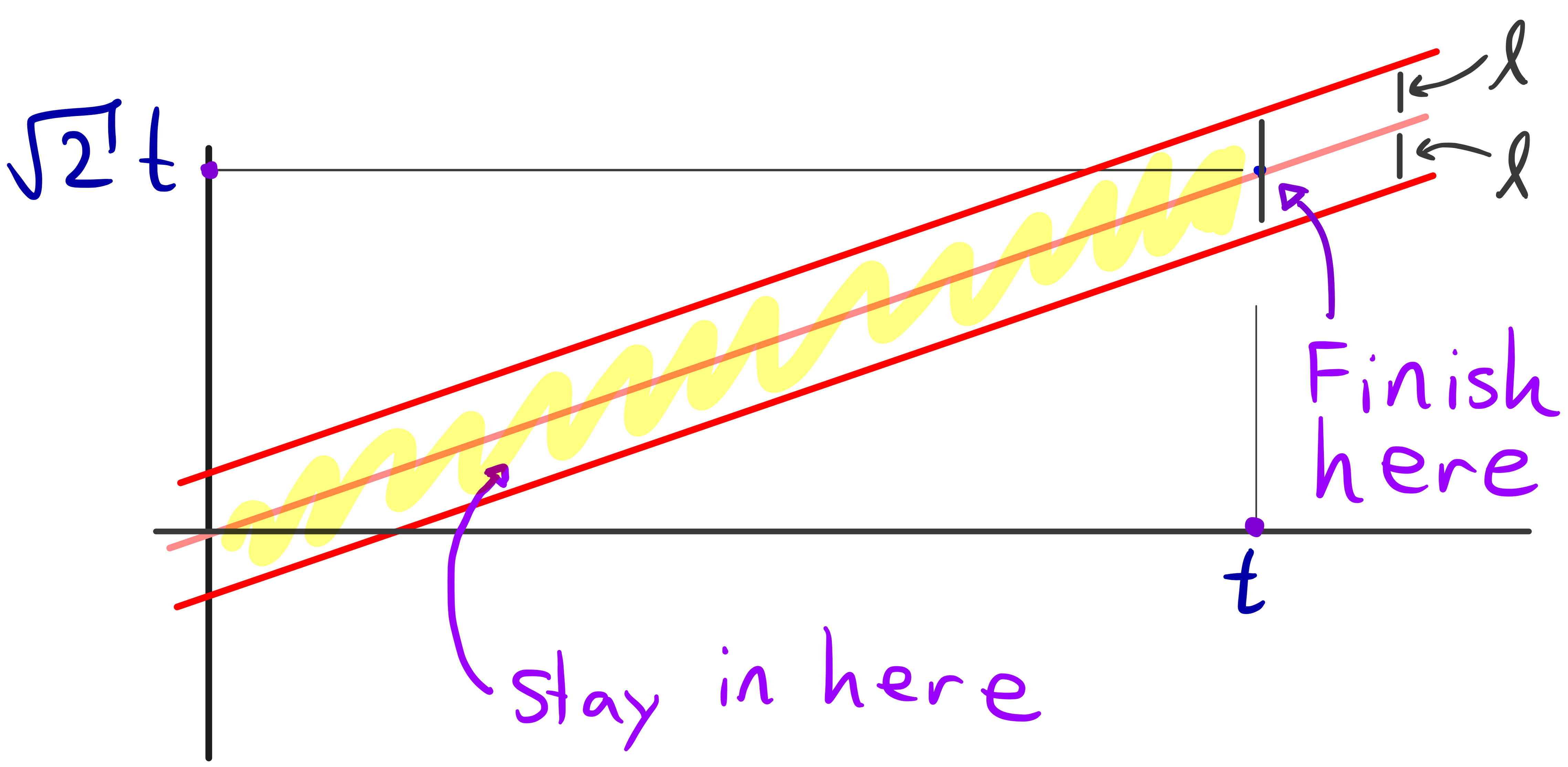
ii) Brownian scaling: $\mathbb{P}\left(\mathrm{Leb}\{s\in[0,t] : |B_s|<\ell\} > t/2\right) = e^{-\Theta(t/\ell^2)}$.

iii) Branching

$$\mathbb{E}\left(\#\{i : |X_i(t)-\sqrt{2}t|<\ell,\ \mathrm{Leb}\{s\in[0,t] : X_{i,t}(s)\geq\sqrt{2}s-\ell\}>t/2\}\right)$$

$$= e^t\cdot\mathbb{P}\left(|B_t-\sqrt{2}t|<\ell,\ \mathrm{Leb}\{s\in[0,t] : |B_s-\sqrt{2}s|<\ell\}>t/2\right)$$

$$\simeq e^t\cdot\mathbb{P}\left(|B_t-\sqrt{2}t|\leq\ell\right)\cdot e^{-\Theta(t/\ell^2)}$$



$$\mathbb{P}\left(|B_t-\sqrt{2}t|\leq\ell\right)$$

$$\simeq \mathbb{P}\left(B_t\geq\sqrt{2}t-\ell\right)$$

$$\simeq \mathbb{P}\left(B_t\geq v_t-\ell\right)\simeq e^{\sqrt{2}\ell-t}.$$

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$$\mathbb{E}\left(\#\{i : |X_i(t)-\sqrt{2}t|<\ell, \mathrm{Leb}\{s\in[0,t] : X_{i,t}(s)\geq \sqrt{2}\,s-\ell\} > t/2\}\right)$$
$$= e^t\cdot \mathbb{P}\left(|B_t-\sqrt{2}t|<\ell, \mathrm{Leb}\{s\in[0,t] : |B_s-\sqrt{2}s|<\ell\} > t/2\right)$$
$$\simeq e^t\cdot \mathbb{P}(|B_t-\sqrt{2}t|\leq \ell)\cdot e^{-\Theta(t/\ell^2)}$$
$$\simeq e^t\cdot e^{\sqrt{2}\ell - t}\cdot e^{-\Theta(t/\ell^2)} \approx e^{\sqrt{2}\ell}\cdot e^{-\Theta(t/\ell^2)}$$
$$= \Theta(1) \text{ when } \ell\approx t/\ell^2 \;;\; \ell\approx t^{1/3}$$

iv) Conclusion

For $c>0$ small enough, obtain
$$\mathbb{E}\left(\#\{i : X_i(t)\geq\sqrt{2}\,t-ct^{1/3}, \mathrm{Leb}\{s\leq t : X_{i,t}(s)\geq cs^{1/3}\} > t/2\}\right) \leq e^{-\delta t^{1/3}}, \text{ some } \delta>0$$

Proof idea III. Upper Bound.

Lemma: Let $g(t) = \sqrt{2}t - ct^{1/3}$, c as in previous lemma.

Then a.s. $\forall\, t > 0$ large $\exists\, s \in [t/4, t]$ s.t. $d(s) < g(s) + 1$

Proof:

Note $\mathbb{E}\,\#\{i : X_i(t) \geq g(t)\} = \int_{g(t)}^{\infty} \mu(t,x)\,dx$

$$\cong \mu(t, g(t)) \cong e^{\sqrt{2}(V_t - g(t))} \cong e^{\sqrt{2}\cdot t^{1/3}}.$$

If $d(s) \geq g(s) + 1$ $\forall\, s \in [t/4, t)$ then $\forall\, x \geq g(t) + 1$

$$\mathcal{J}(t,x) \leq \sum_{\{i : X_i(t) \geq g(t)\}} M_i(t) \lesssim e^{-t/8} \cdot \#\{i : X_i(t) \geq g(t)\}$$

$$\approx e^{-t/8} \cdot e^{\sqrt{2}t^{1/3}} = o(1).$$

So $d(t) \leq g(t) + 1$ $\blacksquare$

Proof idea IV: Lower bound.

Strict fastpoke threshold:

$h(t)$ s.t. whp $\forall t \ \exists i$ s.t. $\forall s \leq t$, $X_{i,t}(s) \geq h(s)$.

If $D(s) < h(s)$ $\forall s < t$ then any fastpoke has mass $\geq e^{-t/2}$.

⊛ If $D(s, \frac{1}{2t}) < h(s)$ $\forall s < t$ then any fastpoke has mass $\geq 1/2$

Fact: Can take $h(t) = \sqrt{2}\, t - O(t^{1/3})$ and have ⊛ hold.

($\Rightarrow$) Either
- $\exists\, s < t$ s.t. $D(s, \frac{1}{2t}) \geq \sqrt{2}\, s - O(s^{1/3})$
- or, $D(t, \frac{1}{2}) \geq \sqrt{2}\, t - O(t^{1/3})$

Now use that density stabilizes quickly to go from $D(s, \frac{1}{2t})$ to $D(s + O(\log t), \frac{1}{2})$

Thank you!