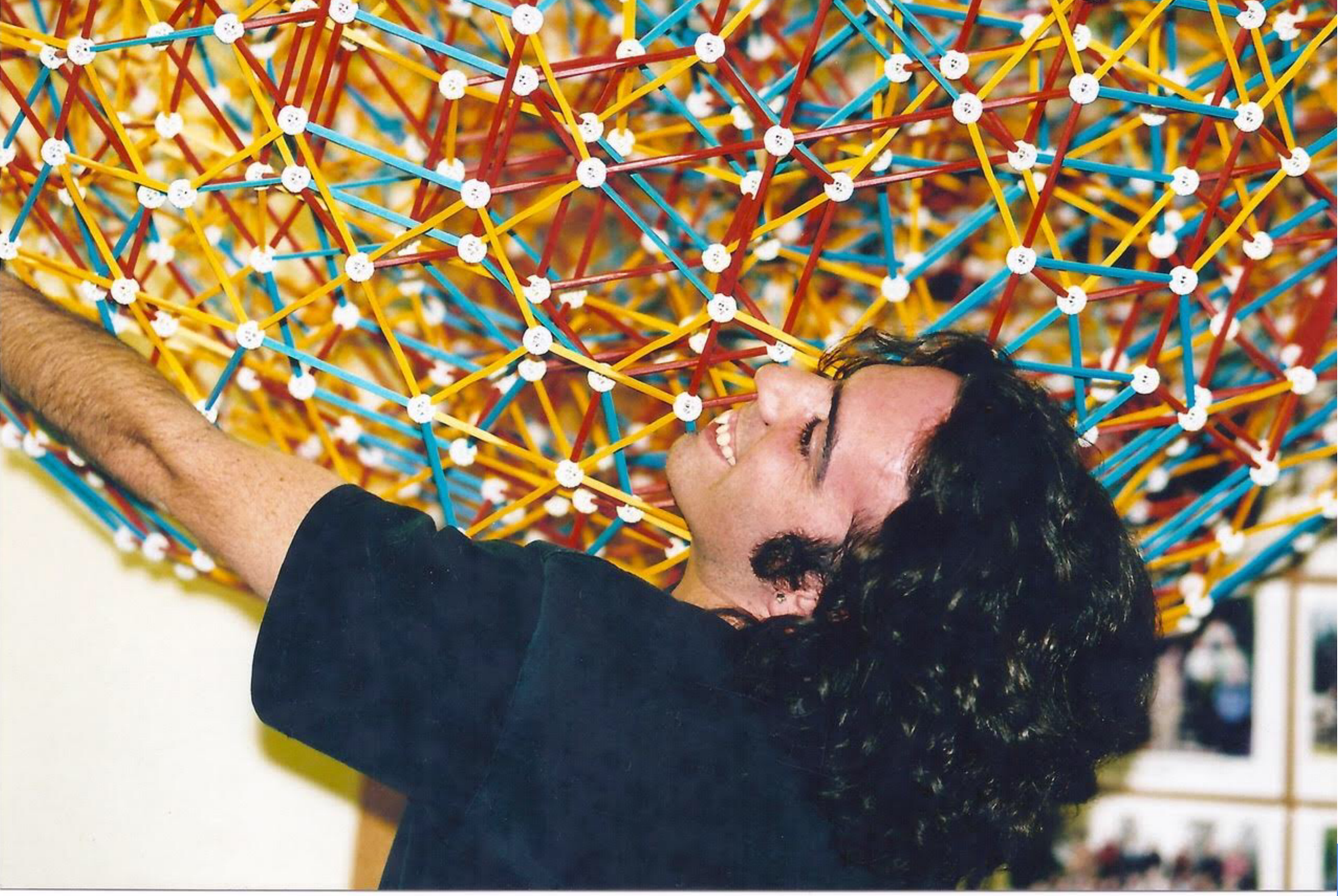


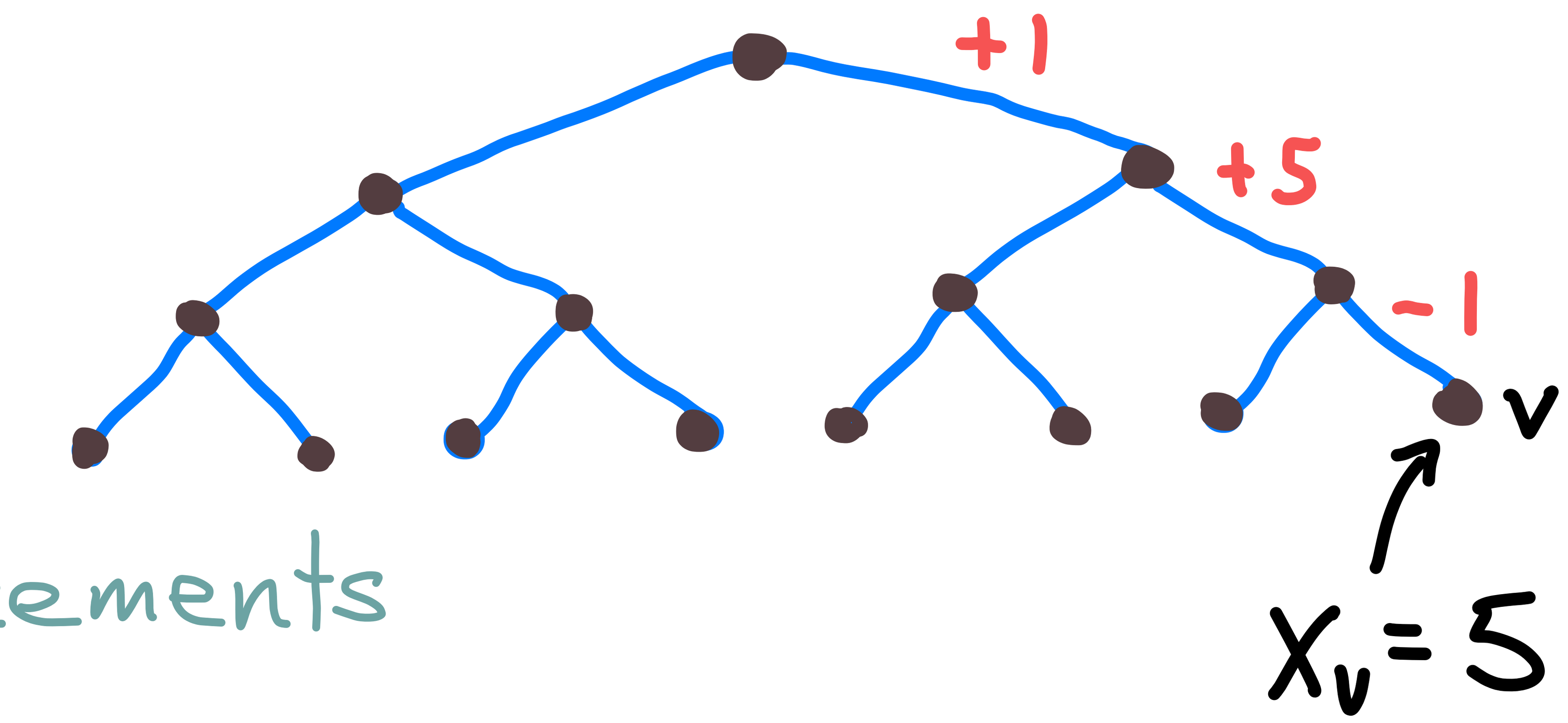
Mini-course part 2: The algorithmic hardness threshold for the continuous random energy model

Louigi Addario-Berry
McGill

Pascal Maillard
Paris Sud

12’th MSJ-SI
August 6, 2019

Branching random walk

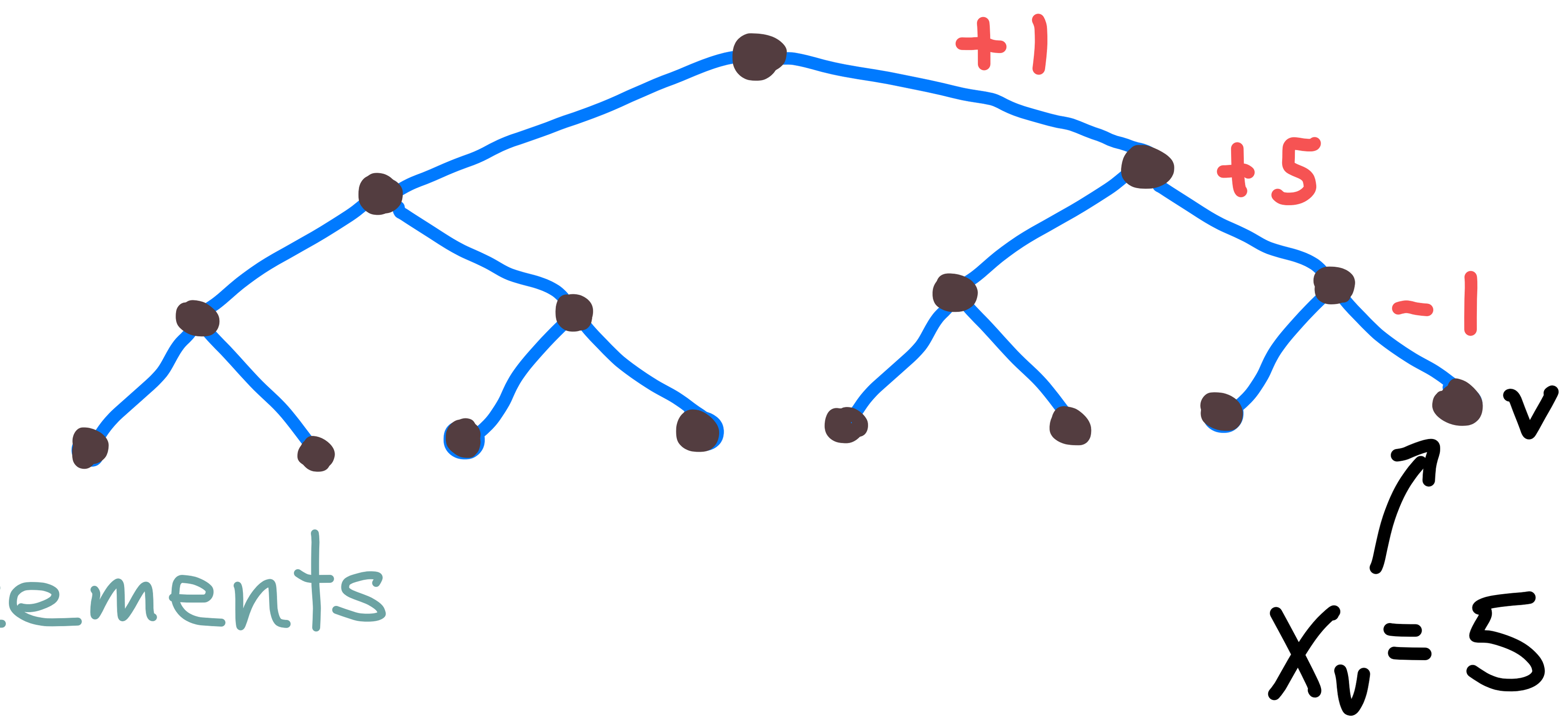


Generation n: T_n (assume binary so $|T_n| = 2^n$)

Position of node v: X_v = sum of ancestral displacements

For now assume displacements are IID with common law μ; later we will weaken this.

Branching random walk



Generation n: T_n (assume binary so $|T_n| = 2^n$)

Position of node v: X_v = sum of ancestral displacements

For now assume displacements are IID with common law μ; later we will weaken this.

Minimum position in generation n: $M_n := \min(X_v, v \in T_n)$

Fairly generic fact: $\exists\, c \in \mathbb{R}$ s.t. $n^{-1} \cdot M_n \xrightarrow{a.s.} c$

and moreover $\mathbb{E} M_n = (1+o(1))\, c\, n$

Hammersley
Kingman 1970's
Biggins

Proof idea a) lower bound.

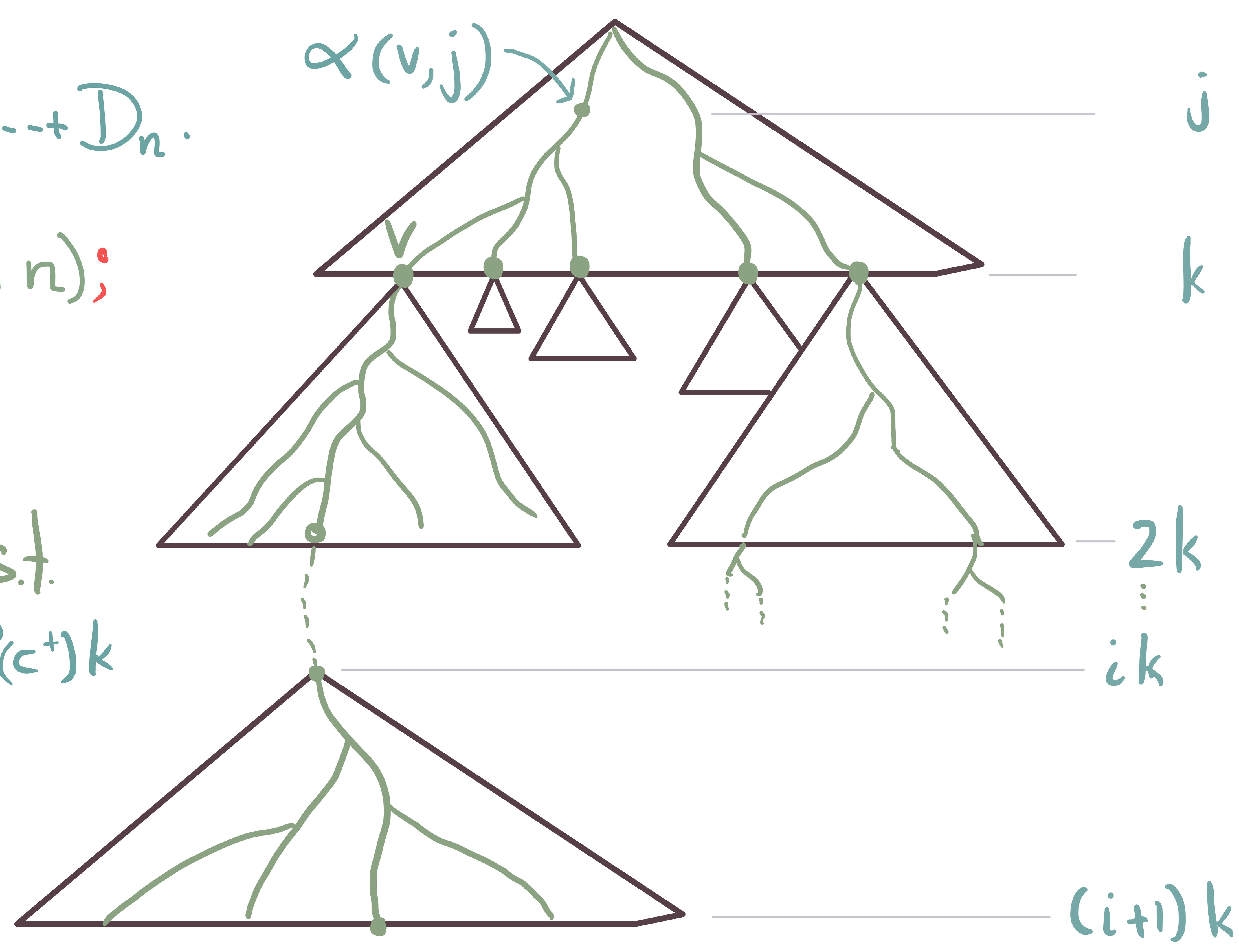
- Let $(D_i, i \geq 1)$ be IID $\sim \mu$, let $S_n = D_1 + \cdots + D_n$.
- For $x < \mathbb{E} D_1$, $\mathbb{P}(S_n \leq xn) = \exp(-(1+o(1)) \cdot f(x)\, n)$; $f(x)$ large dev. rate function.
- Take c so that $f(c) = \log(2)$; then $f(x) > 2$ for $x < c$.
- Then $\forall x < c$, $\mathbb{E}\, \#\{v \in T_n : X_v \leq xn\} = 2^n\, \mathbb{P}(S_n \leq xn) \leq 2^n e^{-(1+o(1)) f(x) n} = O(e^{-\delta n})$,
- Now use Borel-Cantelli. [some $\delta = \delta(x) > 0$]

Branching random walk

Let $(D_i, i \geq 1)$ be IID $\sim \mu$, let $S_n = D_1 + \cdots + D_n$.

For $x < \mathbb{E} D_1$, $\mathbb{P}(S_n \leq xn) = \exp(-(1+o(1)) \cdot f(x)\, n)$;

$f(x)$ large dev. rate function, $f(c) = \log 2$.



b) upper bound: Fix $c^+ > c$, then $\exists\, k \in \mathbb{N}$ s.t.

$$\mathbb{E}\, \#\{v \in T_k : \forall j < k,\ \underbrace{X_{\alpha(v,j)}}_{\text{gen. } j \text{ ancestor of } v} \leq c^+ k\} = 2^k e^{-(1+o(1)) f(c^+) k} > 1$$

Define a renormalized BRW

$T^{(1)} = \{v \in T_k : \forall j < k,\ X_{\alpha(v,j)} \leq c^+ k\}$ $\quad \mathbb{E}\, \# T^{(1)} > 1$.

$T^{(i+1)} = \{v \in T_{(i+1)k} : \forall j \in [ik, (i+1)k],\ X_{\alpha(v,j)} - X_{\alpha(v,ik)} \leq c^+ k\}$

Then $\hat{T} := (T^{(i)}, i \geq 1)$ is a supercritical branching process and on the event $\{\hat{T} \text{ survives}\}$, have $\limsup_{n\to\infty} n^{-1} M_n \leq c^+$.

So $\mathbb{P}\{\limsup_{n\to\infty} n^{-1} M_n \leq c^+\} \geq \mathbb{P}\{\hat{T} \text{ survives}\} =: p(c^+) > 0$.

Branching random walk

Generation n: T_n (assume binary so $|T_n| = 2^n$)

Position of node v: X_v

Minimum position in generation n: M_n

Fairly generic fact: $\exists\, c \in \mathbb{R}$ s.t. $n^{-1} M_n \xrightarrow{a.s.} c$

and moreover $\mathbb{E} M_n = (1+o(1))\, c n$

Fairly generic proof of a.s. conv.: a) lower bound:

$\forall c_- < c$, $\mathbb{E}\, \#\{v \in T_n : X_v \le c_- n\} = O(e^{-\delta n})$, some $\delta(c_-) > 0$

Then use Borel-Cantelli.

b) upper bound: Fix $c^+ > c$, then $\exists\, k \in \mathbb{N}$ s.t.

$\mathbb{E}\, \#\{v \in T_k : \forall j < k,\ X_{\alpha(v,j)} \le c^+ k\} > 1$

Define a renormalized BRW

$T^{(1)} = \{v \in T_k : \forall j < k,\ X_{\alpha(v,j)} \le c^+ k\}$ $\quad \mathbb{E}\, \# T^{(1)} > 1$.

$T^{(i+1)} = \{v \in T_{(i+1)k} : \forall j \in [ik, (i+1)k],\ X_{\alpha(v,j)} - X_{\alpha(v,ik)} \le c^+ k\}$

Then $\hat{T} := (T^{(i)}, i \ge 1)$ is a supercritical branching process

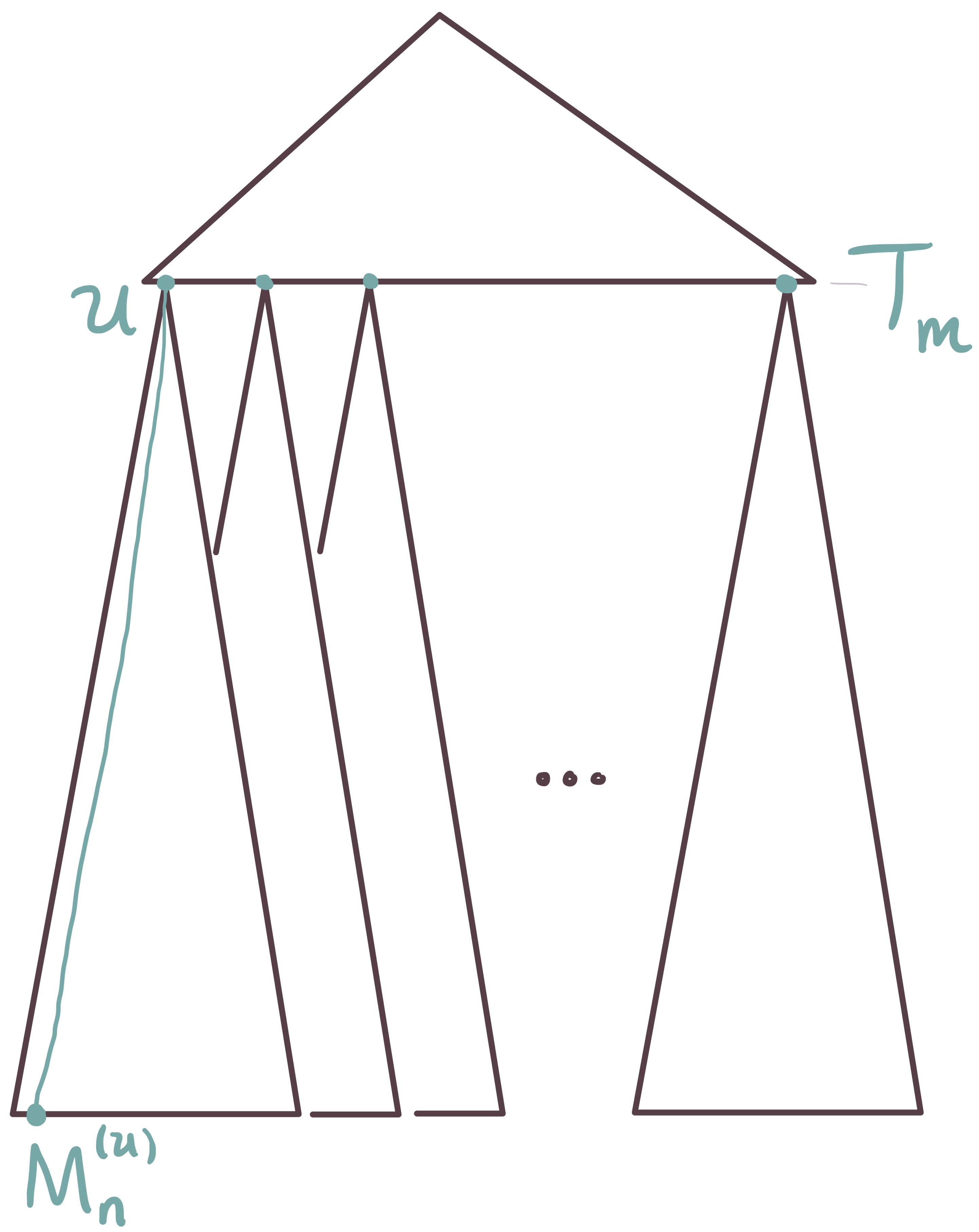
and on the event $\{\hat{T} \text{ survives}\}$, have $\limsup\limits_{n\to\infty} n^{-1} M_n \le c^+$.

So $\mathbb{P}\{\limsup\limits_{n\to\infty} n^{-1} M_n \le c^+\} \ge \mathbb{P}\{\hat{T} \text{ survives}\} =: p(c^+) > 0$.

Now amplify: fix m large, consider all subtrees rooted at $v \in T_m$

For any $u \in T_m$, by the branching property,

$$\mathbb{P}\{\limsup_{n\to\infty} n^{-1} \underbrace{M_n^{(u)}}_{\min\{X_v - X_u :\ v \in T_{n+m},\ \alpha(v,m) = u\}} \le c^+\} \ge p(c^+)$$



And $\forall u \in T_m$, $\limsup\limits_{n\to\infty} n^{-1} M_n^{(u)} \ge \limsup\limits_{n\to\infty} M_n$, so

$$\mathbb{P}\{\limsup_{n\to\infty} n^{-1} M_n \le c^+\} \ge \mathbb{P}\{\exists\, u \in T_m : \limsup_{n\to\infty} n^{-1} M_n^{(u)} \le c^+\}$$

$$\ge 1 - (1 - p(c^+))^{|T_m|}$$

$$\ge 1 - (1 - p(c^+))^{2^m}$$

Branching random walk

Generation n: T_n

Position of node v: X_v

Minimum position in generation n: M_n

Fairly generic fact: $\exists\, c \in \mathbb{R}$ s.t. $n^{-1} \cdot M_n \xrightarrow{a.s.} c$

and moreover $\mathbb{E} M_n = (1+o(1))\, c\, n$

Remark: In fair generality, c is also the expectation threshold, in that

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E}\, \#\{v \in T_n : X_v \leq xn\} \text{ is } \begin{cases} >0, & x > c \\ =0, & x = c \\ <0, & x < c \end{cases}$$

Finding near-minimal states

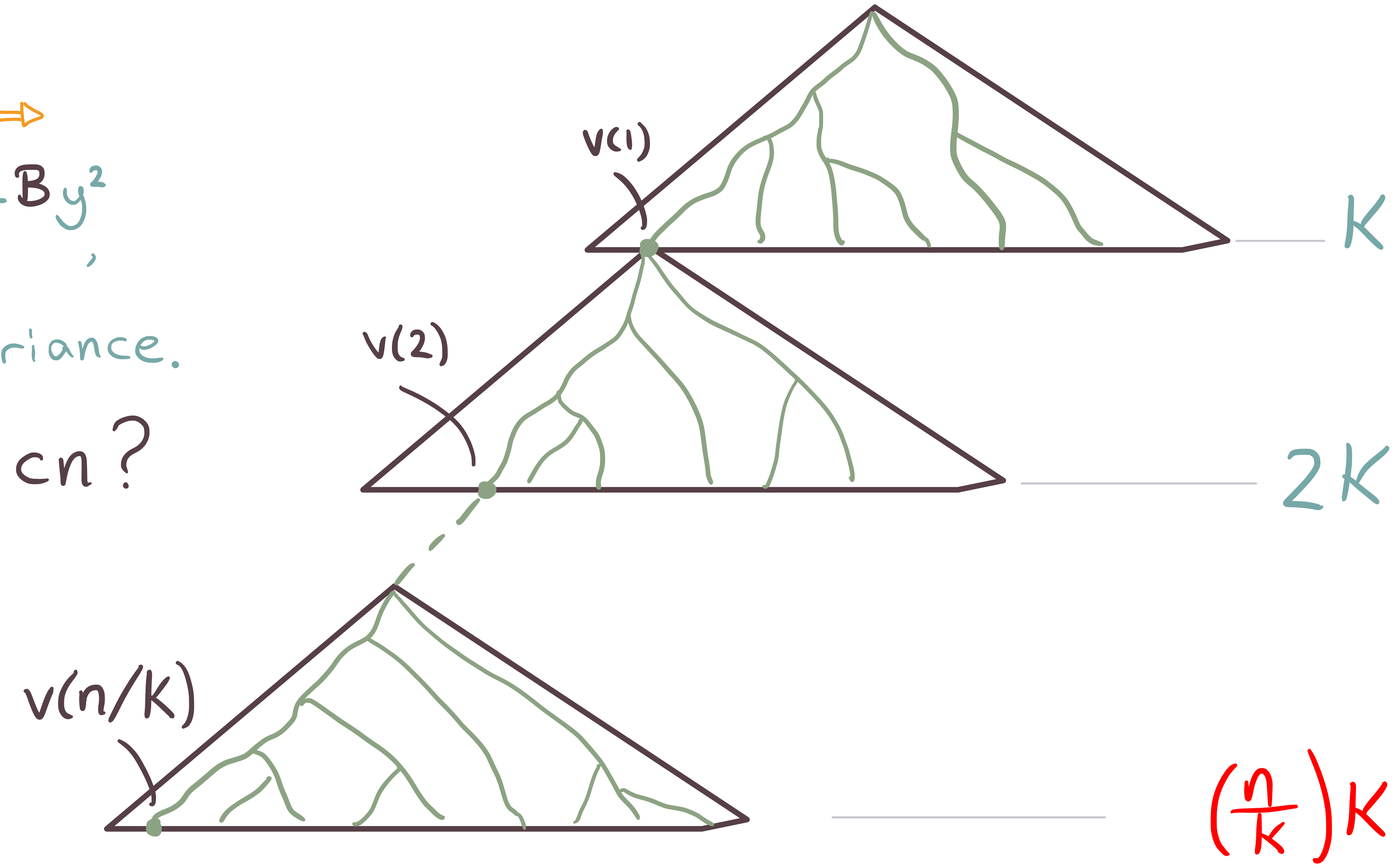
Hereafter assume Gaussian displacements: $\Rightarrow$

$$\exists b, B > 0 \text{ s.t. } \mathbb{P}(|X_v - X_{\text{parent}(v)}| > y) \leq b e^{-By^2},$$

for all nodes v. Write σ^2 for offspring variance.

How can one find nodes $v \in T_n$ with $X_v \approx cn$?

Bootstrap the law of large numbers:



- Given $\varepsilon > 0$, fix $K = K(\varepsilon)$ large enough that $\mathbb{E} M_K < (c+\varepsilon)K$
- Let $v(1) \in T_K$ have minimal position among depth-K nodes:
$$X_{v(1)} = M_K$$
- For $j = 2, \ldots, n/K$, let $v(j) \in T_{jK}$ have minimal position among depth-jK descendants of $v(j-1)$:
$$X_{v(j)} - X_{v(j-1)} = M_K^{(v(j-1))}$$

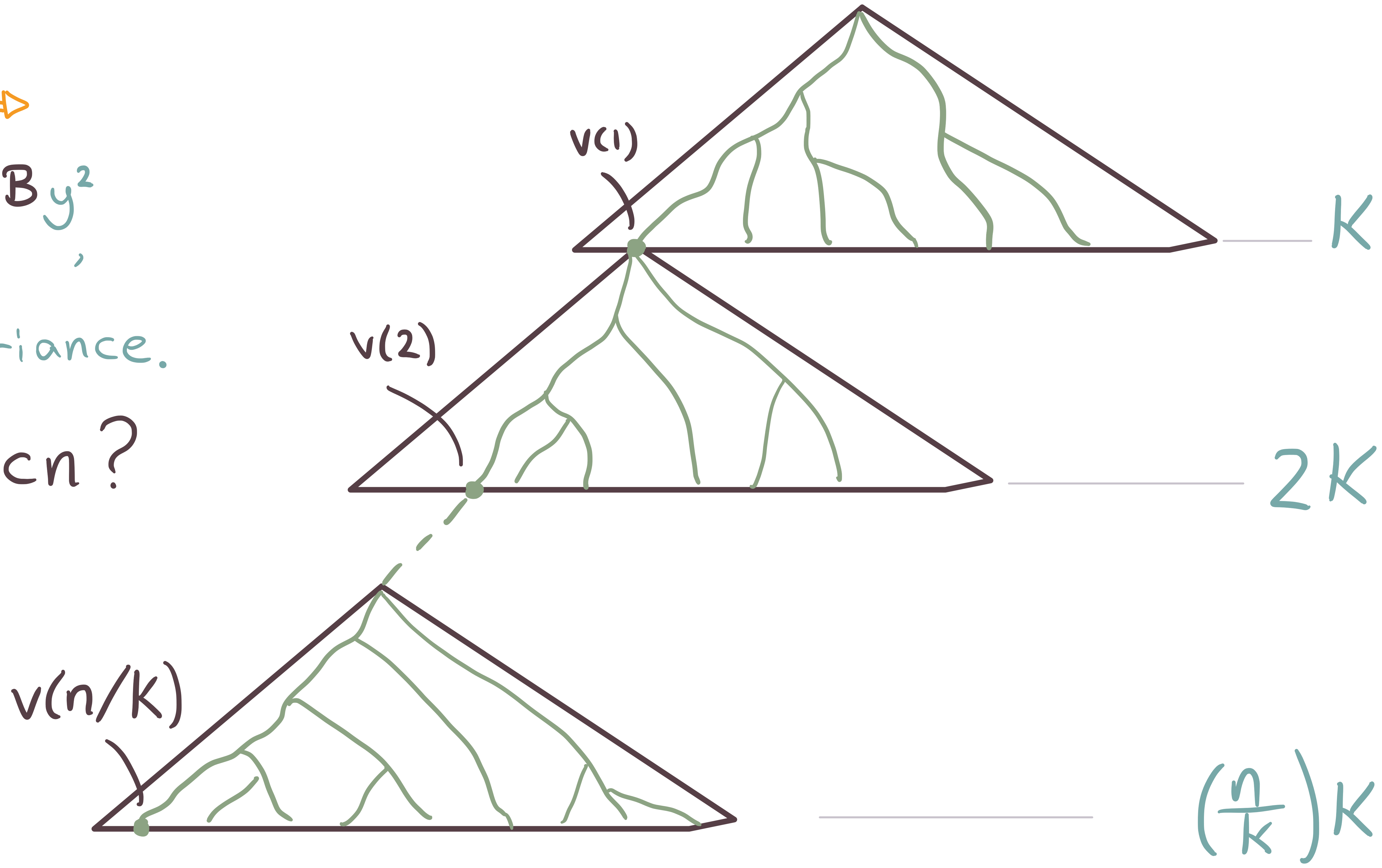
"K-level greedy search"

Finding near-minimal states

Hereafter assume Gaussian displacements: ⇒

$$\exists b, B > 0 \text{ s.t. } \mathbb{P}(|X_v - X_{\text{parent}(v)}| > y) \le b e^{-By^2},$$

for all nodes v. Write σ^2 for offspring variance.



How can one find nodes $v \in T_n$ with $X_v \approx cn$?

Bootstrap the law of large numbers:

- Given $\varepsilon > 0$, fix $K = K(\varepsilon)$ large enough that $\mathbb{E} M_K < (c+\varepsilon) K$
- Let $v(1) \in T_K$ have minimal position among depth-K nodes:
  $$X_{v(1)} = M_K$$
- For $j = 2, \dots, n/K$, let $v(j) \in T_{jK}$ have minimal position among depth-jK descendants of $v(j-1)$:
  $$X_{v(j)} - X_{v(j-1)} = M_K^{(v(j-1))}$$
- Then $\mathbb{E} X_{v(n/K)} = \frac{n}{K} \mathbb{E} X_{v(1)} < (c+\varepsilon) n$

Claim $\operatorname{Var} X_{v(n/K)} \le n \cdot 2^K \sigma^2.$

Proof: By the branching property,

$$\operatorname{Var} X_{v(n/K)} = \frac{n}{K} \operatorname{Var}(X_{v(1)}),$$

and

$$\operatorname{Var} X_{v(1)} = \operatorname{Var} \min\{X_v : v \in T_K\} \le \sum_{v \in T_K} \operatorname{Var} X_v = \sum_{v \in T_K} K \sigma^2 = 2^K \cdot K \sigma^2. \quad \square$$

$$\boxed{\text{Follows that } X_{v(n/K)} \le (c+2\varepsilon) n \text{ with probability } (1-o(1)) \text{ as } n \to \infty.}$$

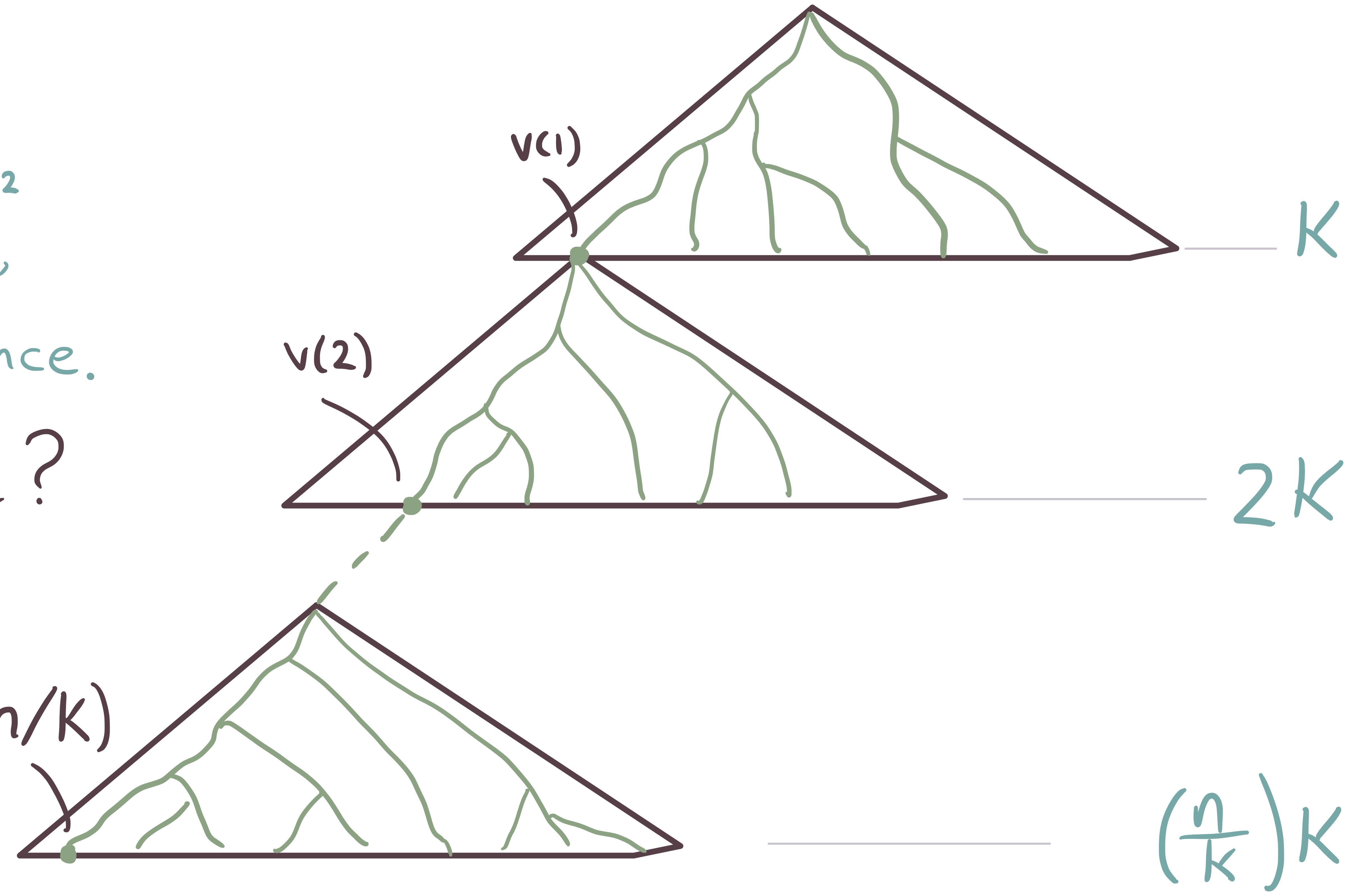
"K-level greedy search finds near-minimal nodes with high prob."

Finding near-minimal states

Hereafter assume Gaussian displacements: ⇒

$$\exists b, B > 0 \text{ s.t. } \mathbb{P}(|X_v - X_{\text{parent}(v)}| > y) \le b e^{-By^2},$$

for all nodes v. Write σ^2 for offspring variance.



How can one find nodes $v \in T_n$ with $X_v \approx cn$?

Bootstrap the law of large numbers:

- Given $\varepsilon > 0$, fix $K = K(\varepsilon)$ large enough that $\mathbb{E} M_K < (c+\varepsilon)K$
- Let $v(1) \in T_K$ have minimal position among depth-K nodes:
  $$X_{v(1)} = M_K$$
- For $j = 2, \ldots, n/K$, let $v(j) \in T_{jK}$ have minimal position among depth-jK descendants of $v(j-1)$:
  $$X_{v(j)} - X_{v(j-1)} = M_K^{(v(j-1))}$$
- Then $\mathbb{E} X_{v(n/K)} = {}^{n}\!/_{K}\, \mathbb{E} X_{v(1)} < (c+\varepsilon)n$

$\boxed{\text{Claim}}$ $\operatorname{Var} X_{v(n/K)} \le n \cdot 2^K \sigma^2.$

Proof: By the branching property,

$$\operatorname{Var} X_{v(n/K)} = {}^{n}\!/_{K} \operatorname{Var}(X_{v(1)}),$$

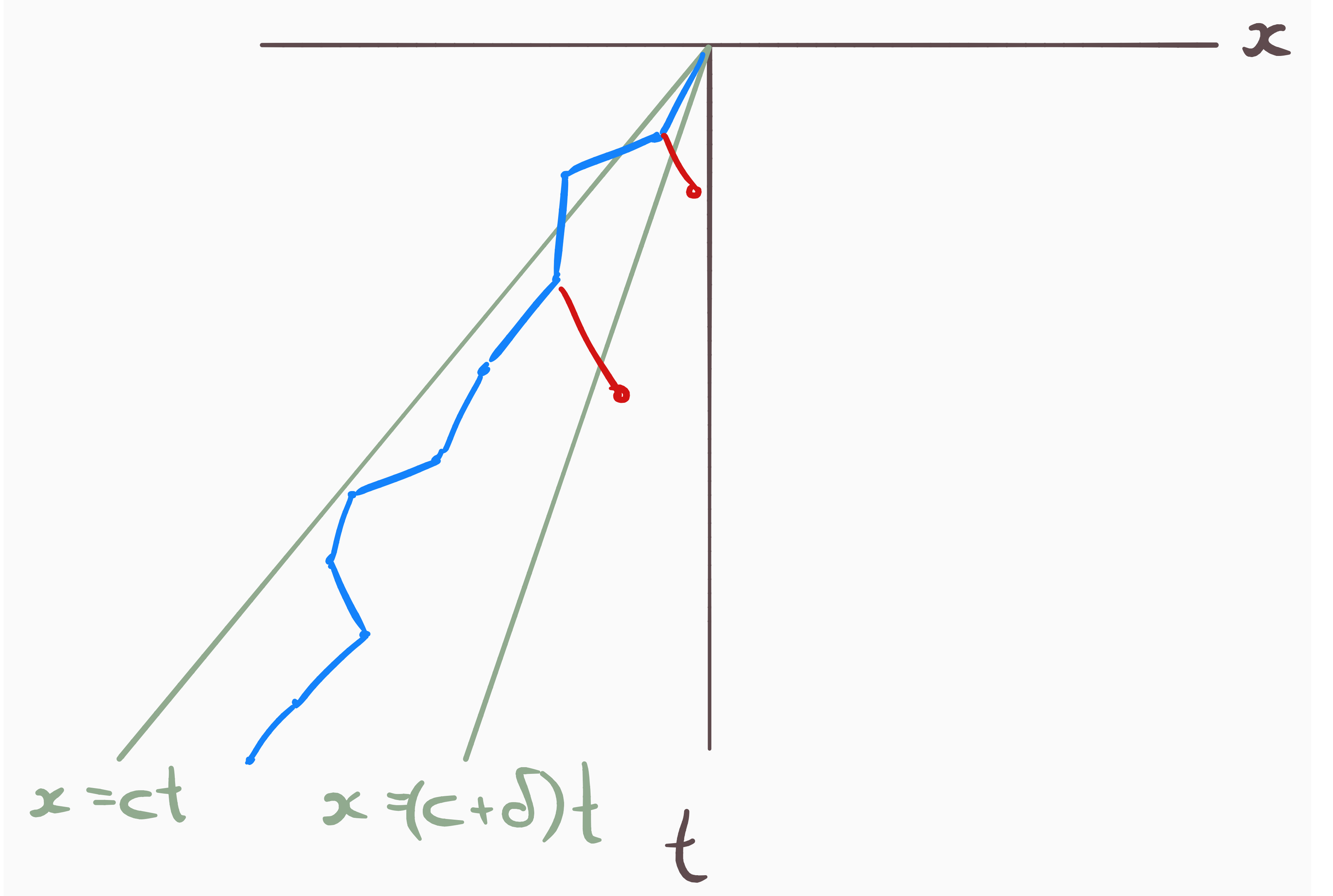
and

$$\begin{aligned} \operatorname{Var} X_{v(1)} &= \operatorname{Var} \min\{X_v : v \in T_K\} \\ &\le \sum_{v \in T_K} \operatorname{Var} X_v \\ &= \sum_{v \in T_K} K\sigma^2 = 2^K \cdot K\sigma^2. \quad \blacksquare \end{aligned}$$

Follows that $X_{v(n/K)} \le (c+2\varepsilon)n$ with probability $(1-o(1))$ as $n \to \infty$.

\# node value queries $= 2^K \cdot n/K = O_\varepsilon(n)$: linear-time algorithm.

Pemantle (2009) Ber(p) steps, $p > \frac{1}{2}$. $\frac{M_n}{n} \longrightarrow c = c(p)$



Write $\rho(\delta, k) := \mathbb{P}(\exists v \in T_k : \forall j \le k,\ X_{a(v,j)} \le (c+\delta)n)$

Theorem (Pemantle 2009) Fix $s > 1$. For any algorithm, for all n suff. large, all ε suff. small with $\varepsilon \gg \frac{1}{n^{2/3}}$, the probability of finding a node $v \in T_n$ with $X_v \le (c+\varepsilon)n$ by using at most $\frac{(s-1)n}{4(1-c)} \cdot \frac{\varepsilon^{1/2}}{\rho(s\varepsilon, \frac{1}{\varepsilon^{3/2}})}$ queries, is $O(\frac{1}{\varepsilon n})$

Note Writing $k = \frac{1}{\varepsilon^{3/2}}$ then $\rho(s\varepsilon, \frac{1}{\varepsilon^{3/2}}) = \mathbb{P}(\exists v \in T_k : \forall j \le k,\ X_{\alpha(v,j)} \le ck + sk^{1/3})$

Corollary (Gantert, Hu, Shi 2011) For $n^{-2/3} \ll \varepsilon \ll 1$, prob. of finding $v \in T_n$ with $X_v \le (c+\varepsilon)n$ in $n \cdot \exp(o((\frac{1}{\varepsilon})^{1/2}))$ steps is $O(\frac{1}{\varepsilon n})$.

Remarks

- For $\varepsilon > 0$ small, fixed, Pemantle proves that an "iterated depth-first search" finds $v \in T_n$ with $X_v \le (c+\varepsilon)n$ in $n \cdot \exp(O((\frac{1}{\varepsilon})^{1/2}))$ with high prob.
- Say a search algorithm does not jump if its set of queried vertices is always connected.

  Conj (Pemantle 2009) For any $k \in \mathbb{N}$, no alg. finds a path with $\ge k$ zeroes in shorter average time than the best alg. which does not jump.

Algorithmically finding extreme values: some examples.

NB: Natural to identify T_n with the hypercube $\{-1,1\}^n$.

Branching random walk: can find $v \in T_n$ with $X_v = (1 \pm \varepsilon) M_n$ in $O_\varepsilon(n)$ time.

Random energy model (R.E.M) $(X_v, v \in \{-1,1\}^n)$ IID $N(0,n)$. Then

- $M_n := \min(X_v, v \in \{-1,1\}^n) = -(1+o_p(1))\sqrt{2\log 2}\, n$
- To find X_v with $|X_v| \geq cn$ requires $\exp\left(\frac{c^2}{2} \cdot n\right)$ queries.
- Perhaps surprising that (small) correlations of BRW change algorithmic complexity so greatly (relative to REM)

Random signed sum $G = (G_1, \ldots, G_n)$ IID $N(0,1)$; for $\sigma \in \{-1,1\}^n$ let $H(\sigma) = \sum_{i=1}^n \sigma(i) G_i = \langle \sigma, G \rangle$

By querying $(\sigma_1, \ldots, \sigma_n)$ and $(\sigma_1, \ldots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \ldots, \sigma_n)$ can determine $\text{sign}(G_i)$; so $2n$ queries suffice to find σ minimizing $H(\sigma)$.

Algorithmically finding extreme values: some examples.

$J = (J_{ij}, 1 \le i,j \le n)$ IID $N(0, \frac{1}{n})$

Sherrington-Kirkpatrick: $H(\sigma) = \langle \sigma, J\sigma \rangle = \sum_{i=1}^{n} J_{ij}\sigma(i)\sigma(j)$

$M_n = \min(X_\sigma, \sigma \in \{-1,1\}^n)$

Theorem (Montanari 2018) Conditional on the truth of the SK replica overlap conjecture, for all $\varepsilon > 0$, with high probability can find $\sigma \in \{-1,1\}^n$ with $H(\sigma) = (1 \pm \varepsilon) M_n$ in $O_\varepsilon(n^2)$ running time.

Spherical SK $M_n = \min(H(\sigma), \sigma \in \mathbb{R}^n, |\sigma| = n^{1/2})$ ⊛

This is just min. eigenvalue of J; can be efficiently computed.

Spherical p-spin $J = (J_{i_1,\ldots,i_p} : (i_1,\ldots,i_p) \in \{1,2,\ldots,n\}^p)$ IID $N(0, \frac{1}{n^{p-1}})$

$H(\sigma) = \sum_{i_1,\ldots,i_p=1}^{n} J_{i_1,\ldots,i_p} \cdot \sigma(i_1) \cdot \ldots \cdot \sigma(i_p)$ M_n as in ⊛

UNKNOWN WHAT CAN BE ACHIEVED ALGORITHMICALLY

CREM and its minima

Setting: continuous random energy model (CREM)

$\mathrm{CREM}(A,n)$

A: Cumulative dist. f$^{\underline{n}}$ of a finite measure on $[0,1]$; so $A(0)=0$, $A(1)\in(0,\infty)$.

n: number of levels.

Gaussian process $(X_v, v\in\mathbb{T}_n)$ indexed by

$\mathbb{T}_n := T_1 \cup T_2 \cup \cdots \cup T_n$

Displacement laws:

If $v\in T_k$ then $X_v - X_{\mathrm{parent}(v)}$ is

$$\mathcal{N}\left(0, n\left(A\left(\tfrac{k}{n}\right)-A\left(\tfrac{k-1}{n}\right)\right)\right).$$

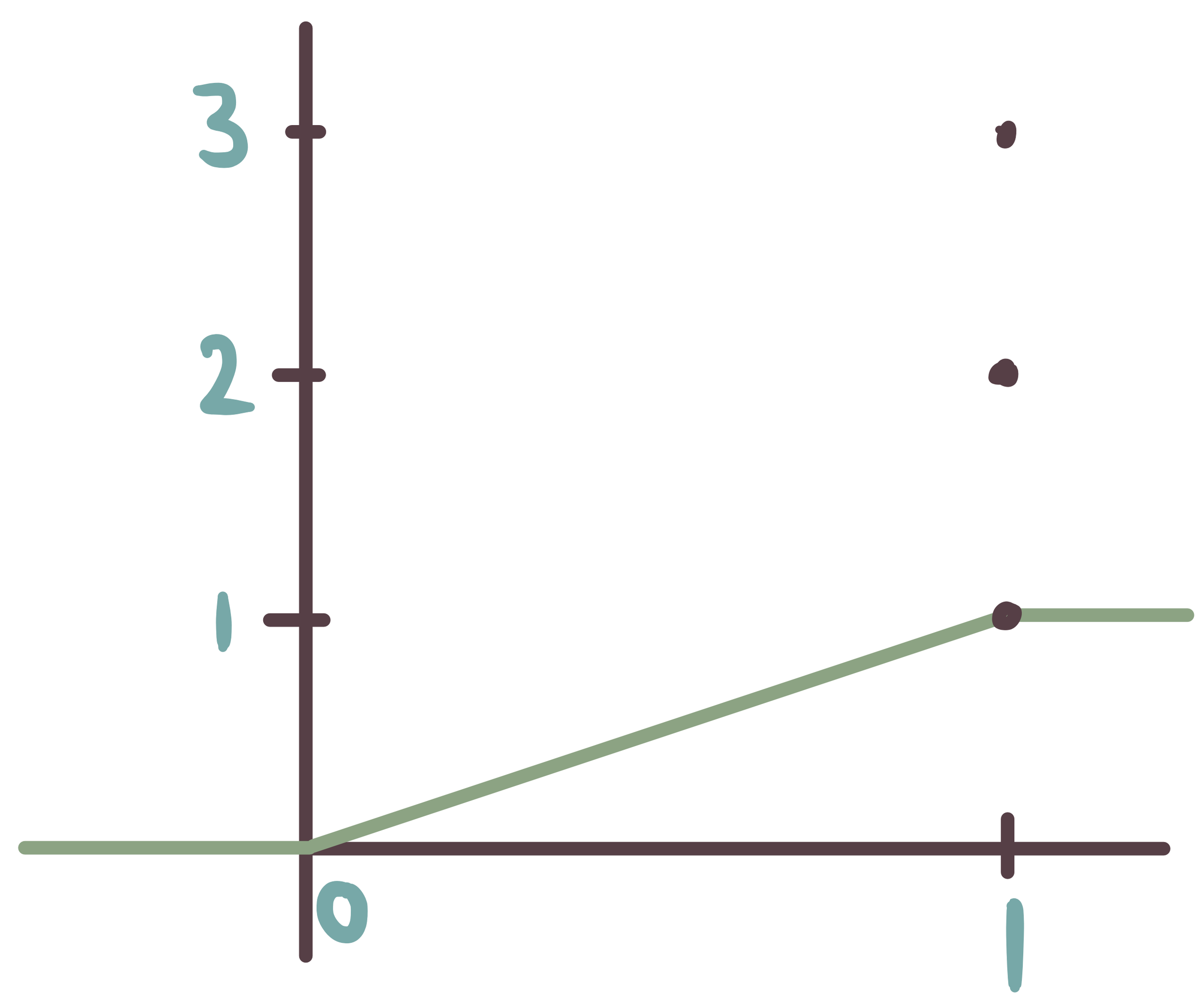
Displacements mutually independent.

Idea: along any root-to-leaf path, observe an inhomogeneous Brownian motion whose infinitesimal variance at time $\approx zn$ is $A'(z)$.

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z) = \begin{cases} 0, & z \leq 0 \\ z, & z \in (0,1) \\ 1, & z \geq 1 \end{cases}$$



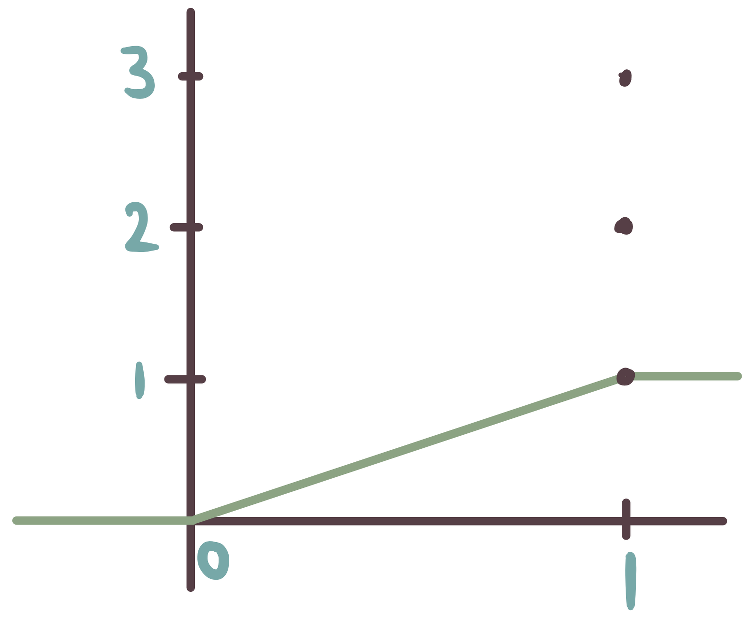
$$\mathbb{E}\,\#\{v \in T_n : X_v \leq -xn\} \approx 2^n \exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(2 \log 2)^{1/2}$$

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z) = \begin{cases} 0, & z \le 0 \\ z, & z \in (0,1) \\ 1, & z \ge 1 \end{cases}$$

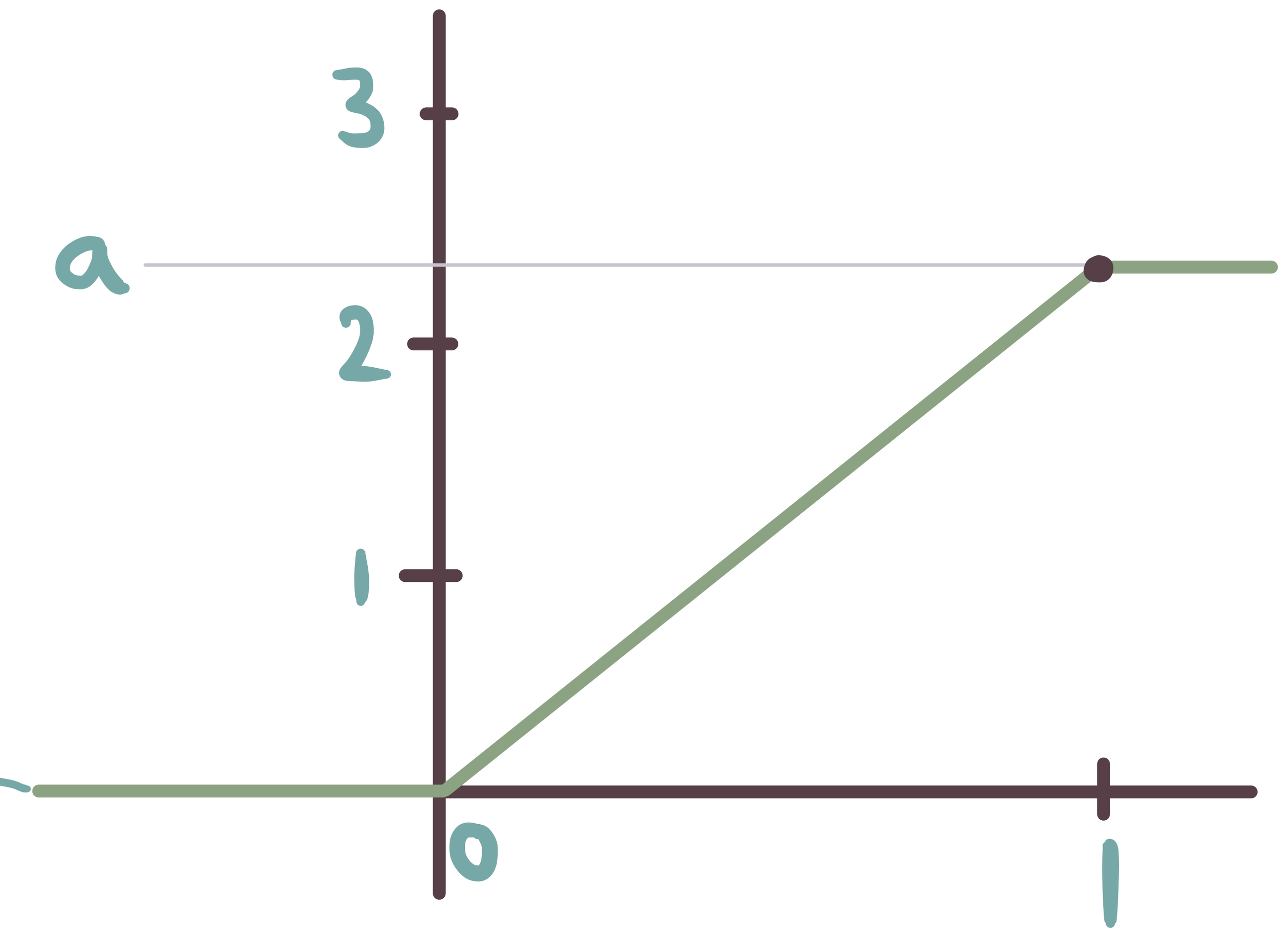


$$\mathbb{E}\,\#\{v \in T_n : X_v \le -xn\} \approx 2^n \exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(2\log 2)^{\frac{1}{2}}$$

Speed-a Gaussian BRW

$$A(z) = \begin{cases} 0, & z \le 0 \\ az, & z \in (0,1) \\ a, & z \ge 1 \end{cases}$$



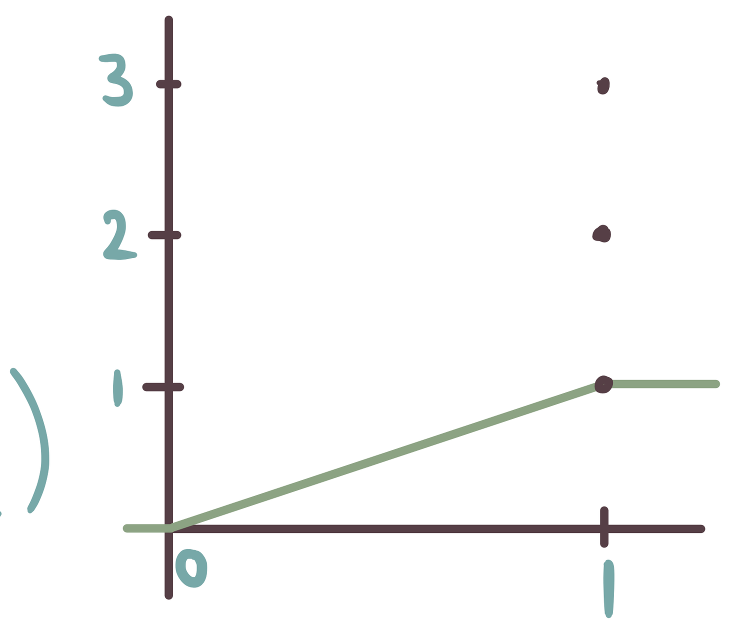
$$\mathbb{E}\,\#\{v \in T_n : X_v \le -xn\} \approx 2^n \exp\left(-\frac{x^2}{a \cdot 2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(a \cdot 2\log 2)^{\frac{1}{2}}$$

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z) = \begin{cases} 0, & z \le 0 \\ z, & z \in (0,1) \\ 1, & z \ge 1 \end{cases}$$

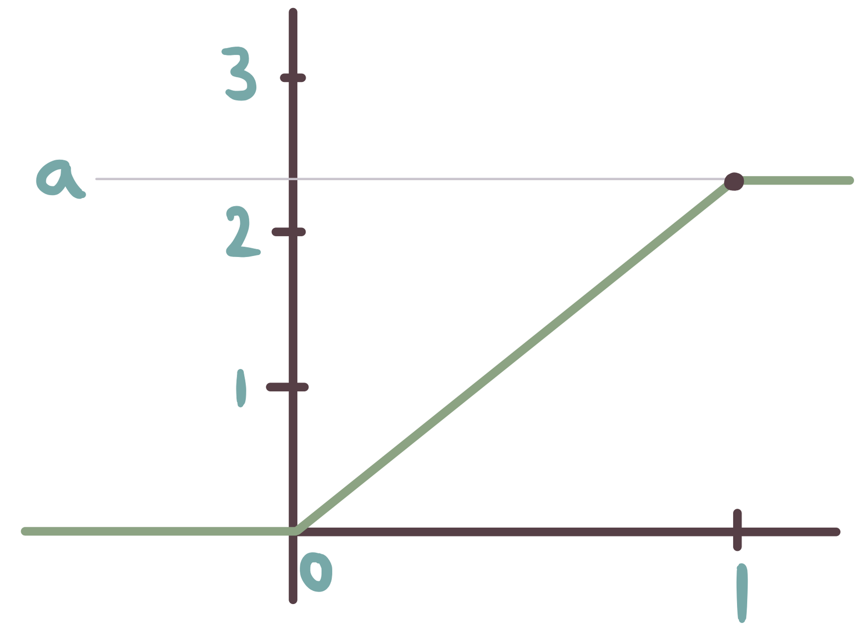


$$\mathbb{E}\,\#\{v \in T_n : X_v \le -xn\} \approx 2^n \exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(2 \log 2)^{\frac{1}{2}}$$

Speed-a Gaussian BRW

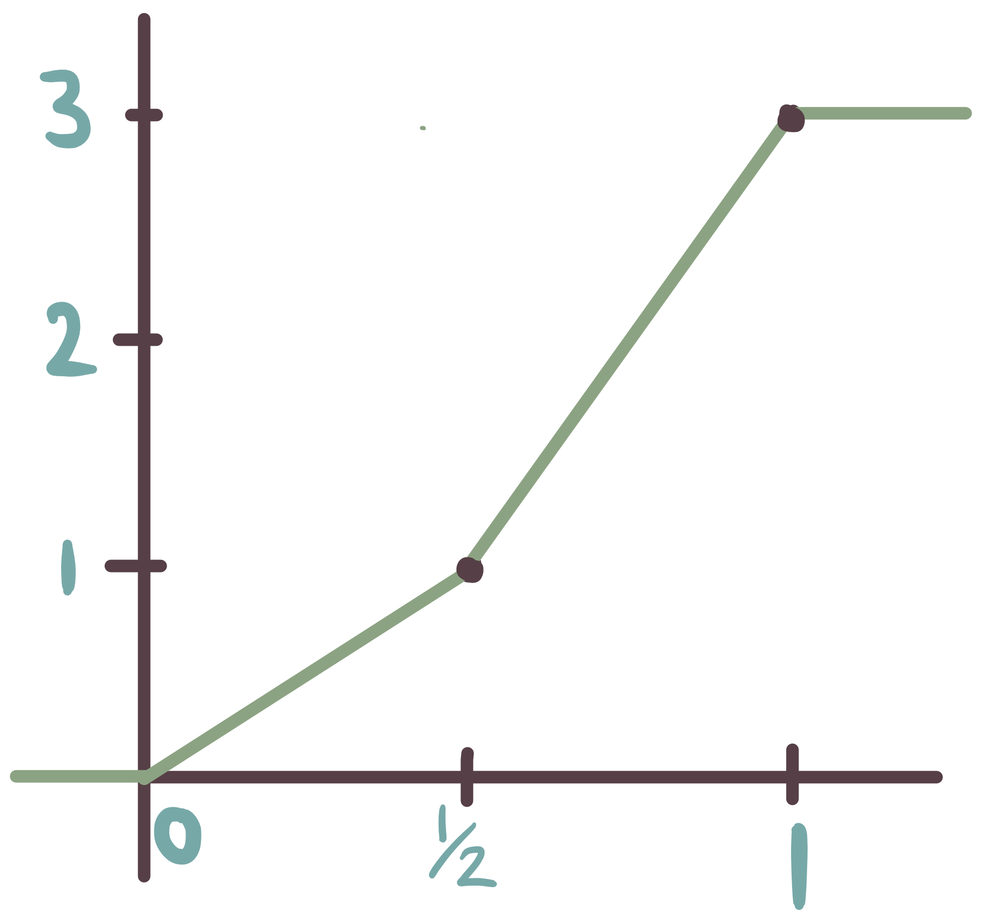
$$A(z) = \begin{cases} 0, & z \le 0 \\ az, & z \in (0,1) \\ a, & z \ge 1 \end{cases}$$



$$n^{-1} M_n \xrightarrow{a.s.} -(a \cdot 2 \log 2)^{\frac{1}{2}}$$

Two-speed convex Gaussian BRW

$$A(z) = \begin{cases} 0, & z \le 0 \\ 2z & z \in (0,\frac{1}{2}) \\ 1+4z, & z \in (\frac{1}{2},1) \\ 3, & z \ge 1 \end{cases}$$



First half

$$\mathbb{E}\,\#\{v \in T_{n/2} : X_v \le -x\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{x^2}{2 \cdot 2(n/2)}\right)$$

$$= 2^{n/2} \exp\left(-\frac{x^2}{2n}\right)$$

Second half

$$\mathbb{E}\,\#\{v \in T_n : X_v - X_{a(v,n/2)} \le -y\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{y^2}{4n}\right)$$

$$\mathbb{E}\,\#\{v \in T_n : X_{a(v,n/2)} \le -\left(\frac{2 \cdot \log 2}{3}\right)^{\frac{1}{2}} n,$$

$$X_v - X_{a(v,n/2)} \le -2\left(\frac{2 \log 2}{3}\right)^{\frac{1}{2}} n\}$$

$$\asymp 2^{n/2} \exp\left(-\frac{\log 2}{3} n\right) \cdot 2^{n/2} \exp\left(-\frac{2 \log 2}{3} n\right)$$

$$\asymp 1$$

$$n^{-1} M_n \longrightarrow -\left(\frac{2 \cdot \log 2}{3}\right)^{\frac{1}{2}} n - 2\left(\frac{2 \log 2}{3}\right)^{\frac{1}{2}} n$$

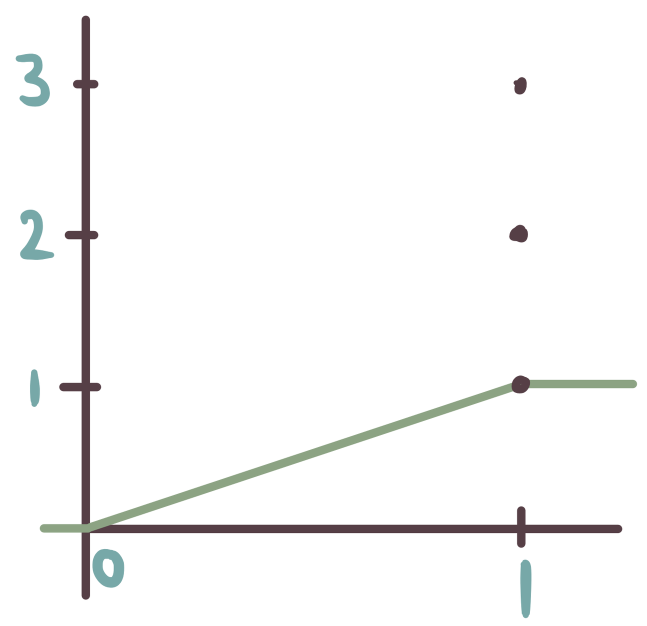
$$= -(3 \cdot 2 \log 2)^{\frac{1}{2}}$$

Same as if $A(x) = 3x$ on $[0,1]$

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z)=\begin{cases}0, & z\le 0\\ z, & z\in(0,1)\\ 1, & z\ge 1\end{cases}$$

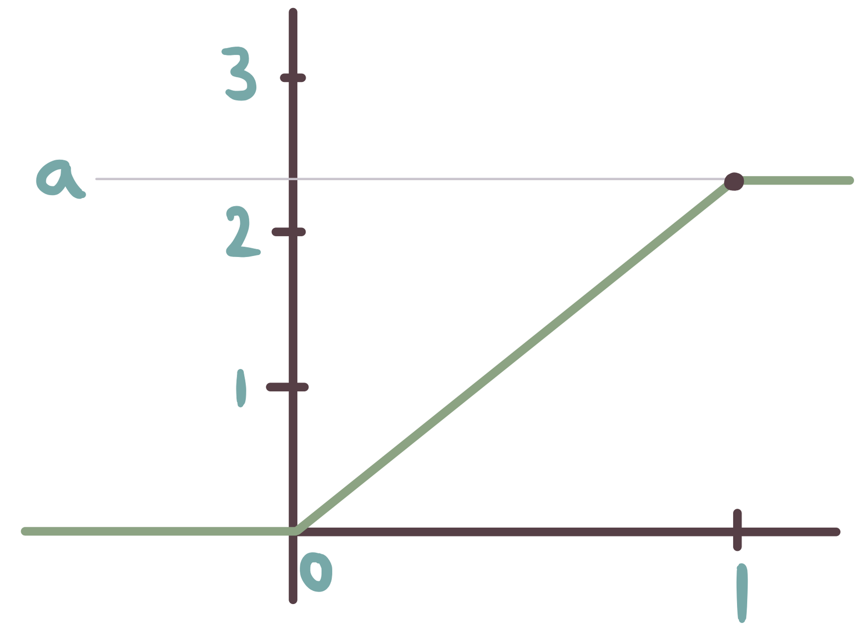


$$\mathbb{E}\,\#\{v\in T_n : X_v\le -xn\}\approx 2^n\exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1}M_n \xrightarrow{a.s.} -(2\log 2)^{\frac12}$$

Speed-a Gaussian BRW

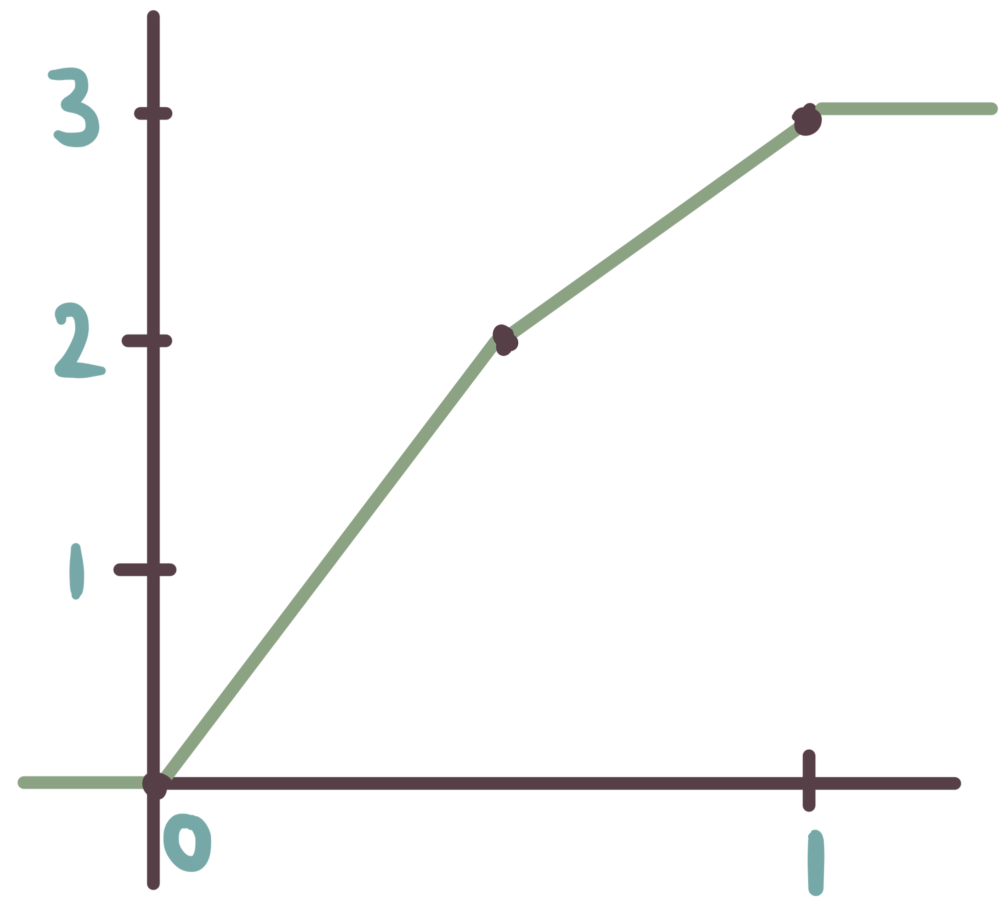
$$A(z)=\begin{cases}0, & z\le 0\\ az, & z\in(0,1)\\ a, & z\ge 1\end{cases}$$



$$n^{-1}M_n \xrightarrow{a.s.} -(a\cdot 2\log 2)^{\frac12}$$

Two-speed concave Gaussian BRW.

$$A(z)=\begin{cases}0, & z\le 0\\ 4z, & z\in(0,\tfrac12)\\ 2+2z, & z\in(\tfrac12,1)\\ 3, & z\ge 1\end{cases}$$



$$\mathbb{E}\,\#\{v\in T_{n/2} : X_v\le -x\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{x^2}{4n}\right)$$

$$\mathbb{E}\,\#\{v\in T_n : X_v - X_{a(v,n/2)}\le -y\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{y^2}{2\cdot n}\right)$$

$$\mathbb{E}\,\#\Big\{v\in T_n : X_{a(v,n/2)}\le -2\left(\tfrac{2\cdot\log 2}{3}\right)^{\frac12}n,\ X_v - X_{a(v,n/2)}\le -\left(\tfrac{2\log 2}{3}\right)^{\frac12}n\Big\}$$

$$\approx 2^{n/2}\exp\left(-\frac{2\log 2}{3}n\right)\cdot 2^{n/2}\exp\left(-\frac{\log 2}{3}n\right)$$

$$\approx 1$$

But $n^{-1}M_n \not\to -(3\cdot 2\log 2)^{\frac12}$, because

$$\mathbb{E}\,\#\Big\{v\in T_n : X_{a(v,n/2)}\le -2\left(\tfrac{2\cdot\log 2}{3}\right)^{\frac12}n\Big\}$$

$$\approx 2^{n/2}\exp\left(-\frac{2\log 2}{3}n\right)\approx 2^{-n/6};$$

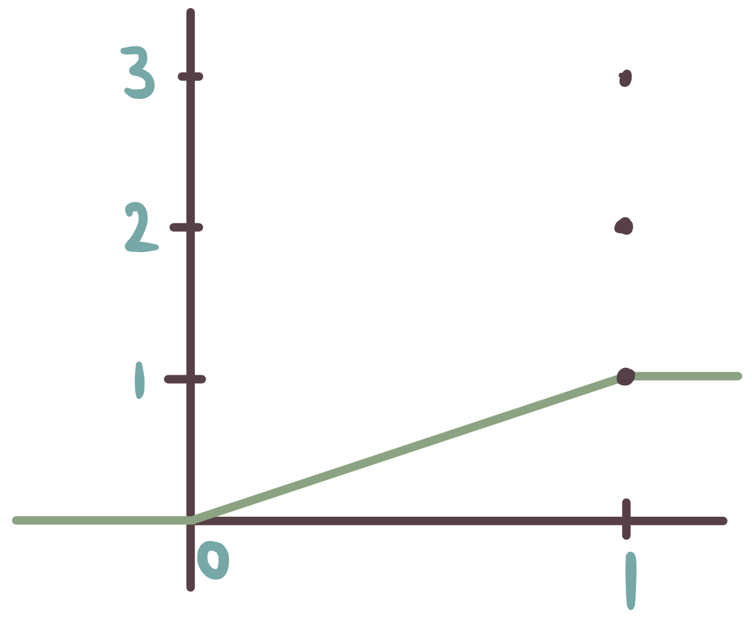
the needed trajectories do not exist.

In fact, here $n^{-1}M_n \to (\sqrt{2}+1)\log 2$.

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z) = \begin{cases} 0, & z \le 0 \\ z, & z \in (0,1) \\ 1, & z \ge 1 \end{cases}$$

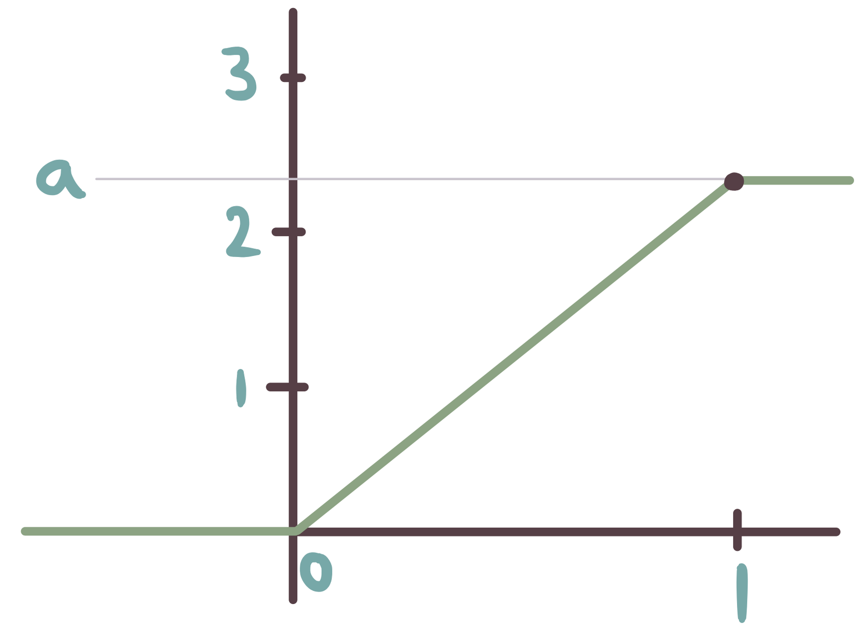


$$\mathbb{E}\,\#\{v \in T_n : X_v \le -xn\} \approx 2^n \exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(2\log 2)^{\frac12}$$

Speed-a Gaussian BRW

$$A(z) = \begin{cases} 0, & z \le 0 \\ a, & z \in (0,1) \\ a, & z \ge 1 \end{cases}$$

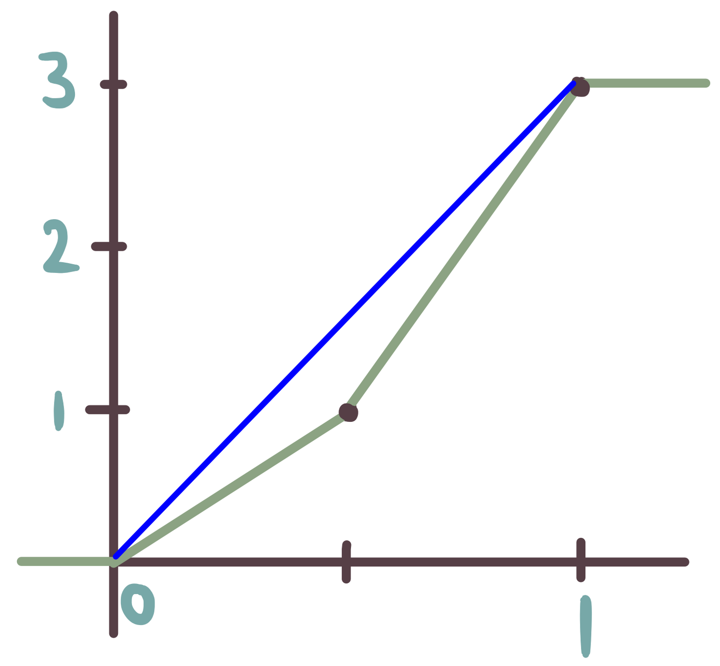


$$\mathbb{E}\,\#\{v \in T_n : X_v \le -xn\} \approx 2^n \exp\left(-\frac{x^2}{a \cdot 2n}\right)$$

$$n^{-1} M_n \xrightarrow{a.s.} -(a \cdot 2\log 2)^{\frac12}$$

Two-speed concave Gaussian BRW

$$A(z) = \begin{cases} 0, & z \le 0 \\ 2z & z \in (0,\frac12) \\ 1+4z, & z \in (\frac12,1) \\ 3, & z \ge 1 \end{cases}$$



$$\mathbb{E}\,\#\{v \in T_{n/2} : X_v \le -x\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{x^2}{2 \cdot n}\right) \quad [2n = 2 \cdot 2 \cdot n/2]$$

$$\mathbb{E}\,\#\{v \in T_n : X_v - X_{a(v,n/2)} \le -y\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{y^2}{4n}\right)$$

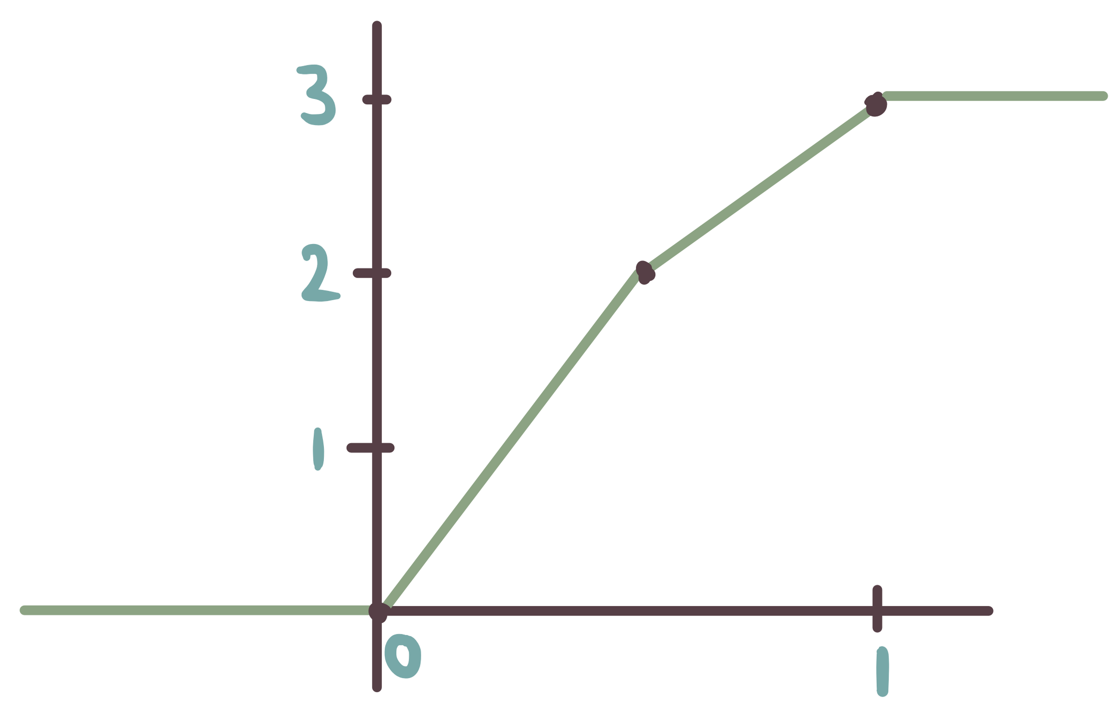
$$\mathbb{E}\,\#\left\{v \in T_n : X_{a(v,n/2)} \le -\left(\frac{2\cdot\log 2}{3}\right)^{\frac12} n,\ X_v - X_{a(v,n/2)} \le -2\left(\frac{2\log 2}{3}\right)^{\frac12} n\right\}$$

$$\asymp 2^n \exp\left(-\frac{\log 2}{3} n - \frac{2\log 2}{3} n\right) \asymp 1$$

$$n^{-1} M_n \to -\left(\frac{2\cdot\log 2}{3}\right)^{\frac12} n - 2\left(\frac{2\log 2}{3}\right)^{\frac12} n = -(3 \cdot 2\log 2)^{\frac12}$$

Two-speed convex Gaussian BRW.

$$A(z) = \begin{cases} 0, & z \le 0 \\ 4z, & z \in (0,\frac12) \\ 2+2z, & z \in (\frac12,1) \\ 3, & z \ge 1 \end{cases}$$



$$\mathbb{E}\,\#\{v \in T_{n/2} : X_v \le -x\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{x^2}{4n}\right)$$

$$\mathbb{E}\,\#\{v \in T_n : X_v - X_{a(v,n/2)} \le -y\tfrac{n}{2}\} \approx 2^{n/2} \exp\left(-\frac{y^2}{2 \cdot n}\right)$$

$$\mathbb{E}\,\#\left\{v \in T_n : X_{a(v,n/2)} \le -2\left(\frac{2\cdot\log 2}{3}\right)^{\frac12} n,\ X_v - X_{a(v,n/2)} \le -\left(\frac{2\log 2}{3}\right)^{\frac12} n\right\}$$

$$\asymp 2^n \exp\left(-\frac{2\log 2}{3} n - \frac{\log 2}{3} n\right) \asymp 1$$

But $n^{-1} M_n \not\to -(3 \cdot 2\log 2)^{\frac12}$, because

$$\mathbb{E}\,\#\left\{v \in T_n : X_{a(v,n/2)} \le -2\left(\frac{2\cdot\log 2}{3}\right)^{\frac12} n\right\} \approx 2^{n/2} \exp\left(-\frac{2\log 2}{3} n\right) \approx 2^{-n/6};$$

the needed trajectories do not exist.

In fact, here $n^{-1} M_n \to (\sqrt{2}+1)\log 2$

CREM: The minimum position

Proposition (Bovier-Kurkova; Mallein; LAB-Maillard)

Suppose A is absolutely continuous wrt Lebesgue measure, and has a Riemann-integrable derivative a.

Let $\hat{A}$ be the concave hull of A, let $\hat{a}$ be the left-derivative of $\hat{A}$.

Then $n^{-1}M_n \xrightarrow{a.s.} -\int_0^1 \sqrt{\hat{a}(t)}\, dt =: -c$

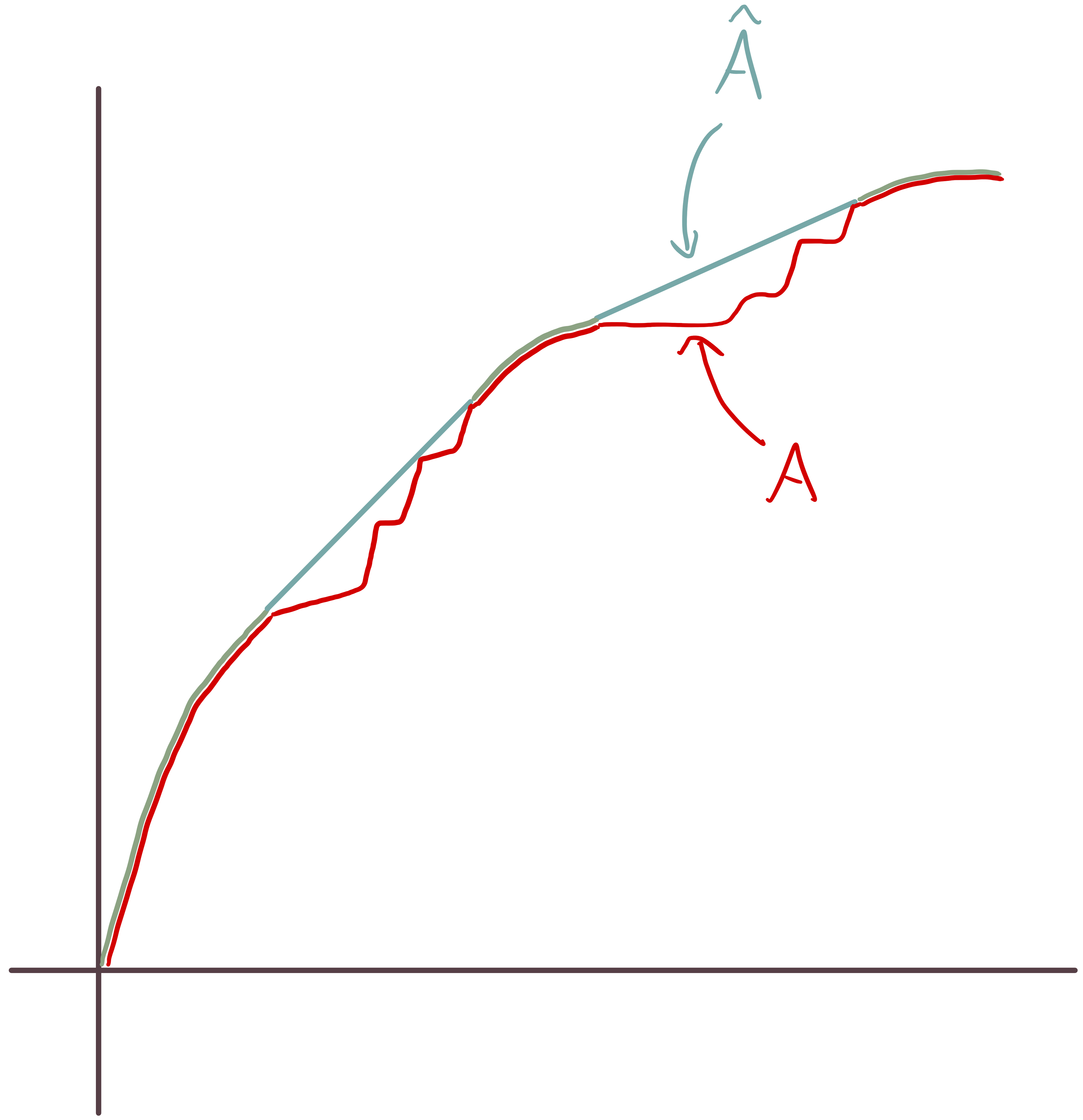
Moreover, $c = \sup\left\{ \int_0^1 v(s)ds : v:[0,1]\to\mathbb{R} \text{ measurable}, \right.$

$\left. \forall t\in(0,1],\ \int_0^t \frac{v(s)^2}{2a(s)}ds \le t\log 2 \right\}.$

The supremum is attained via the f^n $v_{max}:[0,1]\to\mathbb{R}$ with

$$v_{max}(s) = a(s)\cdot\left(\frac{2\log 2}{\hat{a}(s)}\right)^{1/2}.$$

N.B.: The value c only depends on A through $\hat{A}$; but the trajectory v_{max} followed to reach $-cn$ within $\mathbb{T}$ depends sensitively on A.



CREM and its minima: The algorithmic barrier

Def (Mallein): The natural-speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}}\, dt$

(Recall $A' = a$)

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural-speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}} dt$ (Recall $A' = a$)

Idea: Recall that along any root-to-leaf path, we observe an inhomogeneous Brownian motion whose infinitesimal variance at time $\approx tn$ is $A'(t)$.

More precisely: if $v \in T_m$ then $X_v - X_{\text{parent}(v)}$ is $\mathcal{N}(0, n(A(\frac{m}{n}) - A(\frac{m-1}{n})))$.

If $m \approx tn$ then step variance is $n(A(\frac{m}{n}) - A(\frac{m-1}{n})) \approx a(t)$.

Let K be fixed, large. For $v \in T_m$, $w \in T_{m+K}$ with $a(w,m) = v$, then $X_w - X_v \approx N(0, Ka(t))$, so

$$\mathbb{E}\#\{w \in T_{m+K},\ a(w,m) = v,\ X_w - X_v \leq -cK\} \cong 2^K \mathbb{P}(N(0, Ka(t)) \leq -cK)$$

$$\approx \exp\left(K \log 2 - K\frac{c^2}{2a(t)}\right) = 1 \text{ when } c = (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}}$$

So for K large the K-level greedy search from before will typically make a path $v_1, \dots, v_n$ with $X(v_{t \cdot n}) \approx -n Z_A(t)$.

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural-speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}} dt$

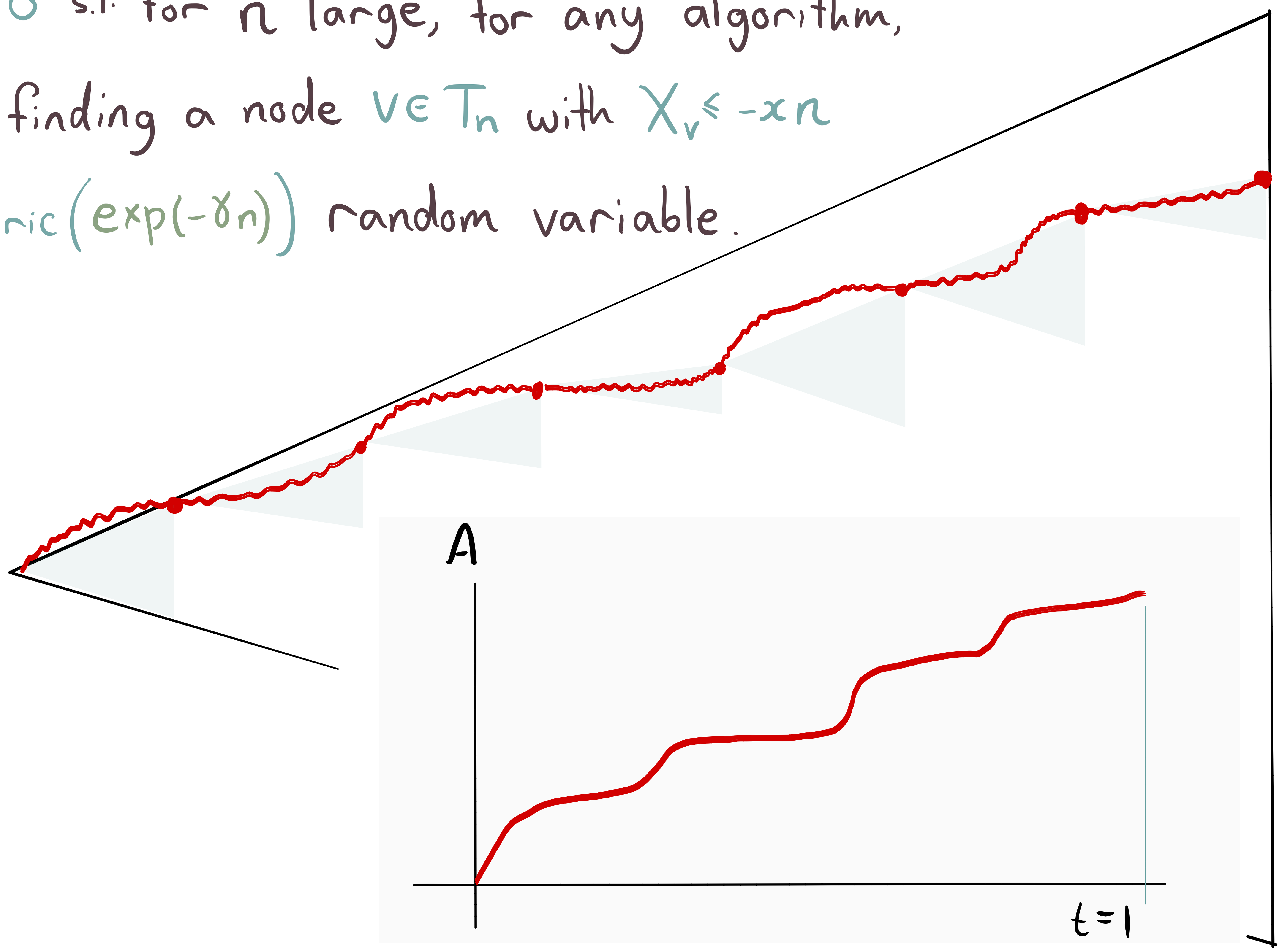
Theorem: (LAB, Maillard)

If A abs. continuous, A' Riemann-integrable, then with $z_* = Z_A(1)$, we have:

1. For all $x < z_*$, there is a linear-time algorithm that finds $v \in T_n$ with $X_v \leq -xn$ with high probability.
2. For all $x > z_*$, there is $\gamma = \gamma(A, x) > 0$ s.t. for n large, for any algorithm, the expected # of queries before finding a node $v \in T_n$ with $X_v \leq -xn$ stochastically dominates a $\text{Geometric}(\exp(-\gamma n))$ random variable.

Proof Idea:

1) The K-level greedy search (for large K) approximately follows a natural-speed path.



CREM and its minima: The algorithmic barrier

Def (Mallein): The natural speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}} dt$

Theorem: (LAB, Maillard)

If A abs. continuous, A' Riemann-integrable, then with $z_* = Z_A(1)$, we have:

1. For all $x < z_*$, there is a linear-time algorithm that finds $v \in T_n$ with $X_v \leq -xn$ with high probability.
2. For all $x > z_*$, there is $\gamma = \gamma(A, x) > 0$ s.t. for n large, for any algorithm, the expected # of queries before finding a node $v \in T_n$ with $X_v \leq -xn$ stochastically dominates a $\text{Geometric}(\exp(-\gamma n))$ random variable.

Proof Idea:

1) In the inhomogeneous setting, the renormalization search follows the natural speed path.

2) For every node $v \in T_n$ with $X_v \leq -xn$, there is a linear-size subsection of the ancestral trajectory of v along which the slope is unnatural (greater in absolute value than the slope of the natural speed path).

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}} dt$

$x > z^* = Z(1)$

Proof Idea:

2) For every node $v \in T_n$ with $X_v \leq -xn$, there is a linear-size subsection of the ancestral trajectory of v along which the slope is unnatural (greater in absolute value than the slope of the natural speed path)

• For large M, $\int_0^1 a(t)^{\frac{1}{2}} dt = \sum_{\ell=1}^{M} \int_{(\ell-1)/M}^{\ell/M} a(t)^{\frac{1}{2}} dt \cong \sum_{\ell=1}^{M} \frac{1}{M^{1/2}} \left(A\left(\frac{\ell}{M}\right) - A\left(\frac{\ell-1}{M}\right)\right)$

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural speed path for A is the function $Z(t)=Z_A(t)=\int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}}\,dt$

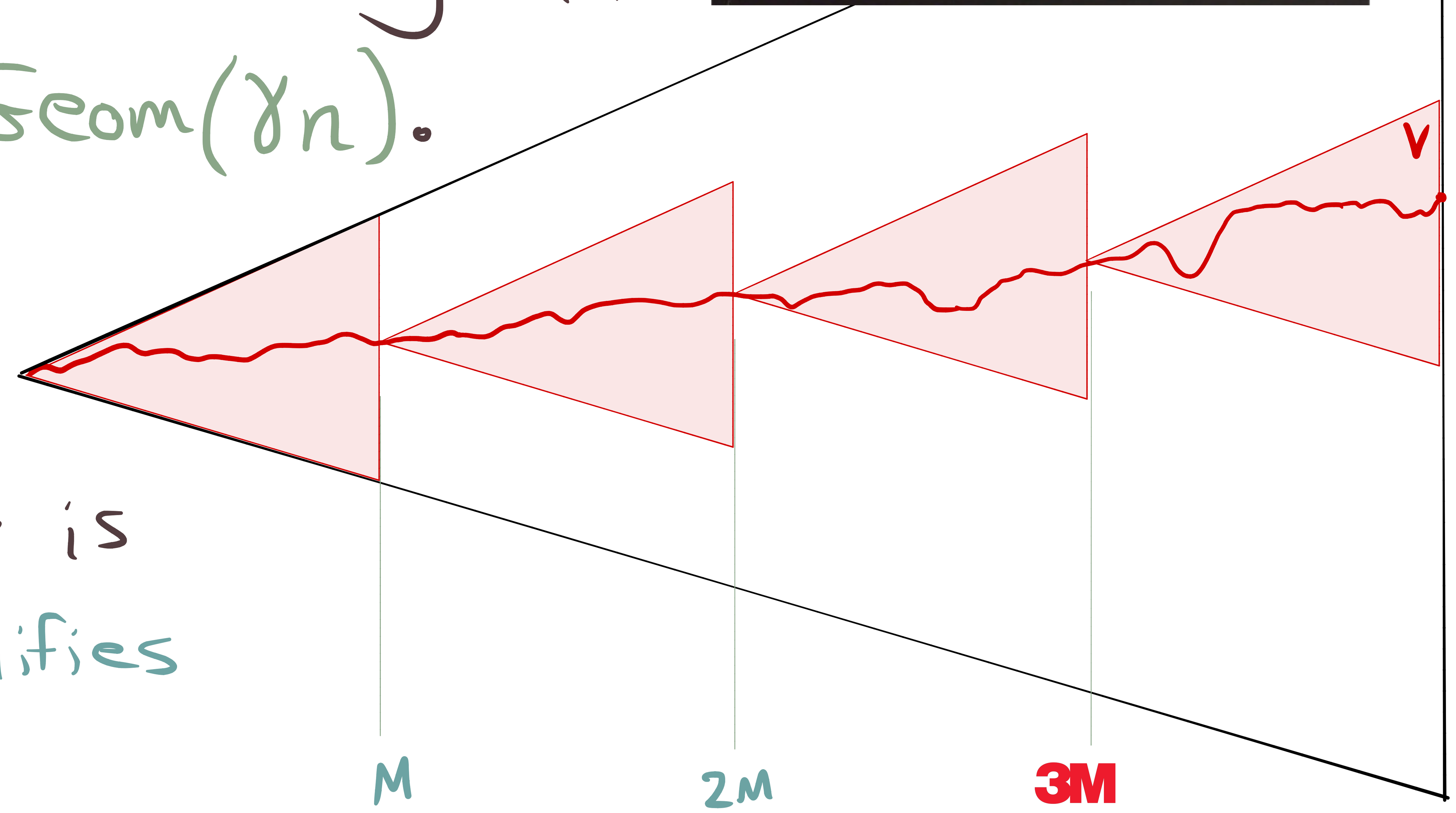
Proof Idea:

2) For every node $v \in T_n$ with $X_v \leq -xn$, there is a linear-size subsection of the ancestral trajectory of v along which the slope is unnatural (greater in absolute value than the slope of the natural speed path)

$x > z^* = Z(1)$

• For large M, $\int_0^1 a(t)^{\frac{1}{2}}\,dt = \sum_{\ell=1}^{M} \int_{(\ell-1)/M}^{\ell/M} a(t)^{\frac{1}{2}}\,dt \simeq \sum_{\ell=1}^{M} \frac{1}{M^{\frac{1}{2}}} \left(A\left(\frac{\ell}{M}\right)-A\left(\frac{\ell-1}{M}\right)\right)$

Lemma 2: $\forall\, \varepsilon, M\ \exists\, \gamma>0$ s.t. for n large, for any algorithm, # queries before finding an (ε, M)-steep subtrajectory is $\gtrsim_{st}$ Geom(γn).



Proof idea: Reveal all displacements in "tower of spindles" above v when v is queried. Gain independence ⇒ simplifies analysis.

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}}\, dt$

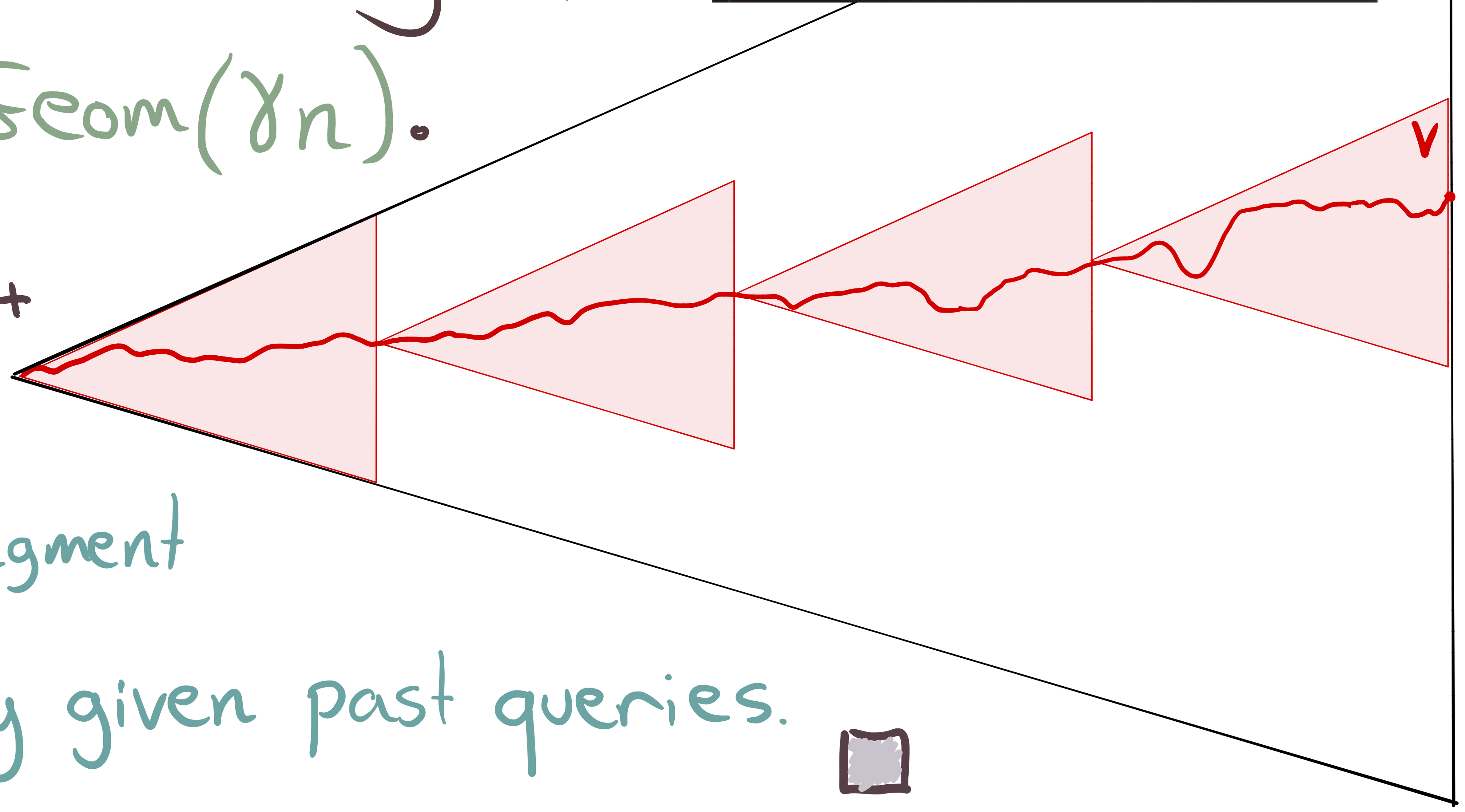
Proof Idea:

$x > z^* = Z(1)$

2) For every node $v \in T_n$ with $X_v \le -xn$, there is a linear-size subsection of the ancestral trajectory of v along which the slope is unnatural (greater in absolute value than the slope of the natural speed path)

• For large M, $\int_0^1 a(t)^{\frac{1}{2}}\, dt = \sum_{\ell=1}^{M} \int_{(\ell-1)/M}^{\ell/M} a(t)^{\frac{1}{2}}\, dt \cong \sum_{\ell=1}^{M} \frac{1}{M^{1/2}} \left(A\left(\frac{\ell}{M}\right) - A\left(\frac{\ell-1}{M}\right)\right)$

Lemma 2: $\forall\, \varepsilon, M \;\; \exists\, \gamma > 0$ s.t. for n large, for any algorithm, # queries before finding an (ε, M)-steep subtrajectory is $\succeq_{st} \mathrm{Geom}(\gamma n)$.



Proof idea: Then the branching property + Gaussian tail estimates

$\Rightarrow$ exponentially unlikely to find a steep segment on any single query, even conditionally given past queries. ∎

Open problems/research directions.

- Extend approach to models with less trivial geometry.
- Understand finer asymptotics of query complexity near the critical point z^*
- Extend analysis to find computational threshold for efficiently approximating Gibbs measures in the CREM.
- Find natural complexity-theoretic assumptions under which this analysis can be extended to prove "hardness".

本日はありがとうございました。