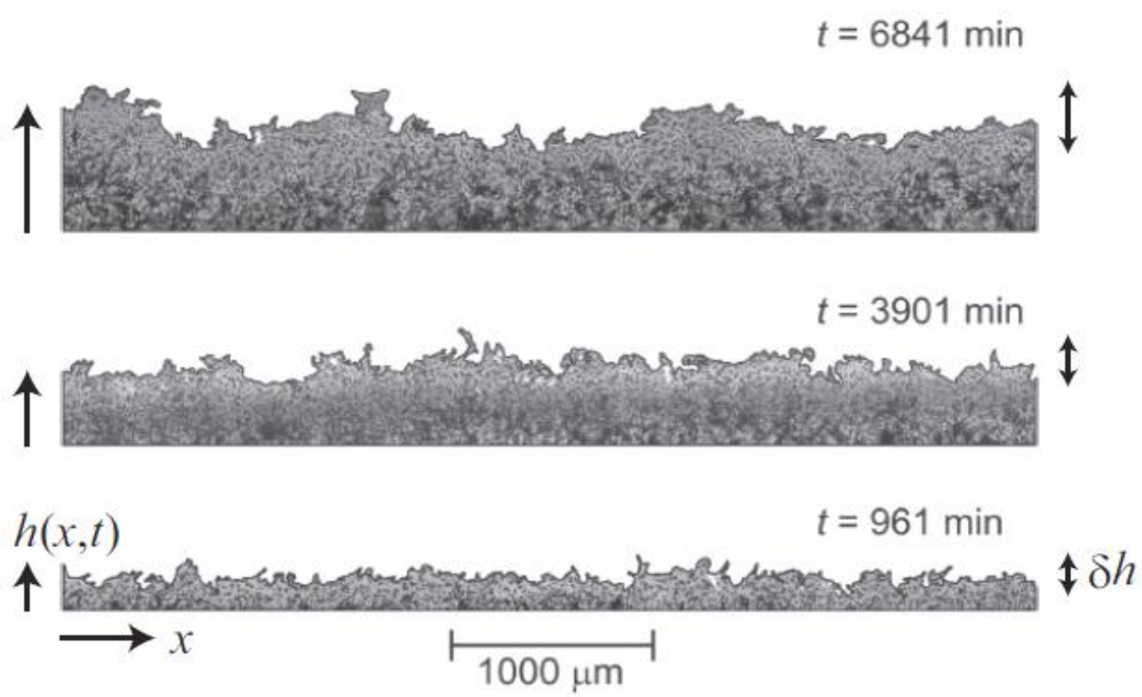
Stochastic six vertex model

Ivan Corwin (Columbia University)

Goals of first hour

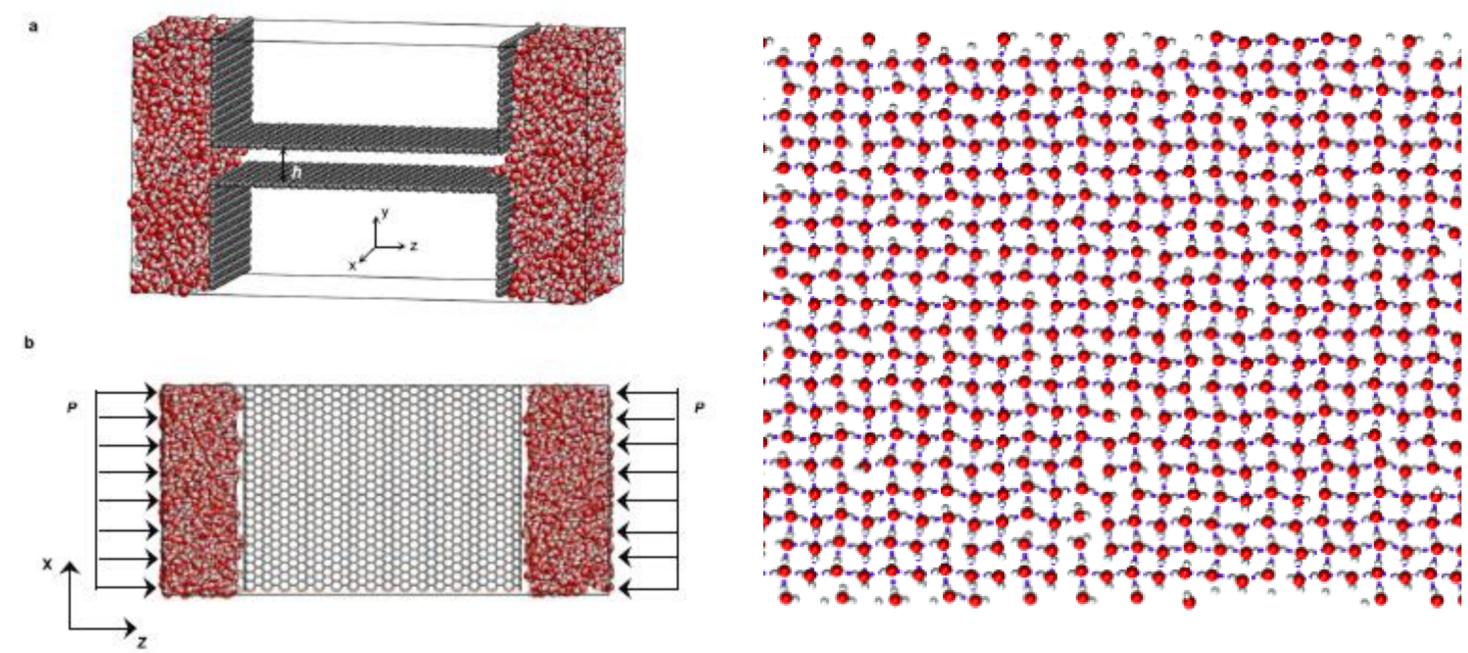
Physical goal: Uncover nonequilibrium Kardar-Parisi-Zhang (KPZ) universality class behavior in the equilibrium six vertex model (6V).

Growth dynamics of cancer cell colonies and their comparison with noncancerous cells. PRE, 2012, Huergo et. al.



Kardar-Parisi-Zhang class

Square ice in graphene nanocapillaries, Nature 2015, Algara-Siller et. al.

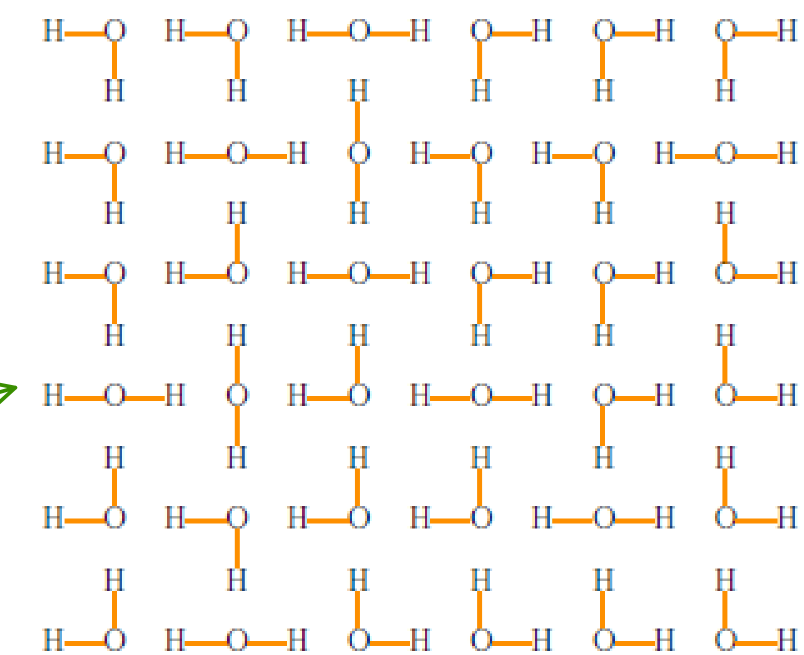


Six vertex model

Mathematical goal: Describe how to analyze the stochastic six vertex model (S6V) via Markov dualities and Bethe ansatz methods.

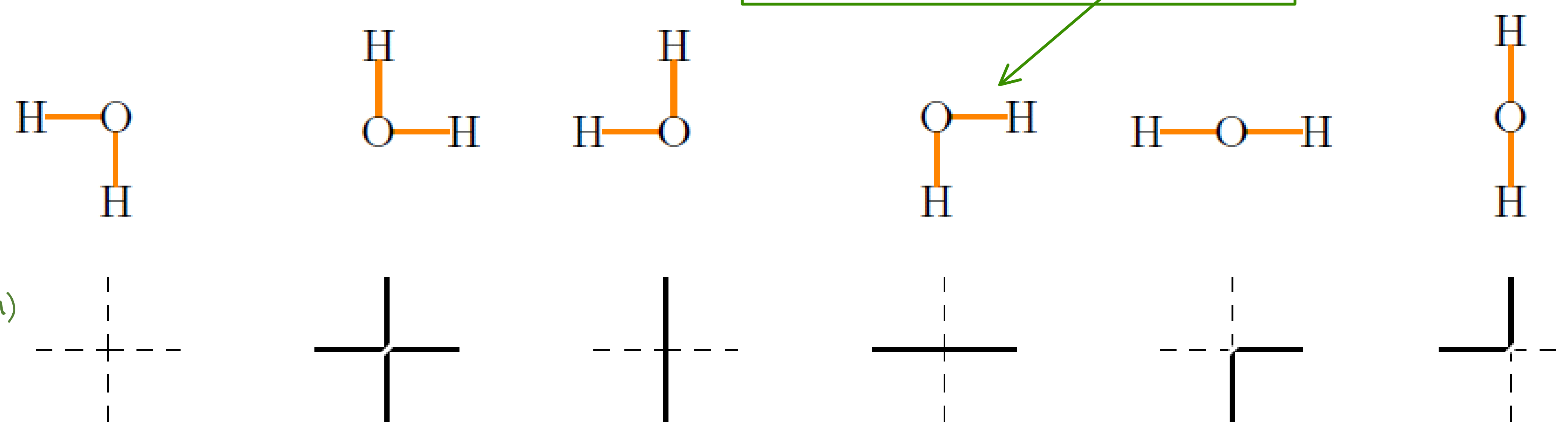
Six vertex model [Pauling '35], [Slater '41], [Lieb '67]

Square-ice model based on six orientations of H_2O. Other molecules (e.g. KH_2PO_4) have unequal binding energy. Led [Slater '41] to the general six vertex model.



Probability proportional to product of weights.

Configurations (two equivalent ways to draw them)



Weights a_1 a_2 b_1 b_2 c_1 c_2

What happens in the large system limit? How do weights matter?

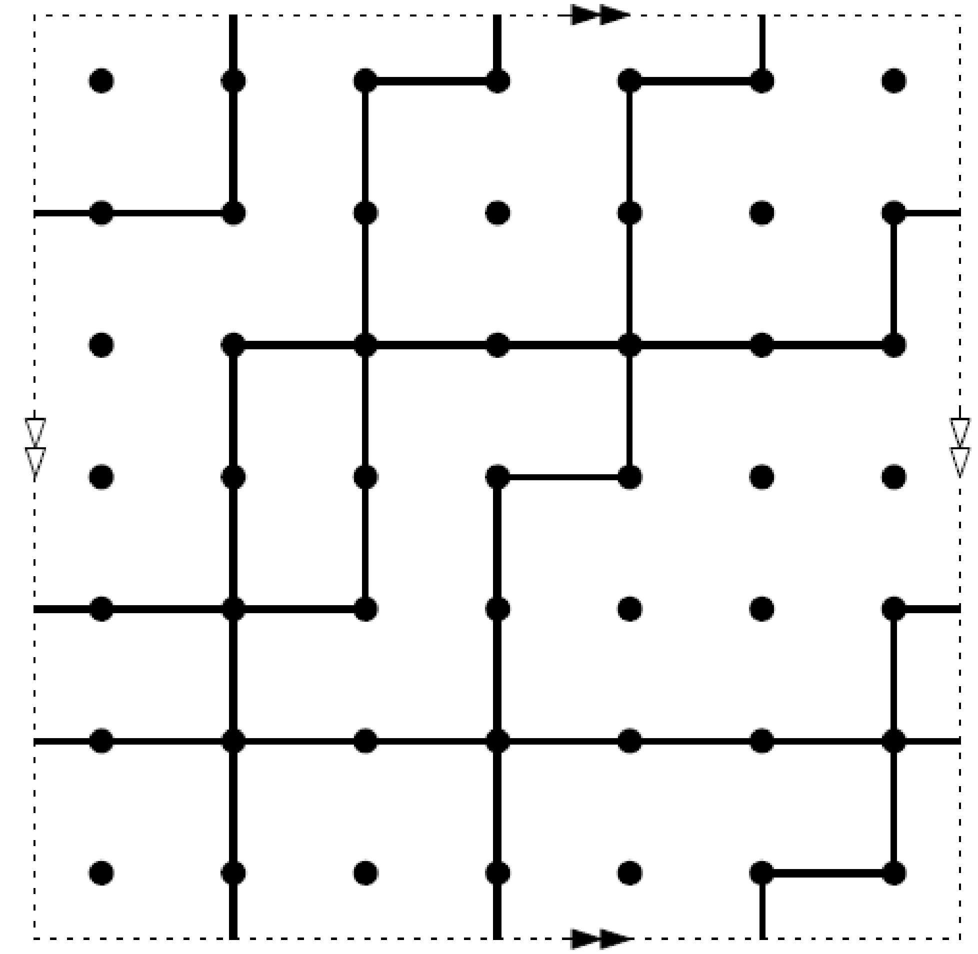
Gibbs states

Infinite volume '**Gibbs states**' and free energies are key to answer these questions.

Given boundary of a subregion, probability of inside proportional to product of weights

Should be limits of the model on a torus.

$e^{-H-V}a$ $e^{H+V}a$ $e^{-H+V}b$ $e^{H-V}b$ c c



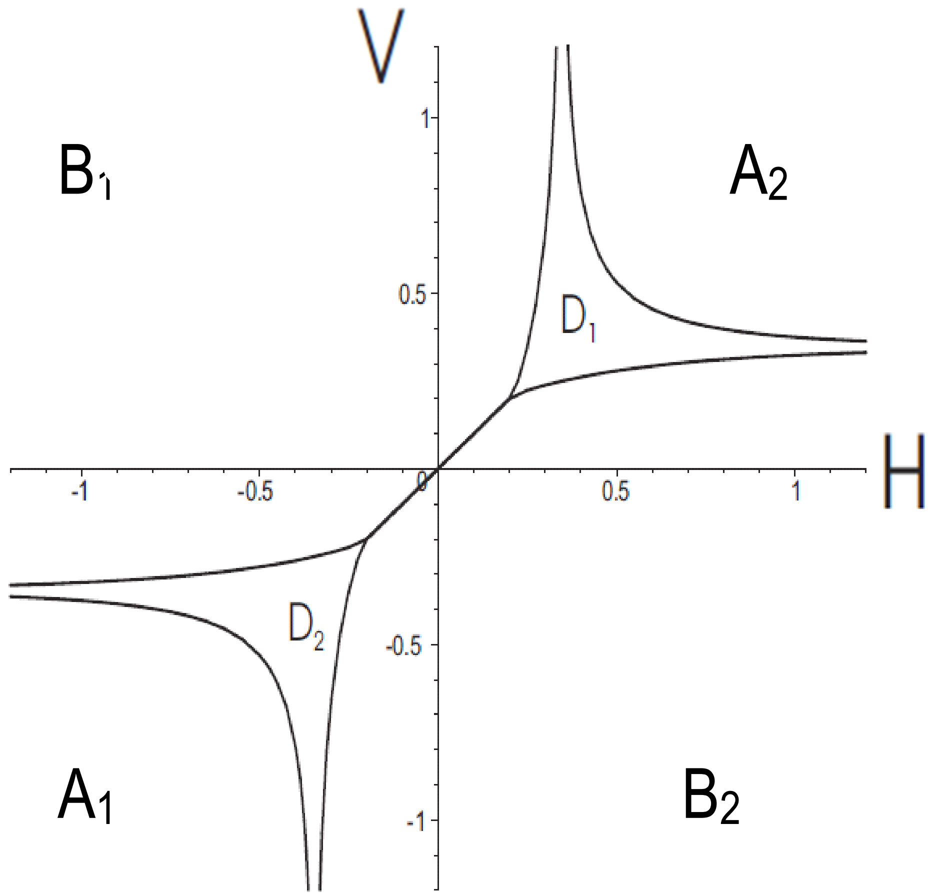
For (a,b,c) fixed, choosing different H and V **external fields** should lead to (possibly) different Gibbs states. The phase diagram of such Gibbs states is mostly conjectural and relies upon a key parameter

$$\Delta := \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2\sqrt{a_1 a_2 b_1 b_2}} = \frac{a^2 + b^2 - c^2}{2ab} \quad .$$

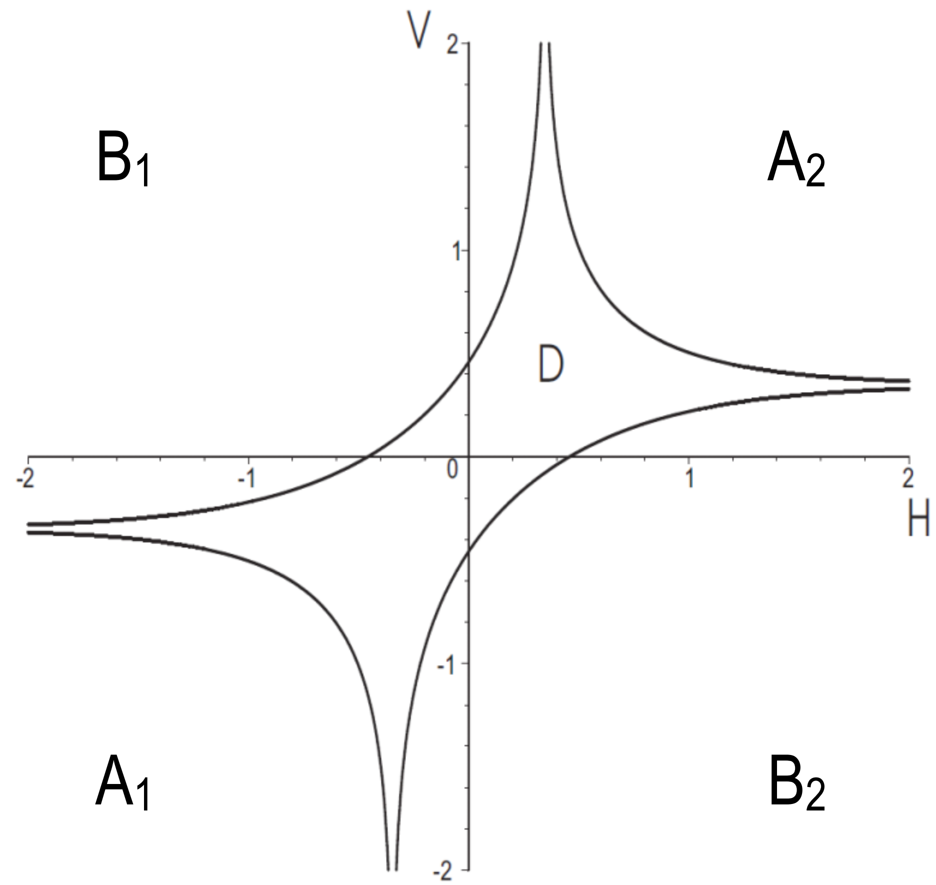
Phase diagrams

From: Lectures on the integrability of the 6-vertex model, 2010, Reshetikhin.

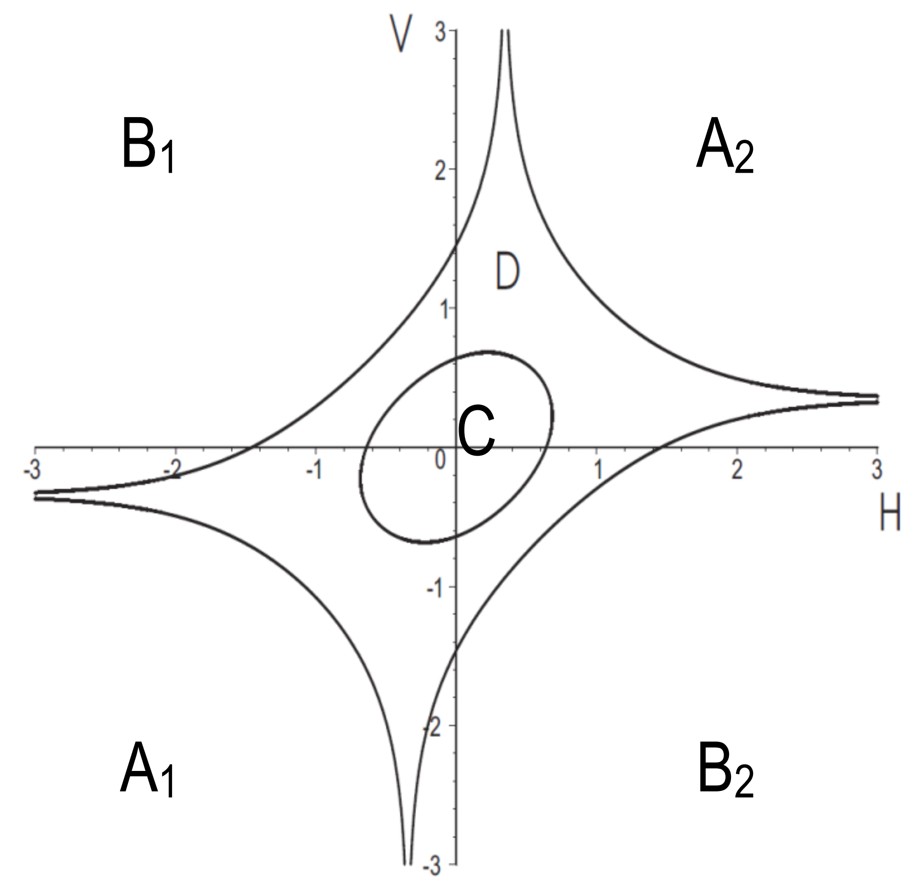
Ferroelectric: $\Delta > 1$



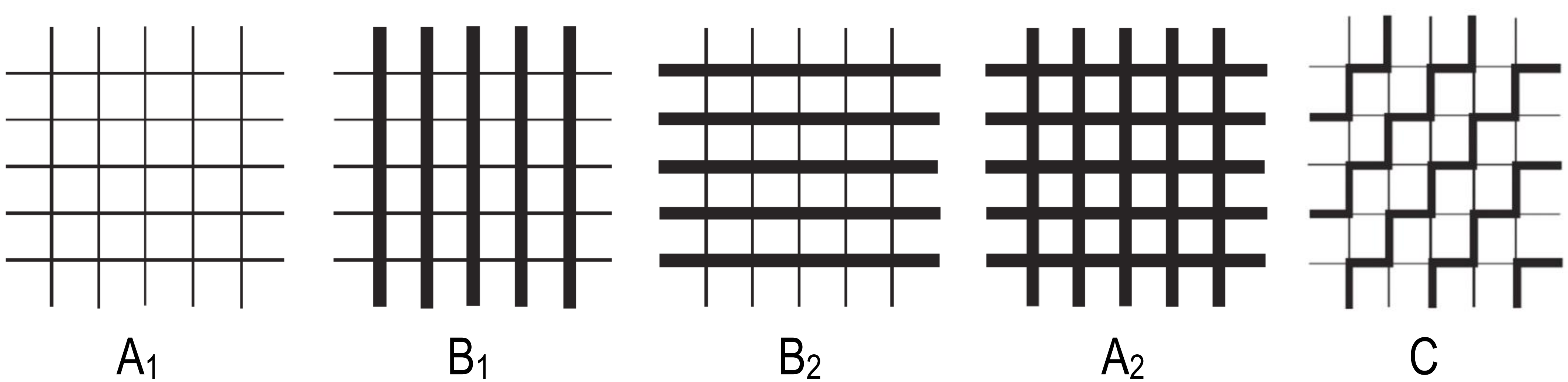
Disordered: $|\Delta| < 1$

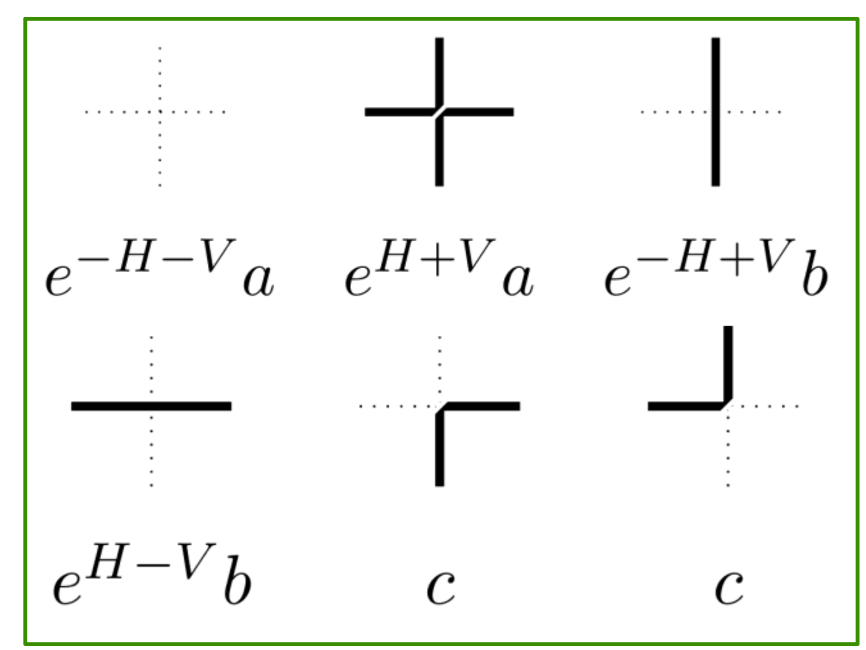


Antiferroelectric: $\Delta < -1$



Ordered Gibbs states:

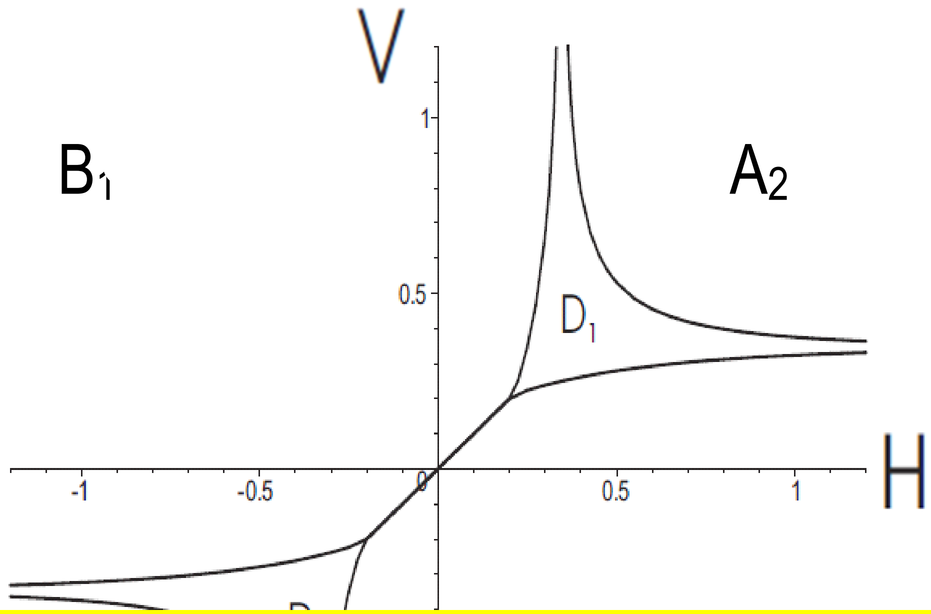


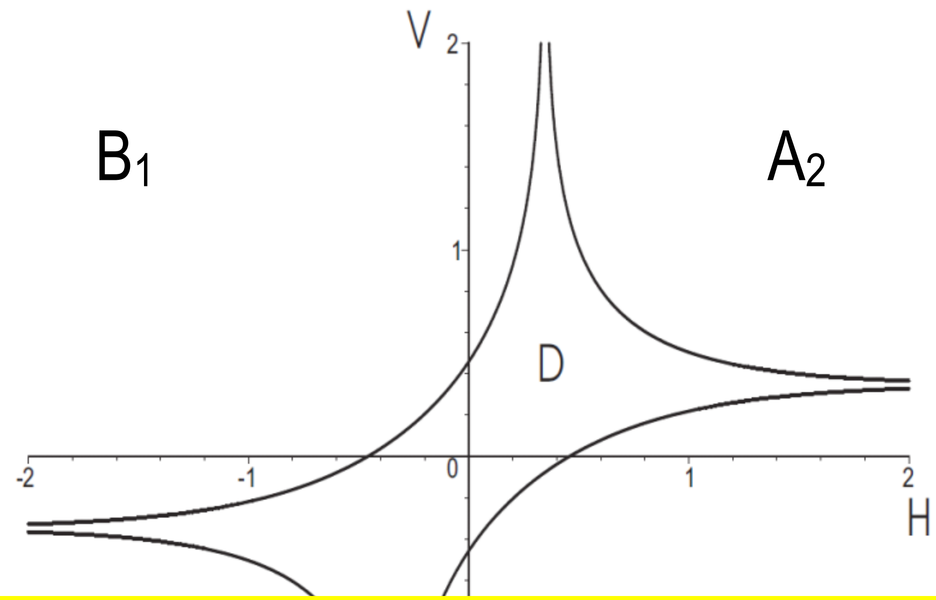


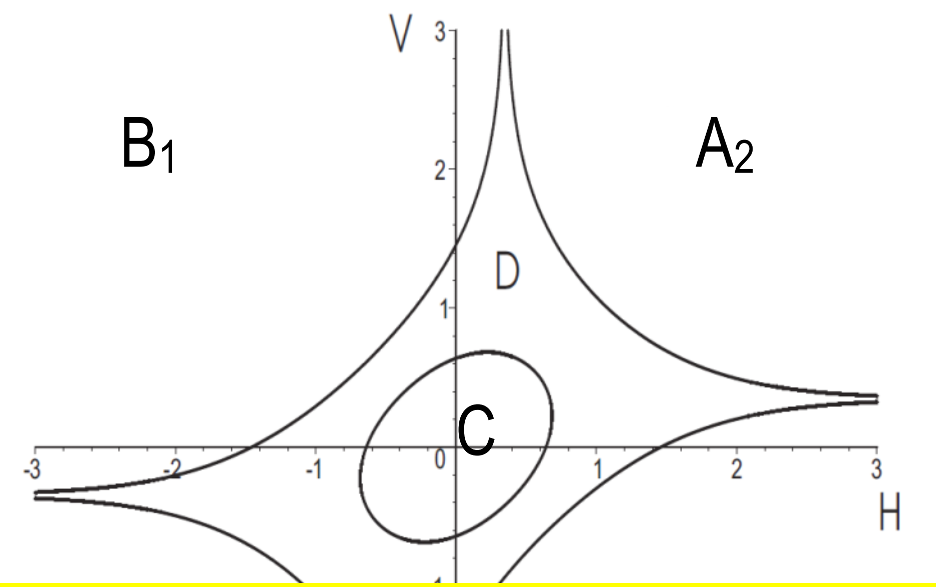
Phase diagrams

From: Lectures on the integrability of the 6-vertex model, 2010, Reshetikhin.

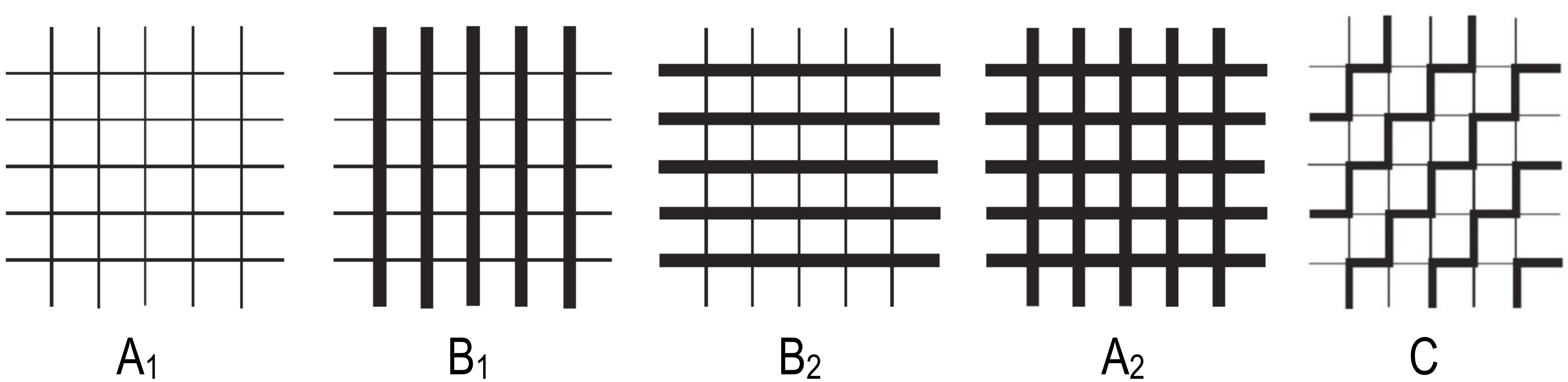
Ferromagnetic: $\Delta > 1$ Disordered: $|\Delta| < 1$ Antiferromagnetic: $\Delta < -1$

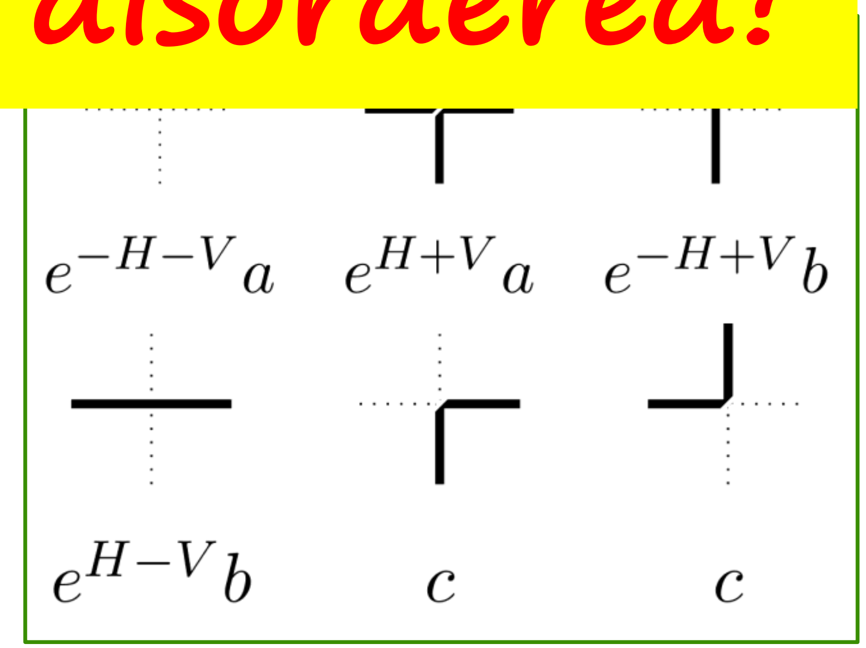






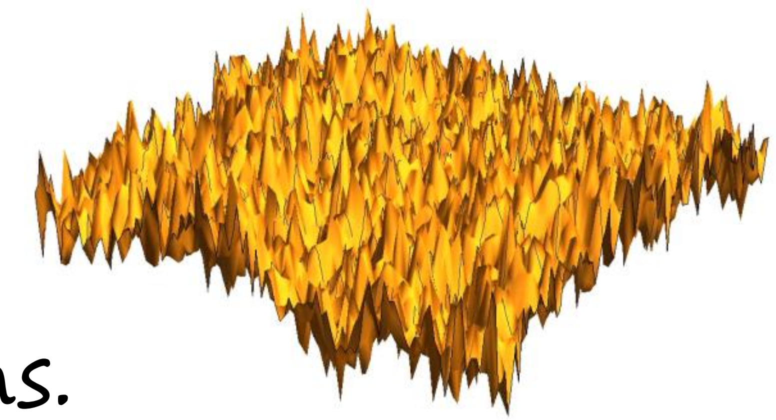
What happens in the disordered phase, or at its boundary? How disordered is disordered?





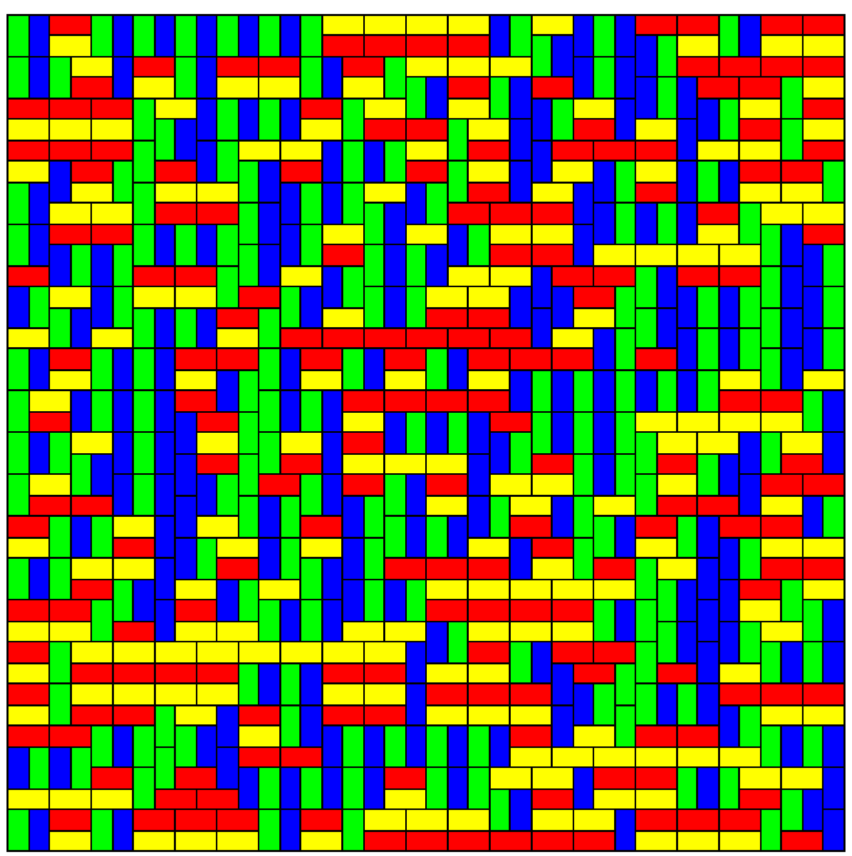
Disordered phase (free fermion case)

Points in disordered phase lead to Gibbs states with various average horizontal and vertical line densities.

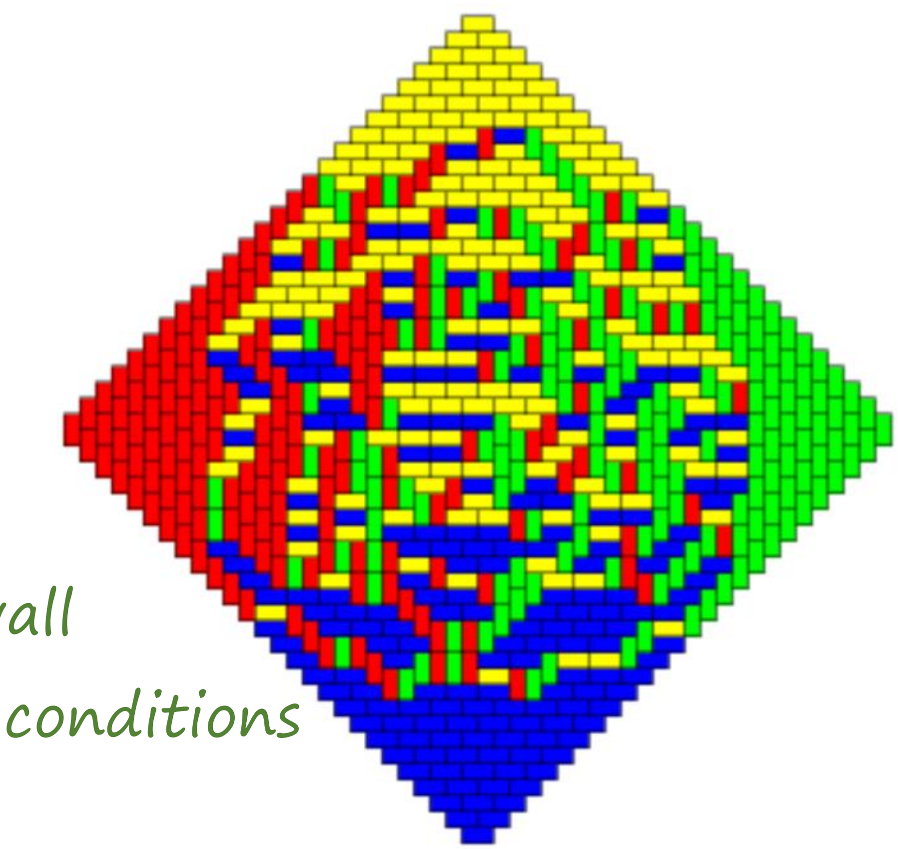


[Nienhuis '84] conjectured that disordered states have Gaussian free field height function fluctuations.

[Kenyon '01] proved results for free fermion case ($\Delta = 0$).



Portion of a Gibbs state for the aztec tiling model

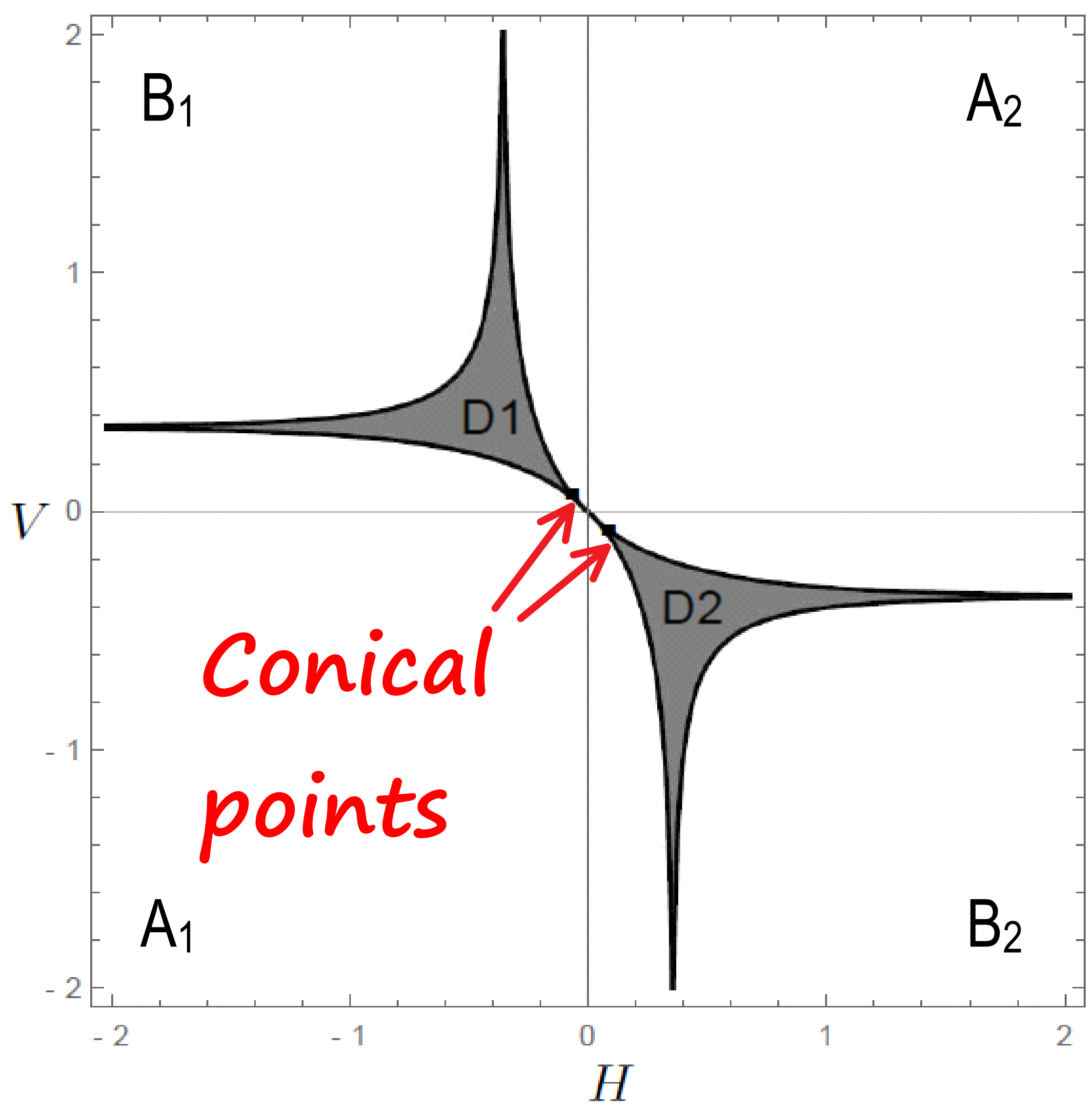


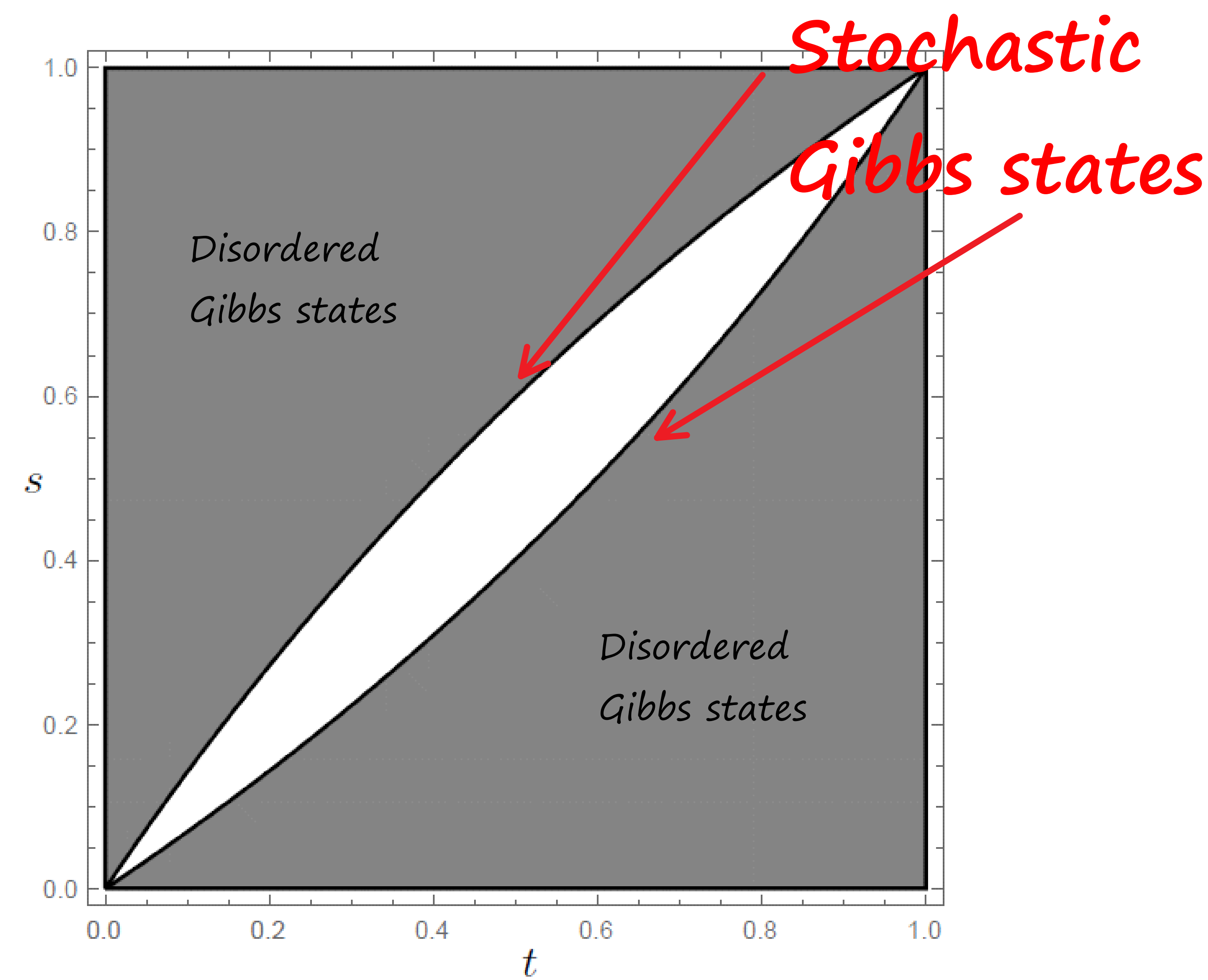
Domain wall boundary conditions

Stochastic point

For $\Delta > 1$ the 'conical points' in the phase diagram correspond with a one-parameter family of explicit 'stochastic' Gibbs states.

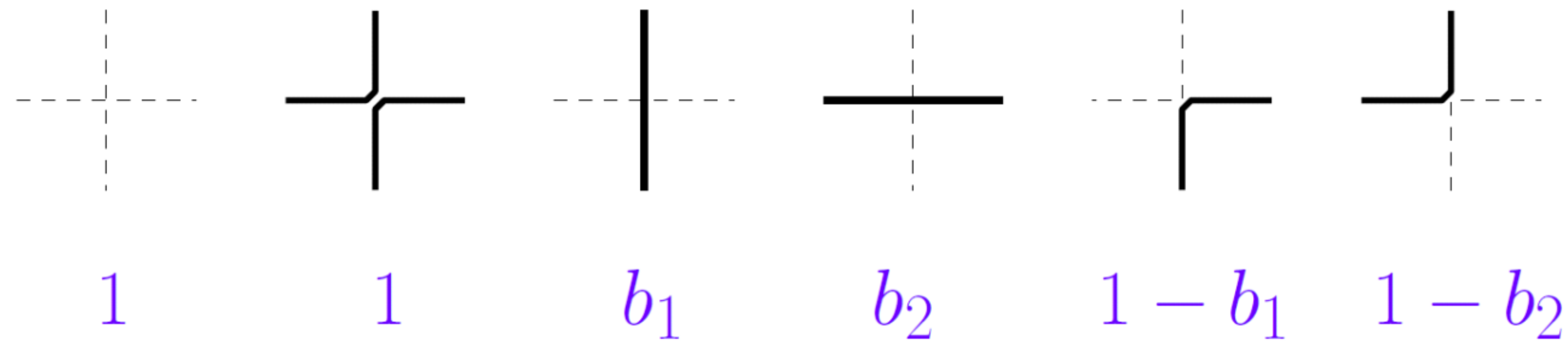
[Jayaprash-Sam '84] [Bukman-Shore '95] [Aggarwal '16]





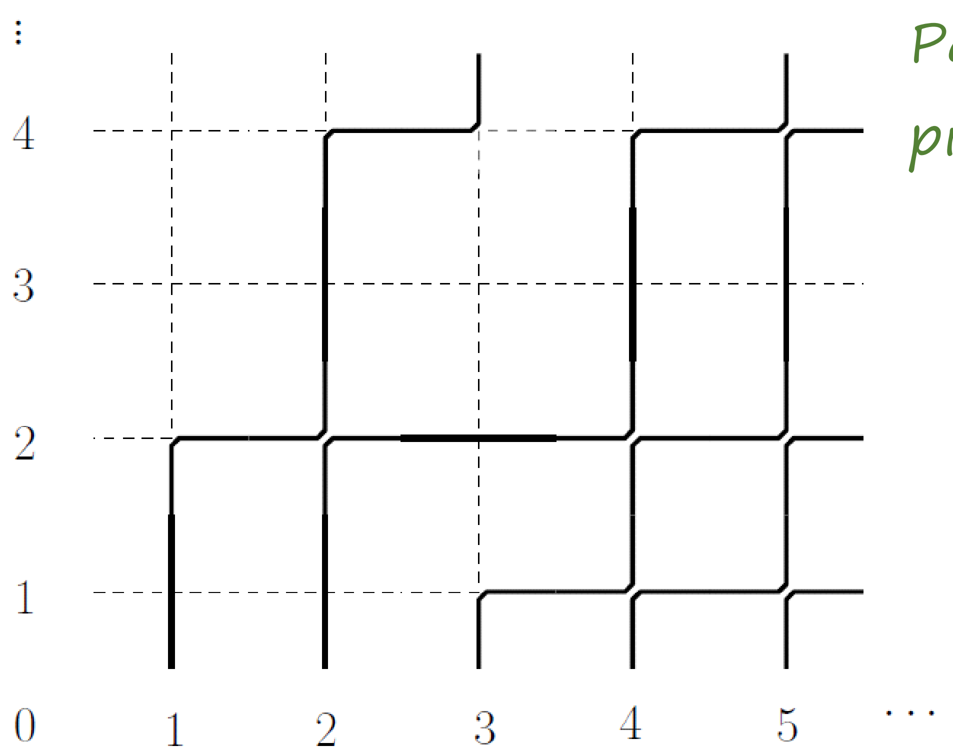
Stochastic six vertex model

On first quadrant, for $\Delta > 1$, special choice of vertex weights



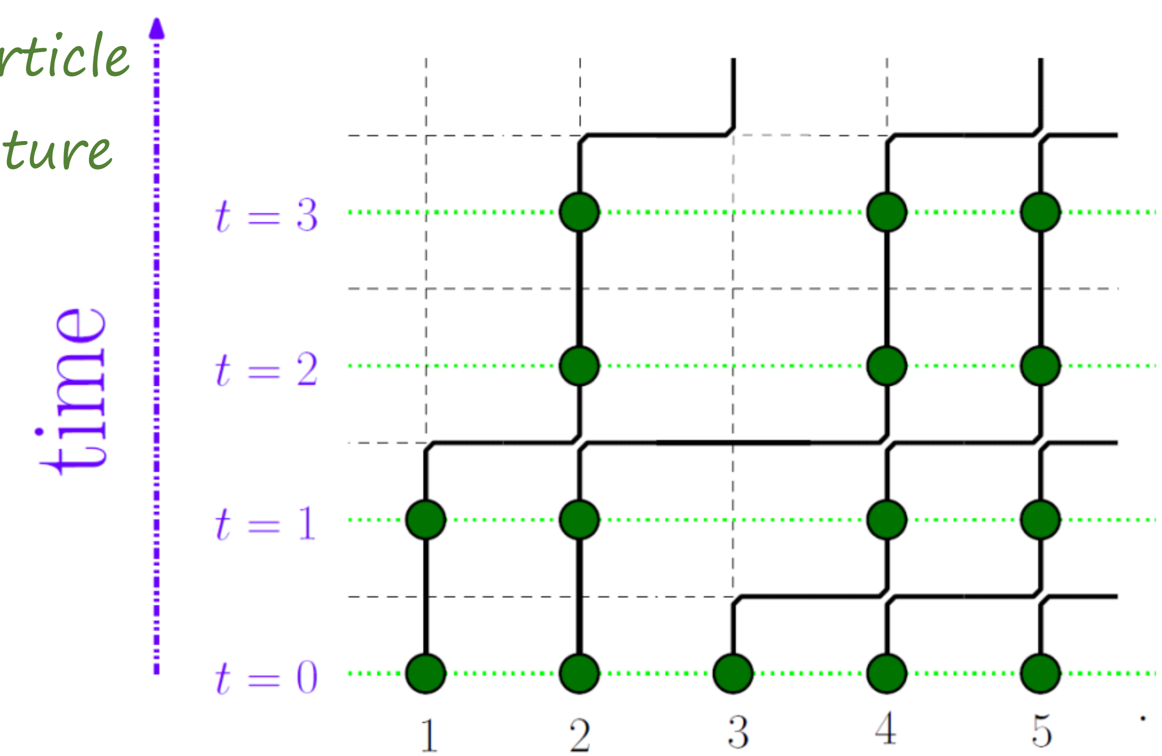
yields stochastic six vertex model (S6V) [Gwa-Spohn '92].

Markov update provides interacting particle system interpretation.



Path picture

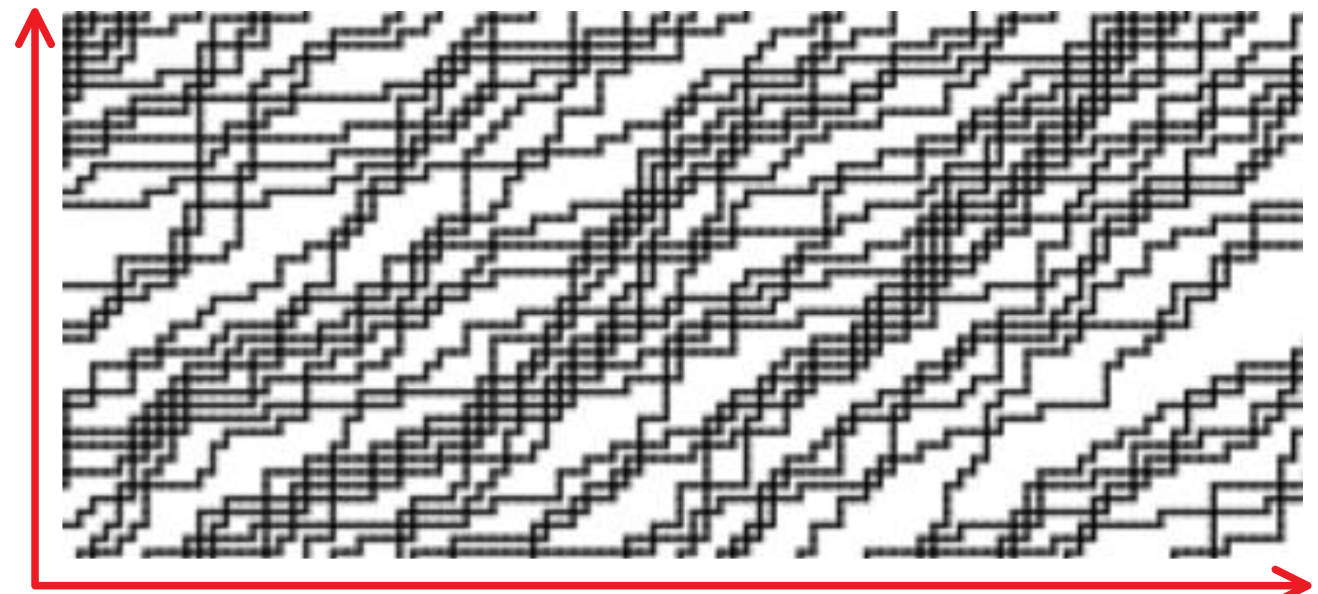
Particle picture



Stochastic Gibbs states

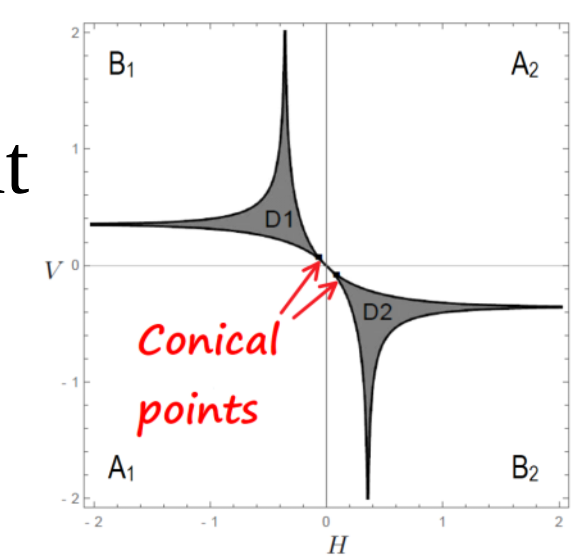
Let $\rho = \frac{1-b_1}{1-b_2}$ and (δ_1 , δ_2) be solutions to $\frac{\delta_1}{1-\delta_1} = \rho \cdot \frac{\delta_2}{1-\delta_2}$.

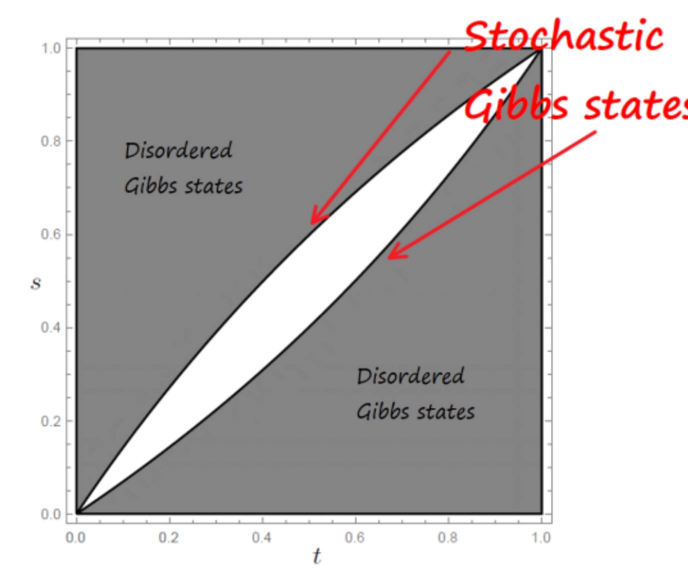
Bernoulli product measure (δ_1 on the y-axis and δ_2 on the x-axis) is stationary [Aggarwal '16] and hence produces an infinite volume 'stochastic Gibbs state'.



Theorem [Bukman-Shore '95] [Aggarwal '16]:
Stochastic Gibbs states (above) are conical point Gibbs states for the symmetric 6V model when

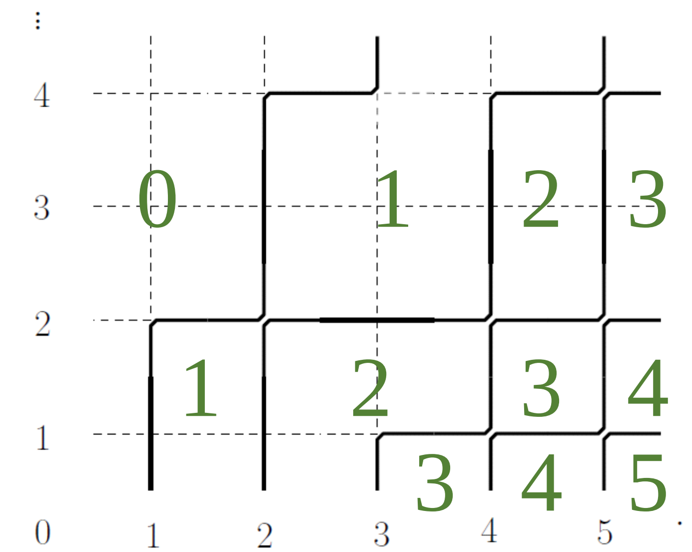
$$b_1 = \frac{b}{a}(\Delta + \sqrt{\Delta^2 - 1}), \quad b_2 = \frac{b}{a}(\Delta - \sqrt{\Delta^2 - 1}) .$$



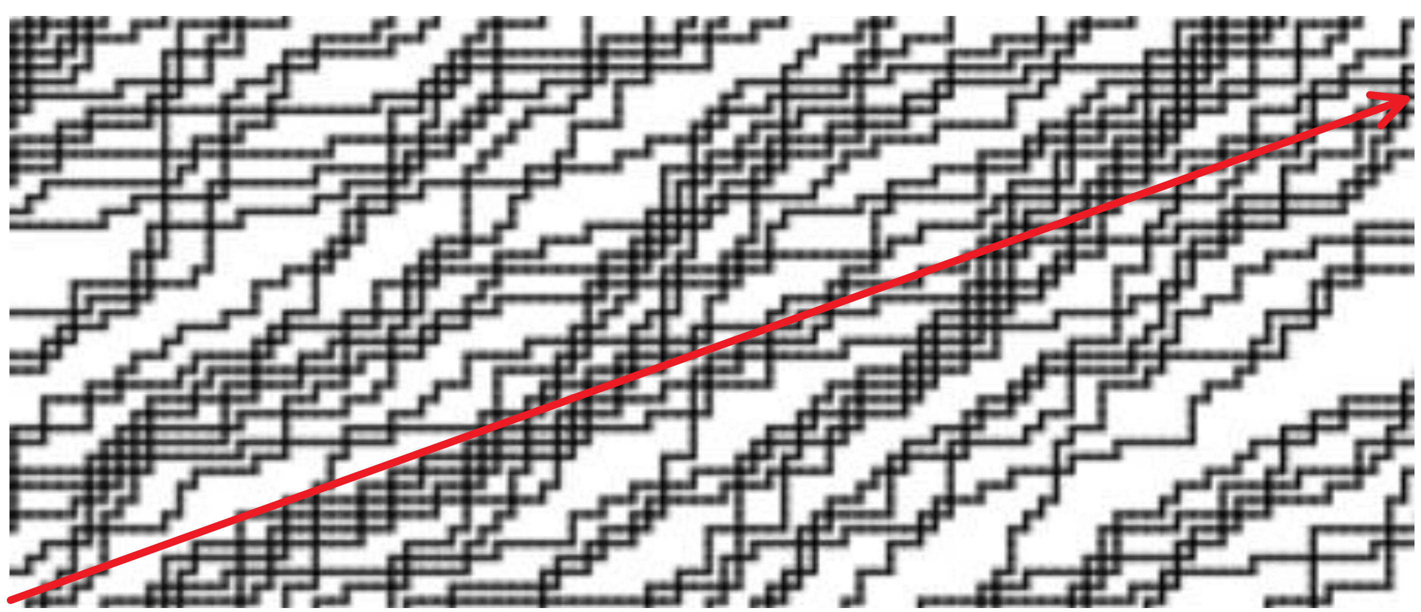


Stationary S6V height fluctuations

Define the height function (zero at the origin):



Theorem [Aggarwal '16]: For fixed b_1, b_2 the stochastic Gibbs state height function fluctuates like distance$^{1/2}$ with Gaussian distribution, except along the 'characteristic direction' where it's like distance$^{1/3}$ with stationary KPZ distribution [Baik-Rain '01].

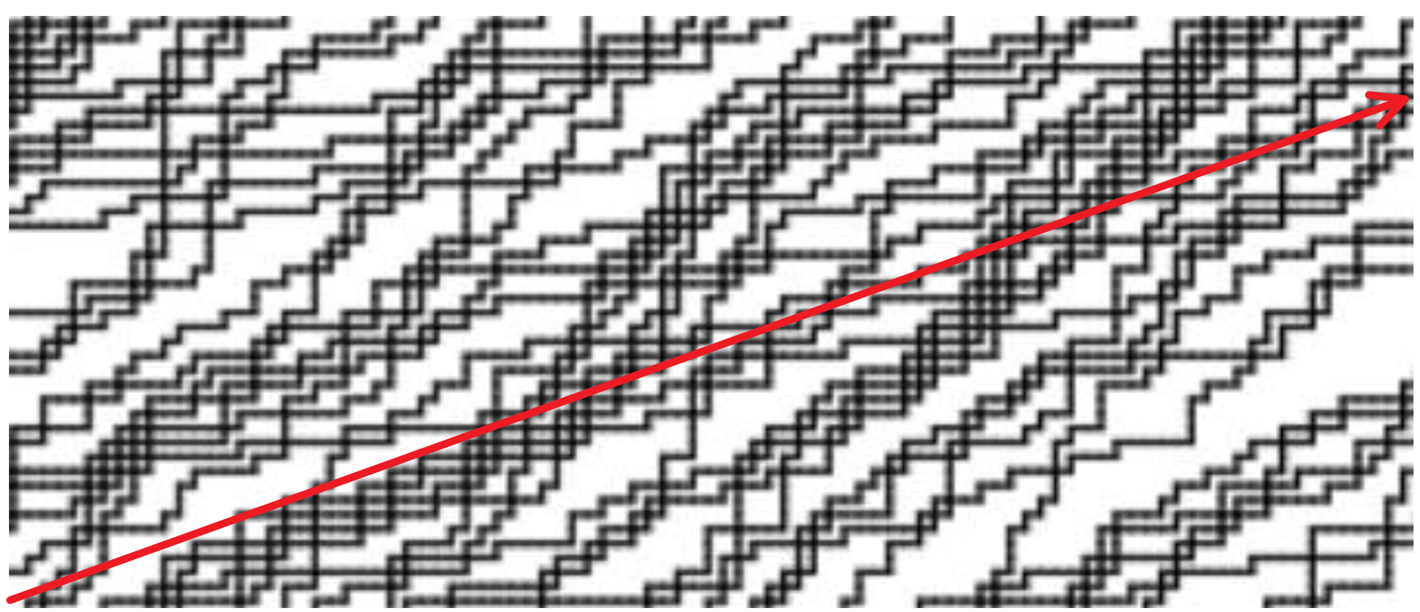


characteristic direction comes from the Hamilton-Jacobi hydrodynamic limit flux (essentially δ_1 as a function of δ_2).

Stationary S6V height fluctuations

Compare to conjectural Gaussian free field disordered phase behavior with logarithmic scale fluctuations.

fluctuates like distance$^{1/2}$ with Gaussian distribution, except along the 'characteristic direction' where it's like distance$^{1/3}$ with stationary KPZ distribution [Baik-Rain '01].



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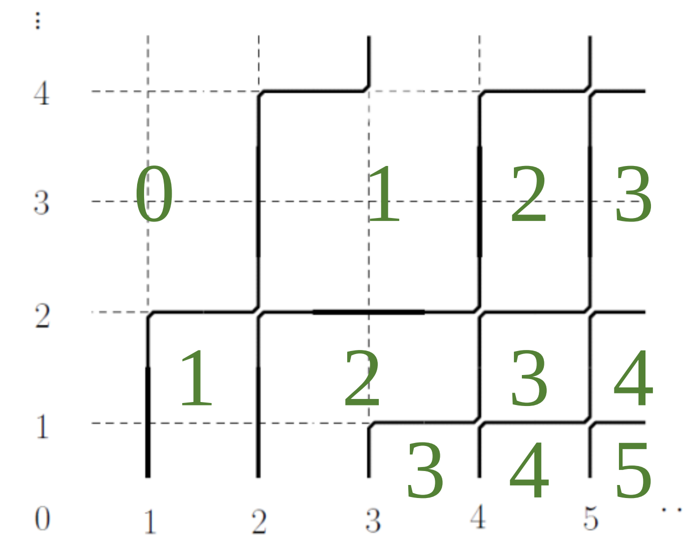
Step initial data S6V height fluctuations



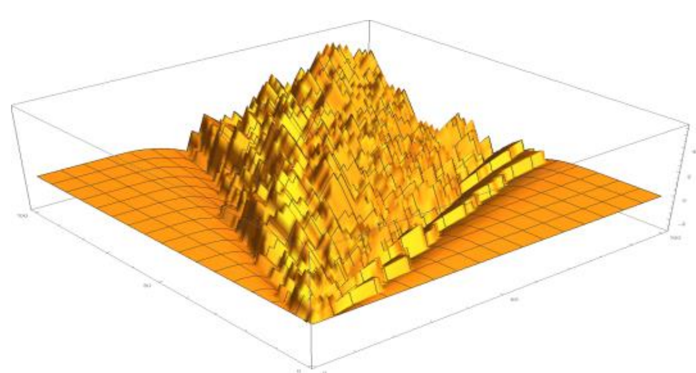
Theorem [Borodin-C-Gorin '14]: For step initial data S6V

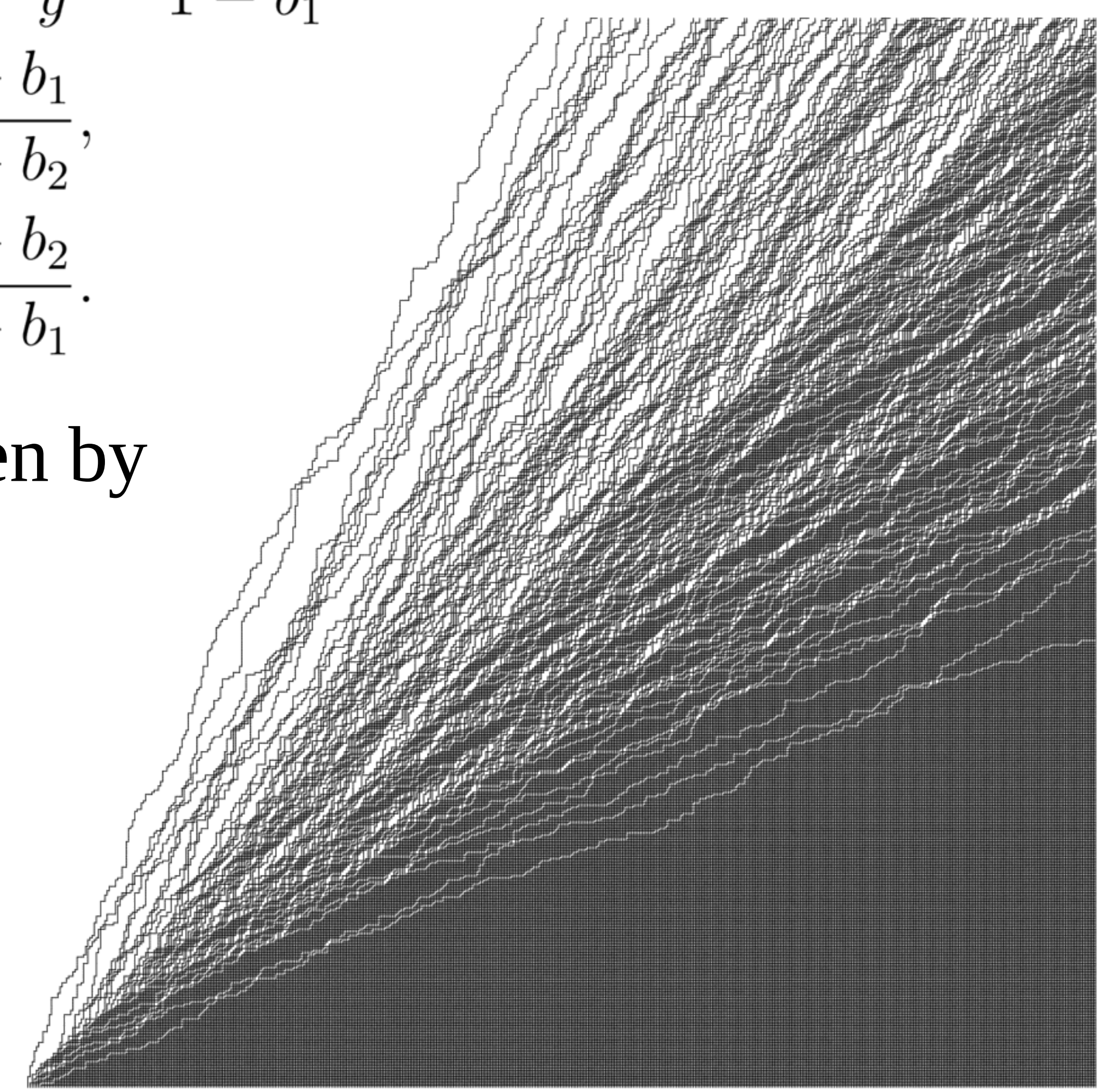
$$\lim_{L\to+\infty} \frac{H(Lx,Ly;\omega)}{L} = \mathcal{H}(x,y)$$ where the limit shape is

$$\mathcal{H}(x,y) = \begin{cases} \dfrac{\left(\sqrt{y(1-b_1)} - \sqrt{x(1-b_2)}\right)^2}{b_1-b_2}, & \dfrac{1-b_1}{1-b_2} < \dfrac{x}{y} < \dfrac{1-b_2}{1-b_1}, \\ 0, & \dfrac{x}{y} \leq \dfrac{1-b_1}{1-b_2}, \\ x-y, & \dfrac{x}{y} \geq \dfrac{1-b_2}{1-b_1}. \end{cases}$$

The fluctuations around the limit shape are given by

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{\mathcal{H}(x,y)L - H(Lx,Ly;\omega)}{\sigma_{x,y}L^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s)$$

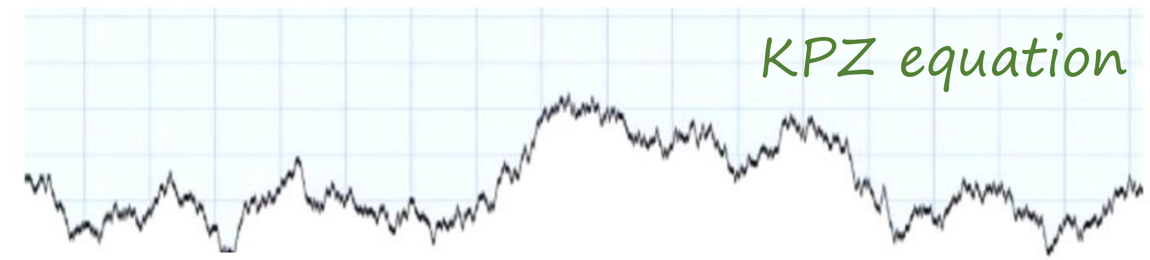




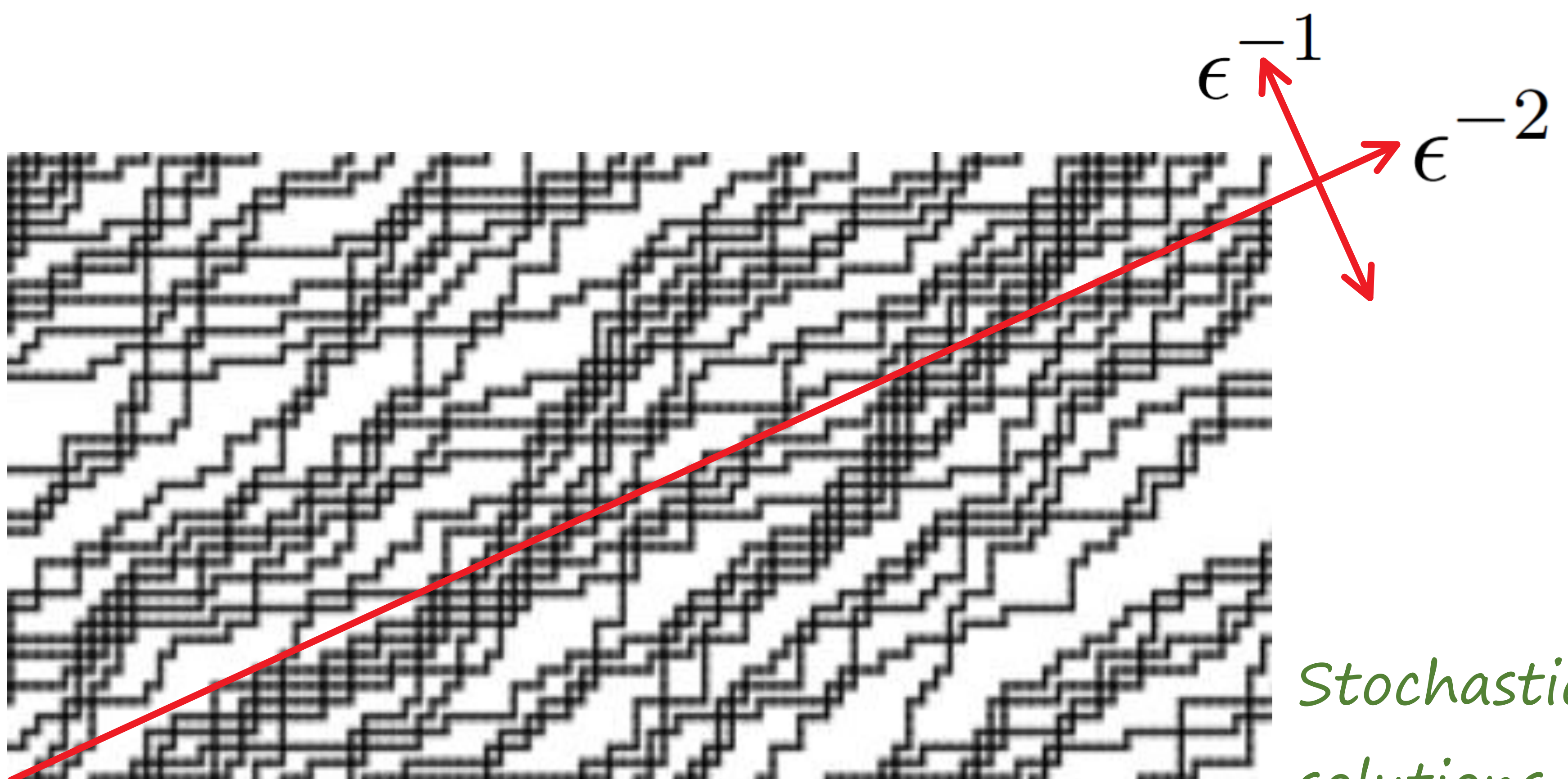
SPDE limit of S6V

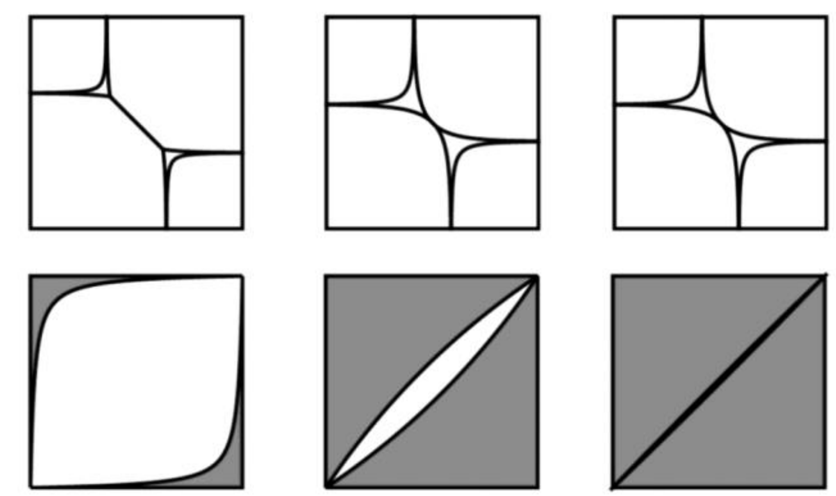
<u>Theorem</u> [C-Ghosal-Shen-Tsai '18]: Let $b_1, b_2 \to b \in (0,1)$ with $\Delta = \dfrac{b_1 + b_2}{2\sqrt{b_1 b_2}} \approx 1 + \epsilon$.

The stationary initial data S6V height function converges (after centering and scaling) along the characteristic directions to the stationary (Brownian initial data) solution to the KPZ equation:



$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\delta\Delta\mathcal{H}(t,x) + \frac{1}{2}\kappa\left(\partial_x \mathcal{H}(t,x)\right)^2 + \sqrt{D}\dot{\mathcal{W}}(t,x).$$





Phase diagram as $\Delta \to 1^+$

Stochastic Gibbs states converge to stationary solutions to the stochastic Burgers equation!

Recap and what's next

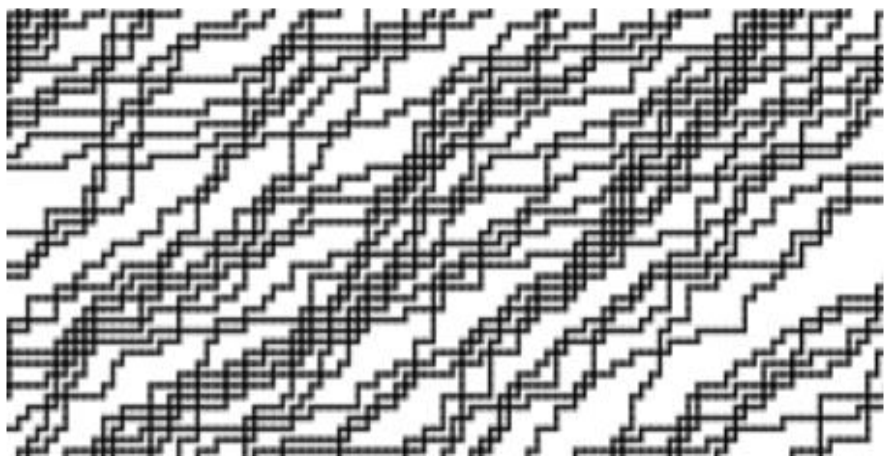
- Gibbs states arise from 6V on torus with external fields. Mapping between field strength and Gibbs state line densities is not simple.
- Disordered states should have GFF and log-correlated fluctuations.
- Stochastic Gibbs states arise at conical point. Fluctuations have 1/3 KPZ exponent in characteristic directions, and the entire field admits a limit when $\Delta \to 1^+$ to the stationary KPZ equation.
- There are other KPZ class / equation convergence results.
- Rest of the talk will focus on two methods (Markov duality and Bethe ansatz) which play important roles in these type of results.

Markov duality

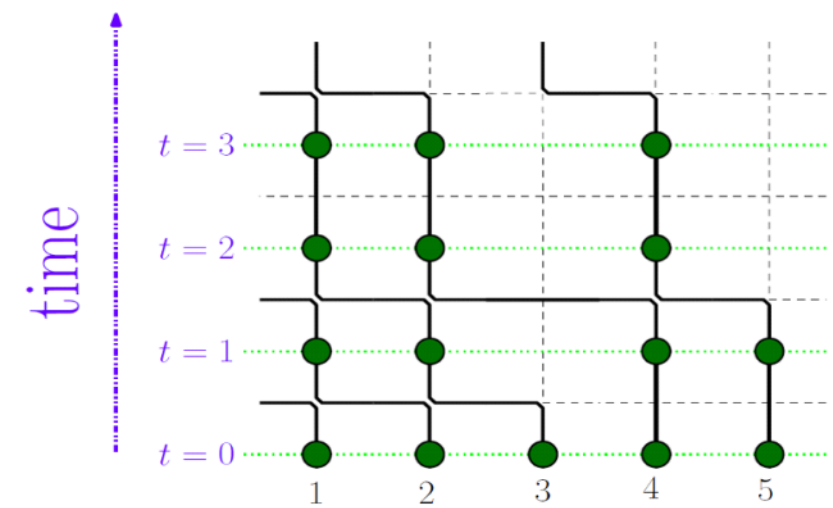
Definition: Two Markov processes $x(t) \in X$ and $y(t) \in Y$ are dual with respect to

$Q : X \times Y \to \mathbb{R}$ if for all $x \in X$, $y \in Y$ and $t \geq 0$: $\mathbb{E}^x\Big[Q\big(x(t), y\big)\Big] = \mathbb{E}^y\Big[Q\big(x, y(t)\big)\Big]$.

$\vec{x}(t)$



$\overleftarrow{y}(t)$

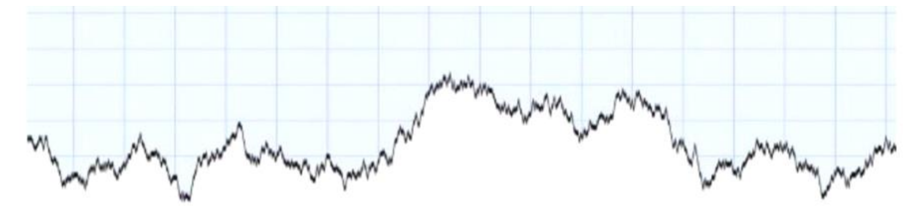


Theorem [C-Petrov '15]: The S6V particle process $\vec{x}(t)$ and the independent, space reversed S6V k-particle process $\overleftarrow{y}(t)$ are dual with respect to $Q(\vec{x}, \overleftarrow{y}) := \prod_{i=1} k\tau^{N_{y_i}(\vec{x})}$,

where $\tau = b_2/b_1 \in (0,1)$ and $N_y(\vec{x}) = \max\{n : x_n(t) \leq y\}$.

Such dualities can be proved directly (as above or in [Borodin-C-Sasamoto '12]...), inductively ([Lin '19]) or based on quantum group symmetries ([Schutz '95], [Carinci-Giardina-Redig-Sasamoto '16], [Kuan '17]...)

Microscopic stochastic heat equation



Definition: The Cole-Hopf solution to the KPZ equation

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\delta\Delta\mathcal{H}(t,x) + \frac{1}{2}\kappa\left(\partial_x\mathcal{H}(t,x)\right)^2 + \sqrt{D}\dot{\mathcal{W}}(t,x)$$

is $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$,where $\mathcal{Z}$ solves the stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\delta\Delta\mathcal{Z}(t,x) + \frac{\kappa}{\delta}\sqrt{D}\mathcal{Z}(t,x)\dot{\mathcal{W}}(t,x)\,.$$

S6V duality implies that $\mathbb{E}\Big[\prod_{i=1}^{k}\tau^{N_{y_i}(\vec{x}(t))}\Big] = \sum_{\overleftarrow{y}'\in\mathbb{Y}^k}\mathbb{P}_{\overleftarrow{S6V}}(\overleftarrow{y}\to\overleftarrow{y}';t)\prod_{i=1}^{k}\tau^{N_{y_i'}(\vec{x}(0))}$.

--> $\tau^{N_y(\vec{x}(t))}$ solves a discrete SHE with an explicit martingale whose quadratic variation involves the k=2 duality function.

Key challenge in convergence to SHE is to control the martingale.

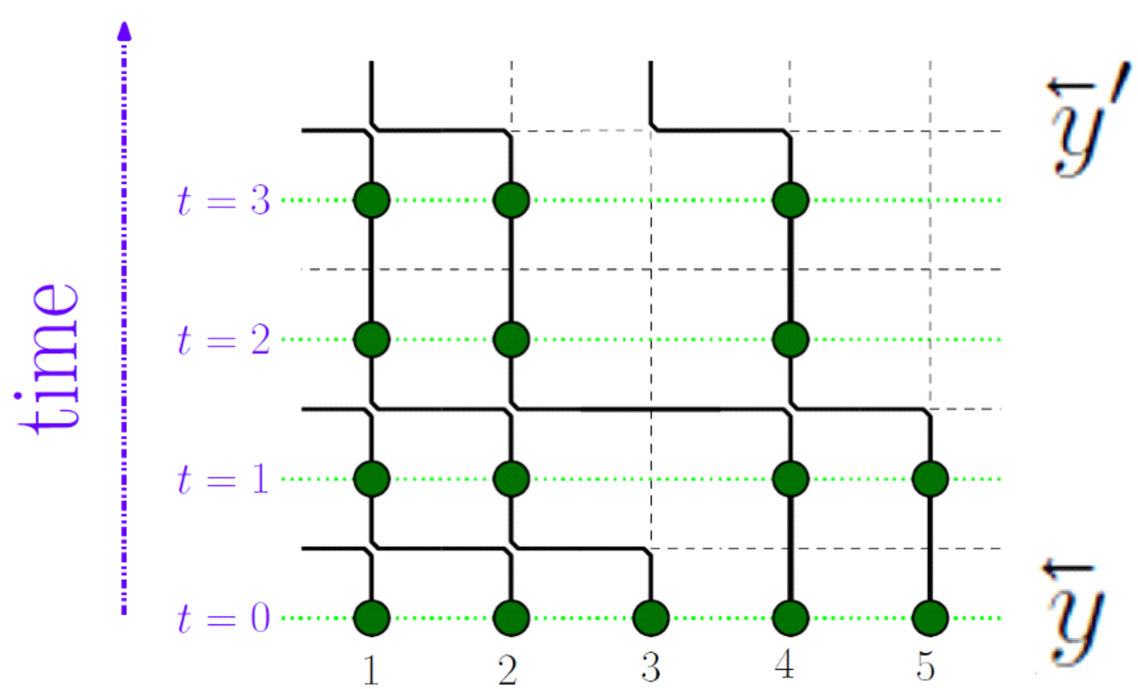
- [Bertini-Giacomin '95] does this for ASEP via complicated identity (doesn't work for S6V).
- [C-Ghosal-Shen-Tsai '17] uses 2-particle duality and Bethe ansatz.

(Coordinate) Bethe ansatz

[Borodin-C-Gorin '14]: Explicit formulas for transition probabilities for k-particle S6V (in spirit of [Tracy-Widom '07] and [Lieb '67])

$$\mathbb{P}_{\overleftarrow{S6V}}(\overleftarrow{y} \to \overleftarrow{y}'; t) = \frac{1}{(2\pi \mathbf{i})^k} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \prod_{1 \le i < j \le k} \mathfrak{C}(z_i, z_j, \sigma) \prod_{i=1}^{k} z_{\sigma(i)}^{y_i - y'_{\sigma(i)} - 1} \mathfrak{D}(z_i, t) dz_i$$

where $\mathfrak{C}(z_i, z_j, \sigma) := \frac{1 - (1 + \tau^{-1}) z_{\sigma(i)} + \tau^{-1} z_{\sigma(i)} z_{\sigma(j)}}{1 - (1 + \tau^{-1}) z_i + \tau^{-1} z_i z_j}$ and $\mathfrak{D}(z, t) := \left(\frac{b_1 + (1 - b_1 - b_2) z^{-1}}{1 - b_2 z^{-1}} \right)^t$.



Explicit formulas like these are also starting points for KPZ universality asymptotics

Plancherel theory

Left/right eigenfunctions diagonalize k-particle S6V transition matrix:

For $\vec{y} = (y_1 > \cdots > y_k)$ and $\vec{z} = (z_1, \ldots, z_k) \in \left(\mathcal{C} \setminus \{1, \tau^{-1}\}\right)^k$ define

$$\psi^{\ell}_{\vec{z}}(\vec{y}) = \sum_{\sigma \in S_k} \prod_{1 \le b < a \le k} \frac{z_{\sigma(a)} - \tau z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1 - \tau^{-1} z_{\sigma(j)}}{1 - z_{\sigma(j)}} \right)^{y_j},$$

and

$$\psi^{r}_{\vec{z}}(\vec{y}) = \sum_{\sigma \in S_k} \prod_{1 \le b < a \le k} \frac{z_{\sigma(a)} - \tau^{-1} z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1 - \tau^{-1} z_{\sigma(j)}}{1 - z_{\sigma(j)}} \right)^{-y_j}.$$

Plancherel theory [Borodin-C-Petrov-Sasamoto '14]:

The forward transform $(\mathcal{F}f)(\vec{z}) := \sum_{y_1 > \cdots > y_k} f(\vec{y}) \psi^{r}_{\vec{z}}(\vec{y})$ and the backward transform

$$(\mathcal{J}g)(\vec{y}) := \mathrm{const} \cdot \oint^{|w_j| = R \in (1, \tau^{-1}),\, 1 \le j \le k} \cdots \oint \det\left[\frac{1}{\tau w_i - w_j} \right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{(1 - w_j)(1 - \tau^{-1} w_j)} \psi^{\ell}_{\vec{w}}(\vec{y}) g(\vec{w}) d\vec{w}$$

are mutual inverses so that $\mathcal{F}\mathcal{J} = \mathrm{Id}$ and $\mathcal{J}\mathcal{F} = \mathrm{Id}$.

Some extensions

- Symmetric functions:
  - [Borodin '14], [Borodin-Petrov '16], [Borodin-Wheeler '17] prove Cauchy identities, Pieri and branching rules for 'spin Hall-Littlewood' and 'spin q-Whittaker' functions.
- Fusion:
  - [C-Petrov '15] (building on [Kulish-Reshetikhin-Sklyanin '81] and [Mangazeev '14]) introduce higher-spin stochastic vertex models (Beta RWRE arises from this).
- Elliptic:
  - [Borodin '16] lifts to elliptic level 'dynamic S6V' and 'dynamic ASEP'. [Borodin-C '17] prove duality for dynamic ASEP and [C-Ghosal-Matetski '19] prove SPDE limit. [Aggarwal '16] gives higher-spin dynamic models.
- High rank:
  - [Kuniba-Mangazeev-Maruyama-Okado '16] introduce high rank models and [Kuan '17] proves their duality. [Borodin-Wheeler '18] develop their symmetric function theory.
- And much more…

Summary

- Six vertex model has an interesting (mostly conjectural) phase diagram.
- Disordered Gibbs states are expected to have GFF fluctuations.
- Using the stochastic six vertex model, we can construct the one-parameter family of stochastic Gibbs states which arise at the conical point in the phase diagram.
- Stochastic Gibbs states show KPZ universality class fluctuations along their characteristic direction, and when $\Delta \to 1^+$ they converge to the stationary solutions to the KPZ equation.
- Duality and Bethe ansatz are key tools in proving both results.

Stochastic six vertex model

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Goals of second hour

- Prove a stochastic heat equation (SHE) Laplace transform formula:

$$\mathbb{E}_{\mathrm{SHE}}\left[e^{-u\mathcal{Z}(2\tau,0)e^{\tau/12}}\right] = \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty} \frac{1}{1+ue^{\tau^{1/3}\mathbf{A}_k}}\right]$$

  where $\mathcal{Z}(t,x)$ is the fundamental solution to the SHE:

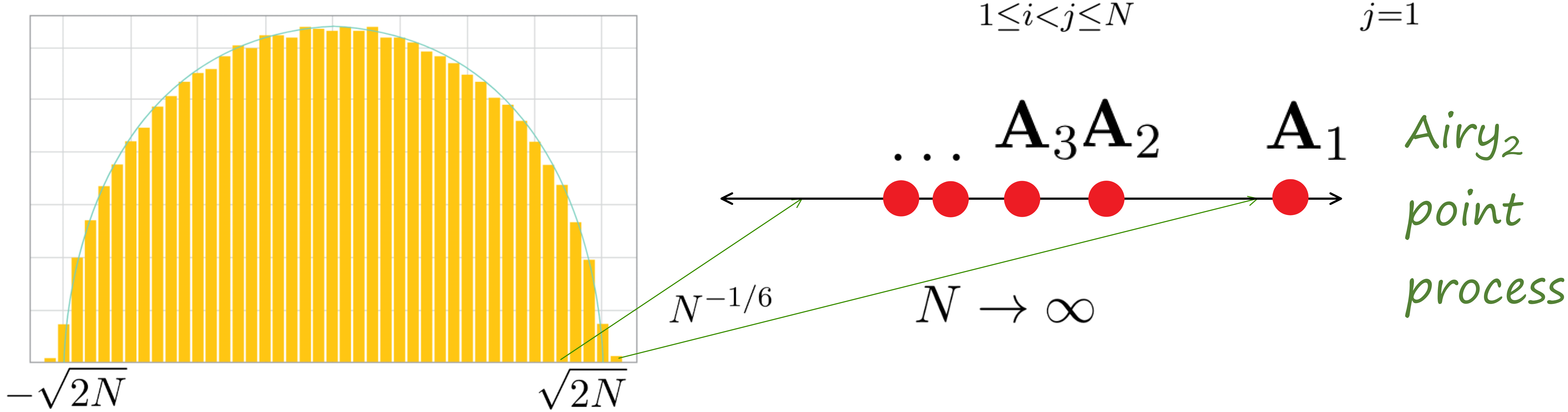
$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\Delta\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\dot{\mathcal{W}}(t,x), \qquad \mathcal{Z}(0,x) = \delta_{x=0}$$

  and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots,$ is the Airy$_2$ point process.

- **Tomorrow**: Tsai will use this identity as the parting point to derive large deviations and tails for the KPZ equation.
- **Today**: We will derive this result using a combination of two tools:
  - Yang-Baxter equation and Macdonald processes

Airy$_2$ point process

- Edge limit of GUE: $\mathbb{P}_{\mathrm{GUE}(N)}(\mathbf{A}_1,\ldots,\mathbf{A}_N) \propto \prod_{1\le i<j\le N} (\mathbf{A}_i - \mathbf{A}_j)^2 \prod_{j=1}^{N} e^{-\mathbf{A}_j^2}\, \mathrm{d}\mathbf{A}_j$



- Airy$_2$ point process has other characterizations:
  - A determinantal point process with a simple explicit kernel.
  - The spectrum of the stochastic Airy operator.
  - The limit of many other point processes, e.g. the Schur measure that we will encounter later.

How to prove the identity?

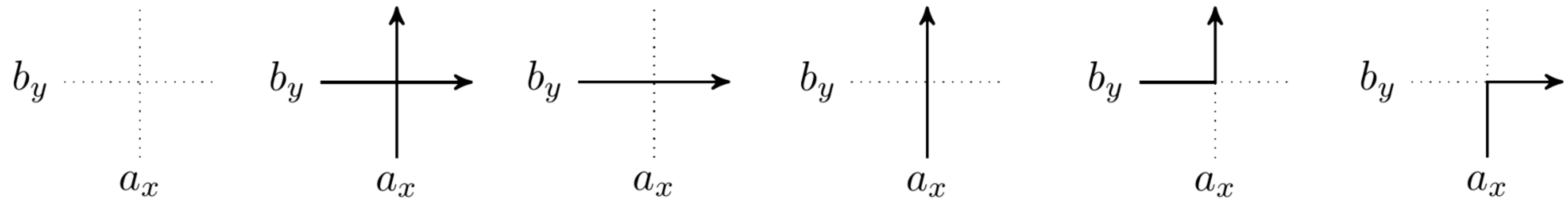
$$\mathbb{E}_{\mathrm{SHE}}\left[e^{-u\mathcal{Z}(2\tau,0)e^{\tau/12}}\right] = \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty}\frac{1}{1+ue^{\tau^{1/3}\mathbf{A}_k}}\right]$$

- [Borodin-Gorin '16] proved this as an easy corollary of the SHE Fredholm determinant formula [Sasamoto-Spohn '10], [Calabrese-Le Doussal-Rosso '10], [Dotsenko '10], [Amir-C-Quastel '10].
- Where does such a formula come from? Can compute SHE moments via 'replica trick' (a version of Markov duality). Moments DO NOT determine the distribution of the SHE, so that route is not rigorous!
- Proof: We lift to a discrete regularization, the S6V model and use a non-trivial relationship between S6V and measures on partitions.

There are other approaches (including duality) I will not discuss...

Inhomogeneous stochastic six vertex model

- Consider an inhomogeneous version of S6V with weights

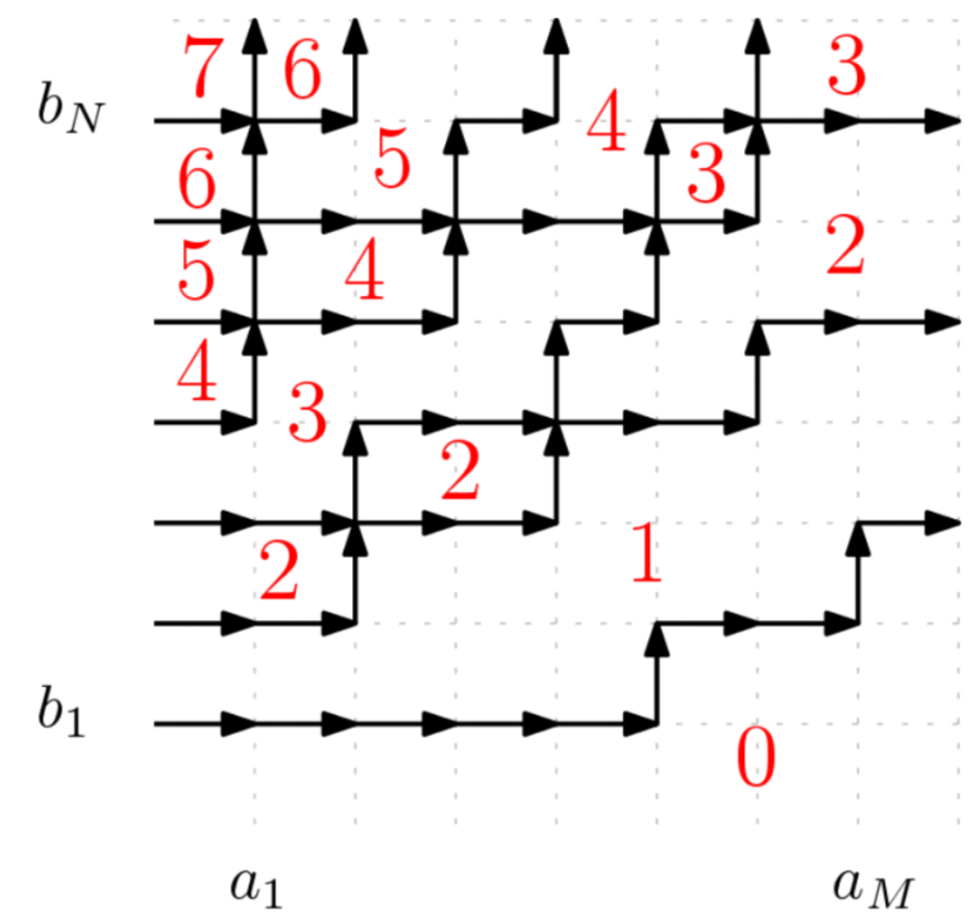


$$1 \qquad 1 \qquad \frac{1-a_xb_y}{1-ta_xb_y} \qquad \frac{t(1-a_xb_y)}{1-ta_xb_y} \qquad \frac{(1-t)a_xb_y}{1-ta_xb_y} \qquad \frac{1-t}{1-ta_xb_y}$$

with a_x for column x and b_y is for row y such that a_xb_y and t in [0,1).

Fix step initial data and define a height function H(x,y) as shown

(this picture is flipped versus earlier for convenience)



Recall $(a;t)_n := \prod_{i=0}^{n-1}(1-at^i)$

A lifting of the identity

$$\mathbb{E}_{\mathrm{S6V}}\left[\frac{1}{(-ut^{H(x,y)};t)_\infty}\right] \stackrel{\mathbf{2}}{=\!=} \mathbb{E}_{\mathrm{HL}}\left[\frac{1}{(-ut^{\ell(\lambda)};t)_\infty}\right] \stackrel{\mathbf{3}}{=\!=} \mathbb{E}_{\mathrm{Schur}}\left[\prod_{k\notin\{\lambda_i-i\}_{i\geq 1}} \frac{1}{1+ut^{k+x}}\right]$$

$\mathbf{1}$ $\downarrow$ Limit as $t \nearrow 1$ and $x,y \nearrow \infty$ $\downarrow$ $\mathbf{4}$

$$\mathbb{E}_{\mathrm{SHE}}\left[e^{-u\mathcal{Z}(2\tau,0)e^{\tau/12}}\right] \longleftrightarrow \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty}\frac{1}{1+ue^{\tau^{1/3}a_k}}\right]$$

The identity!

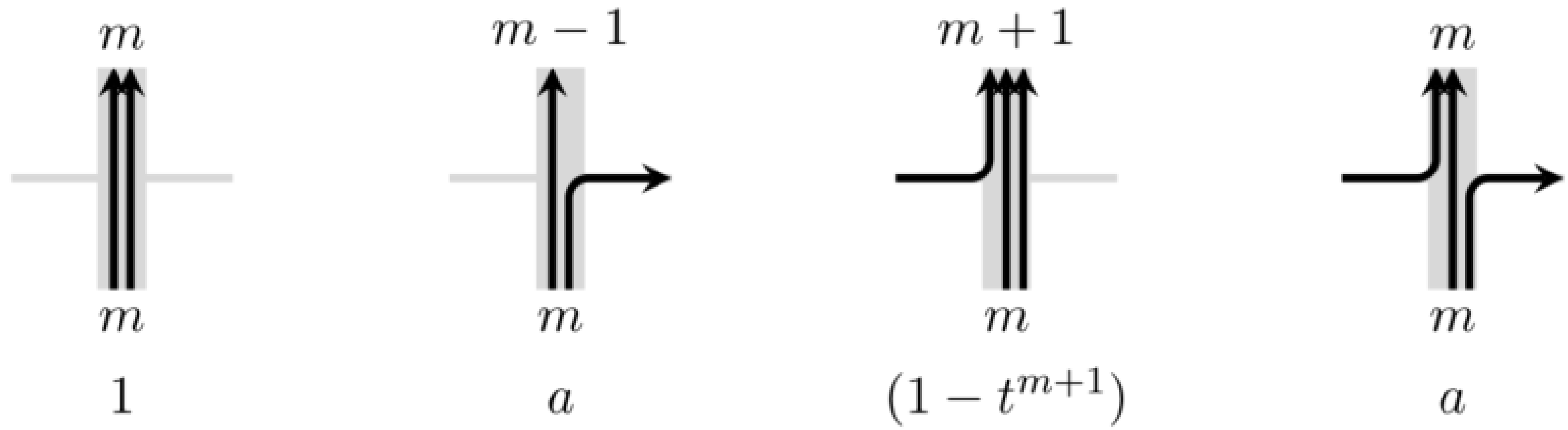
Limit 1: Convergence result from S6V to KPZ/SHE that we already saw along with limit of the Pochhammer symbol to an exponential.

Identities 2 and 3: We will focus on these. Identity 2 uses ideas from [Betea-Wheeler-Zinn-Justin 14] + [Borodin-Bufetov-Wheeler 16].

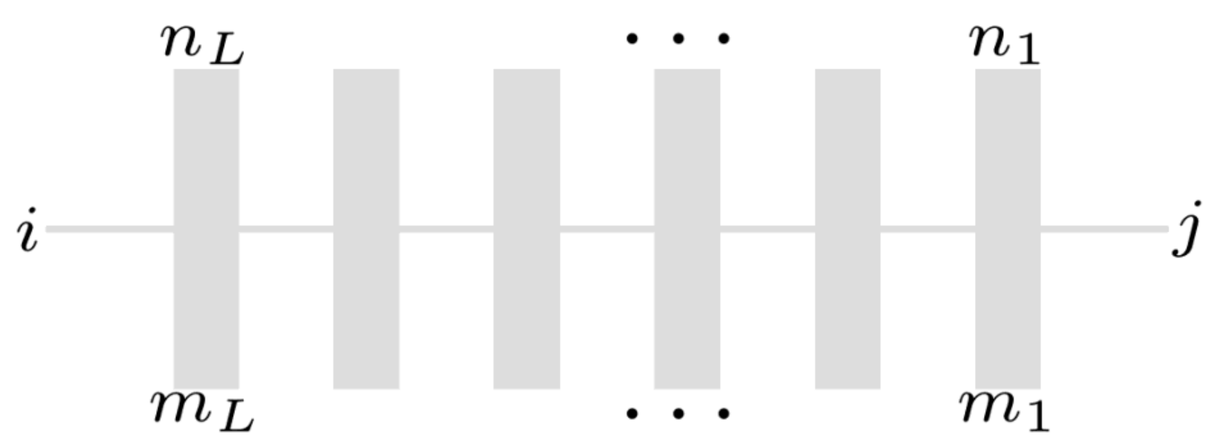
Limit 4: Follows from convergence of explicit determinantal kernels.

t-Boson vertex model

Consider the following vertex weights [Tsilevich '06] with arbitrary vertical lines and 0/1 horizontal lines, subject to line conservation:



Here t and a or in [0,1). We can put together weights like this



where the internal line configurations are summed over.

Skew Hall-Littlewood polynomials

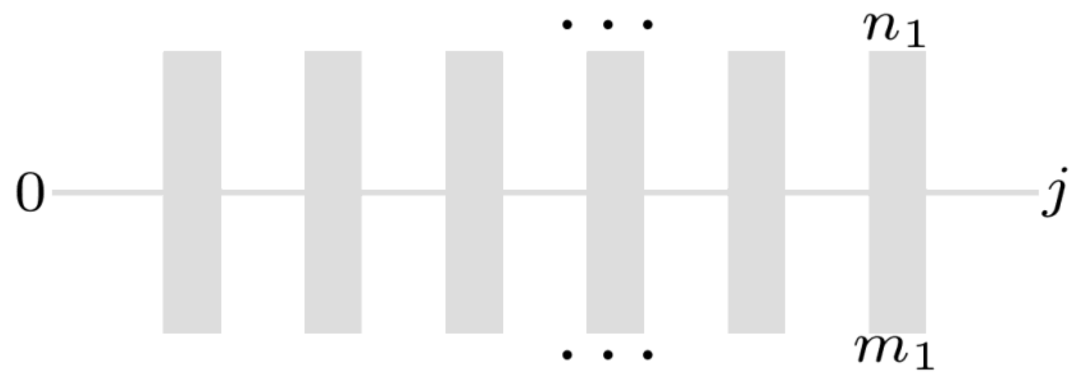
Taking $L \to \infty$, the weight of 

is non-zero only when there is no incoming arrow on the right.

Partition notation:

$\lambda = 1^{m_1} 2^{m_2} \cdots$

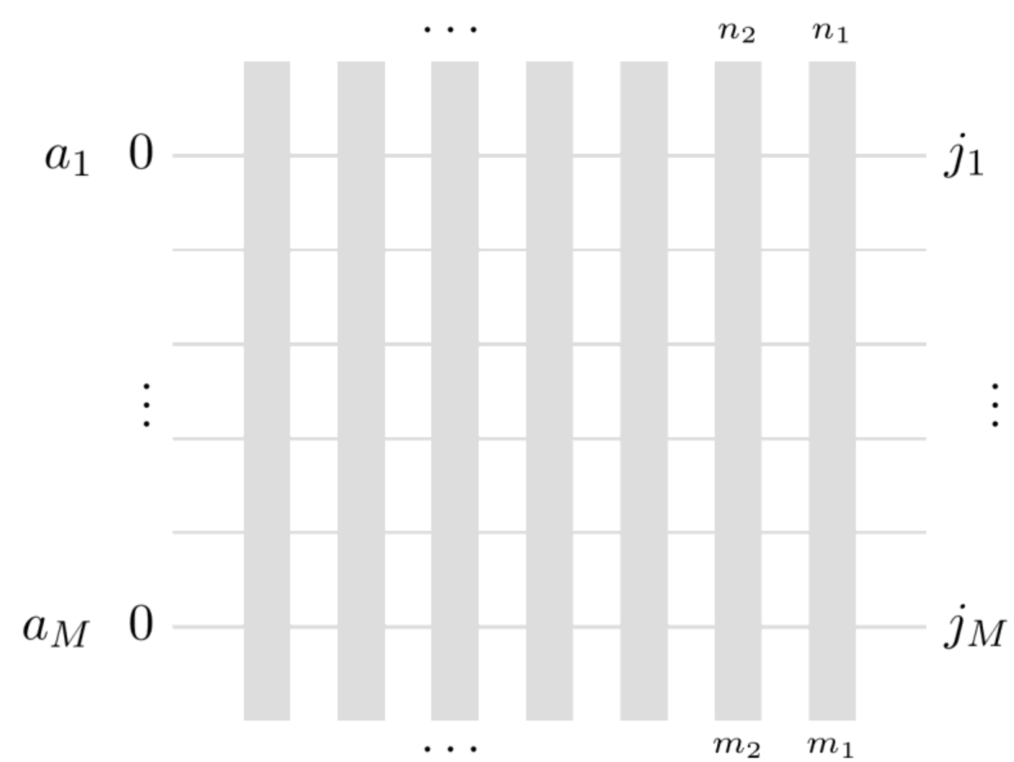
$\mu = 1^{n_1} 2^{n_2} \cdots$



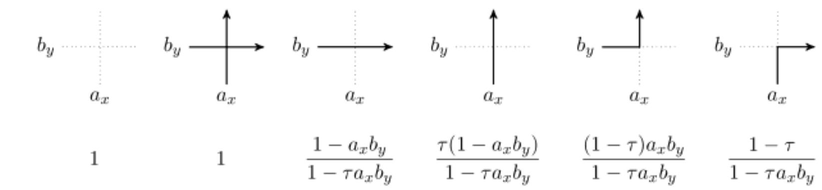
Line conservation implies partition lengths satisfy $\ell(\lambda) = \ell(\mu) + 1$.

Skew Hall-Littlewood polynomial $P_{\lambda/\mu}(a)$ = weight from λ to μ .

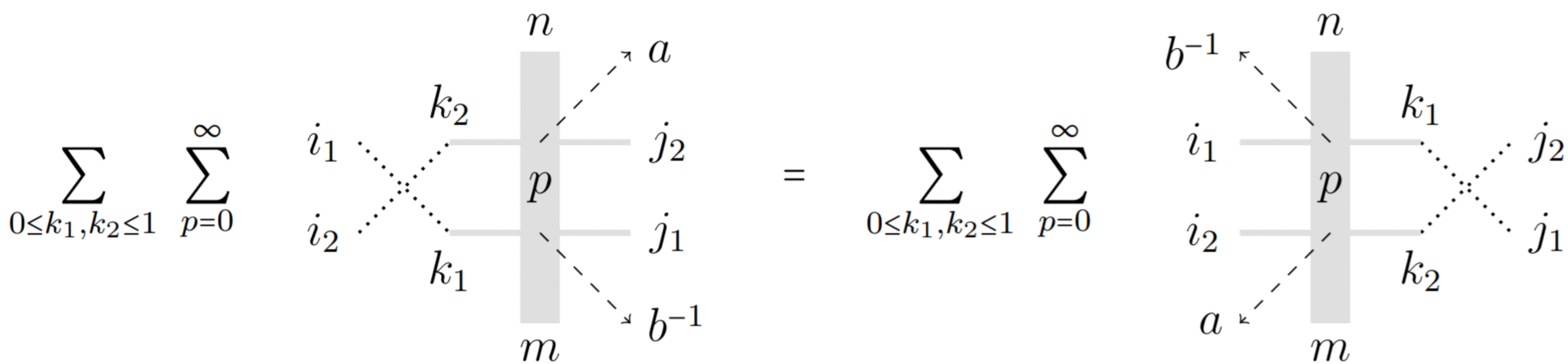
Multivariable skew Hall-Littlewood polynomial $P_{\lambda/\mu}(a_1, \ldots a_M)$ involves stacking rows with variables $a_1,\ldots,a_M$ and summing weights over all possible internal configurations and $j_1,\ldots,j_M$.



Yang-Baxter equation



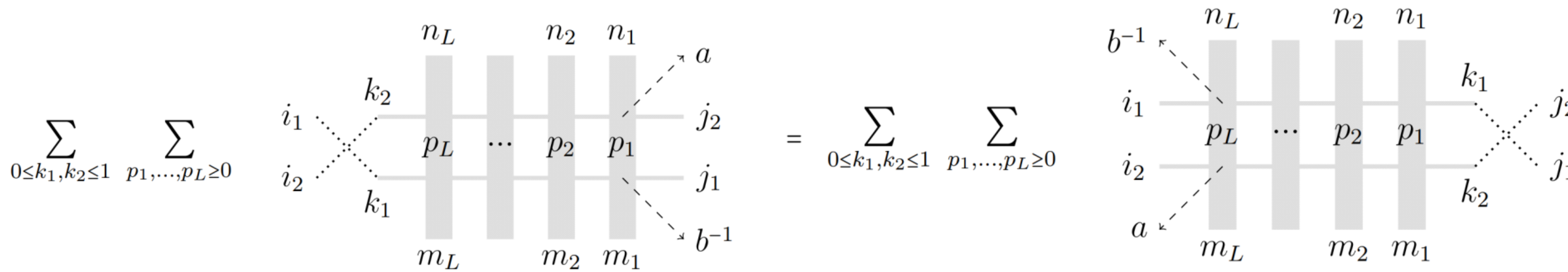
The relationship between Hall-Littlewood polynomials and the S6V model can be seen from the Yang-Baxter equation:



External lines are fixed and internal lines are summed over.

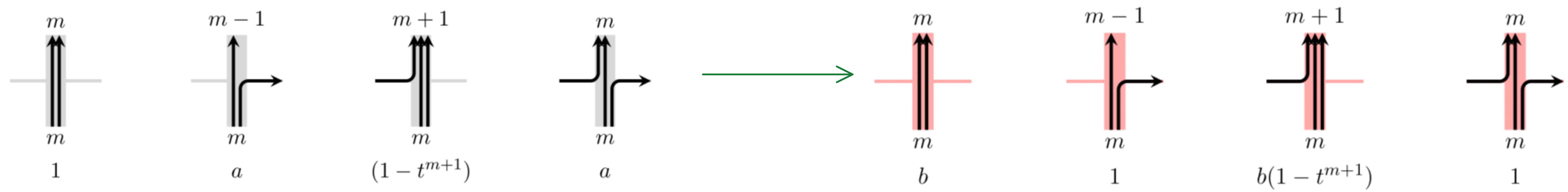
Spectral variables flip and the X's are S6V vertices rotated by 45°.

Iterating:

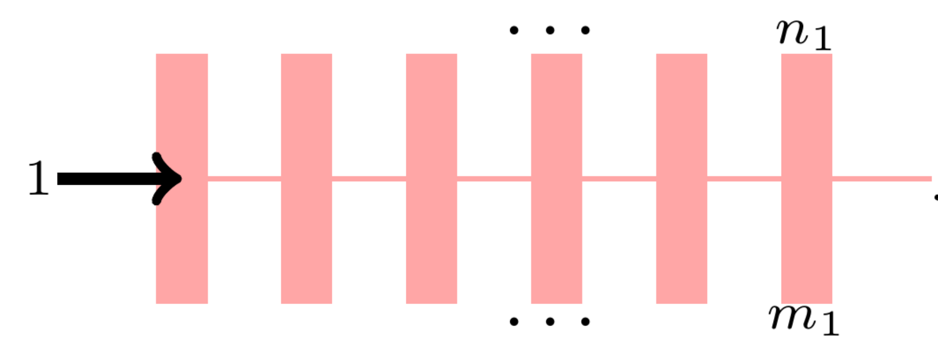


Q polynomials and limiting Yang-Baxter equation

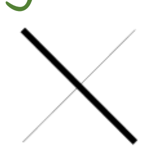
YBE involves a and b^{-1}. We introduce a dual set of t-Boson weights (in salmon) by replacing a by b^{-1} and multiplying through by b.

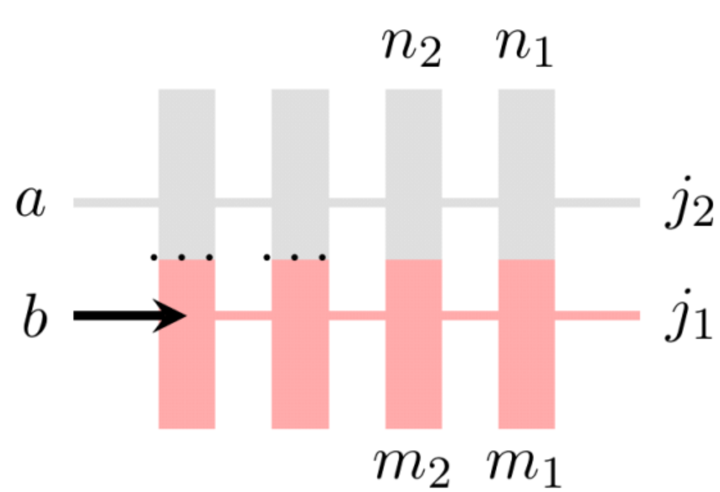


Composing and take $L \to \infty$ only non-zero with right incoming line:



If we take the $L \to \infty$ limit of the YBE, we arrive at:

Weight of  → $\left(\frac{1-ab}{1-tab}\right)$



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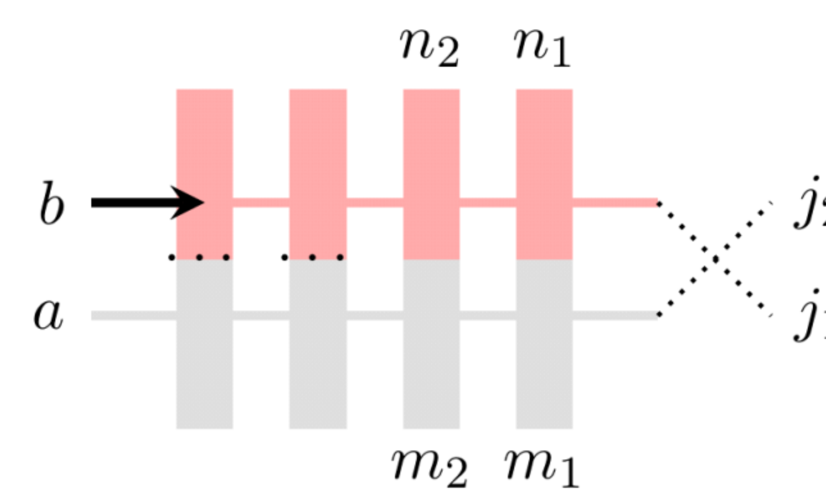


Hall-Littlewood process and Cauchy identity

Fixing parameters $a_1,\ldots,a_M$ and $b_1,\ldots b_N$, define the Hall-Littlewood process to be the probability measure on sequences of partitions:

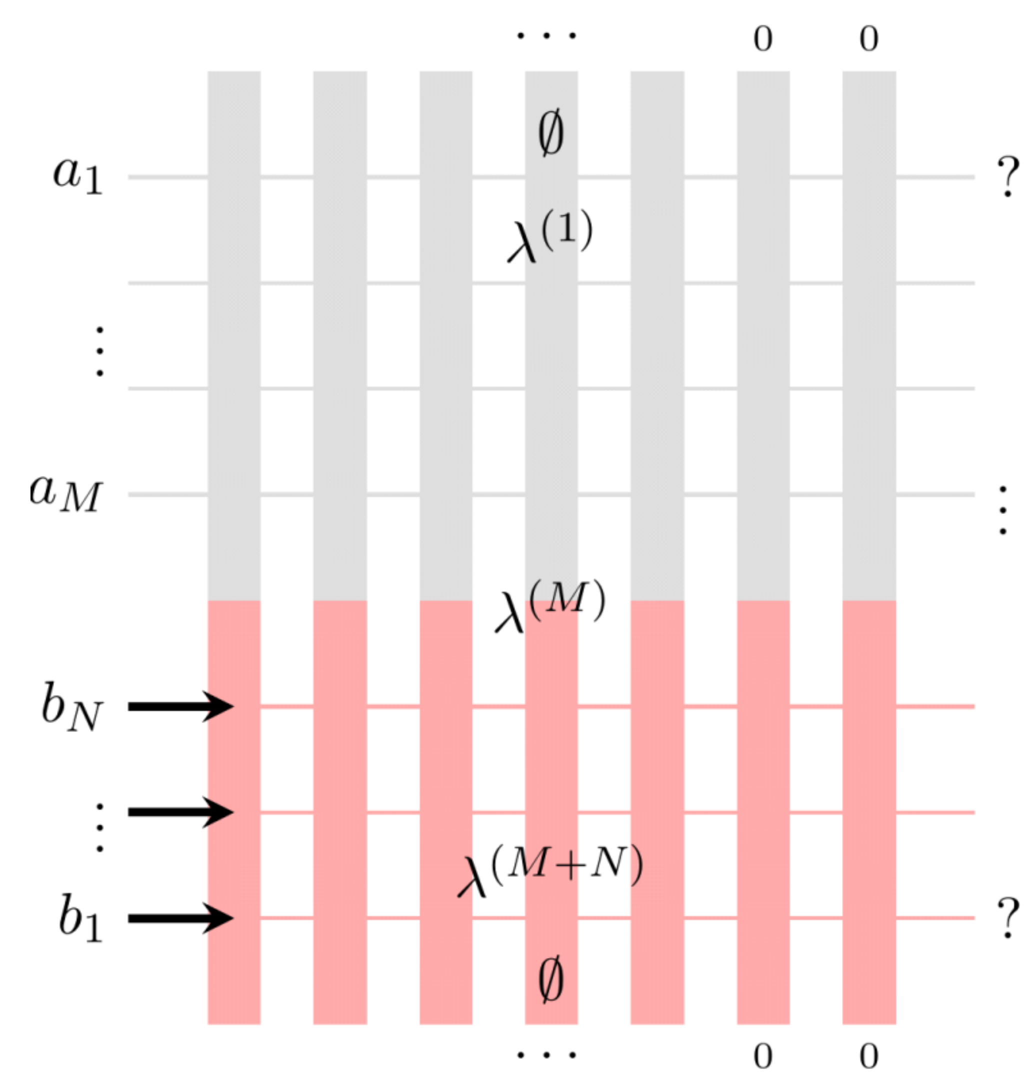
$$\mathbb{P}(\lambda^{(1)},\ldots,\lambda^{(M+N)}) = \frac{P_{\lambda^{(1)}}(a_1)P_{\lambda^{(2)}/\lambda^{(1)}}(a_2)\cdots P_{\lambda^{(M)}/\lambda^{(M-1)}}(a_M)Q_{\lambda^{(M)}/\lambda^{(M+1)}}(b_N)\cdots Q_{\lambda^{(M+N)}}(b_1)}{Z(a;b)}$$

Normalization is given by the Cauchy-Littlewood identity:

$$Z(a,b) = \prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1-ta_ib_j}{1-a_ib_j}$$

In terms of the t-Boson vertex model, the Hall-Littlewood process is the weight of this configuration:

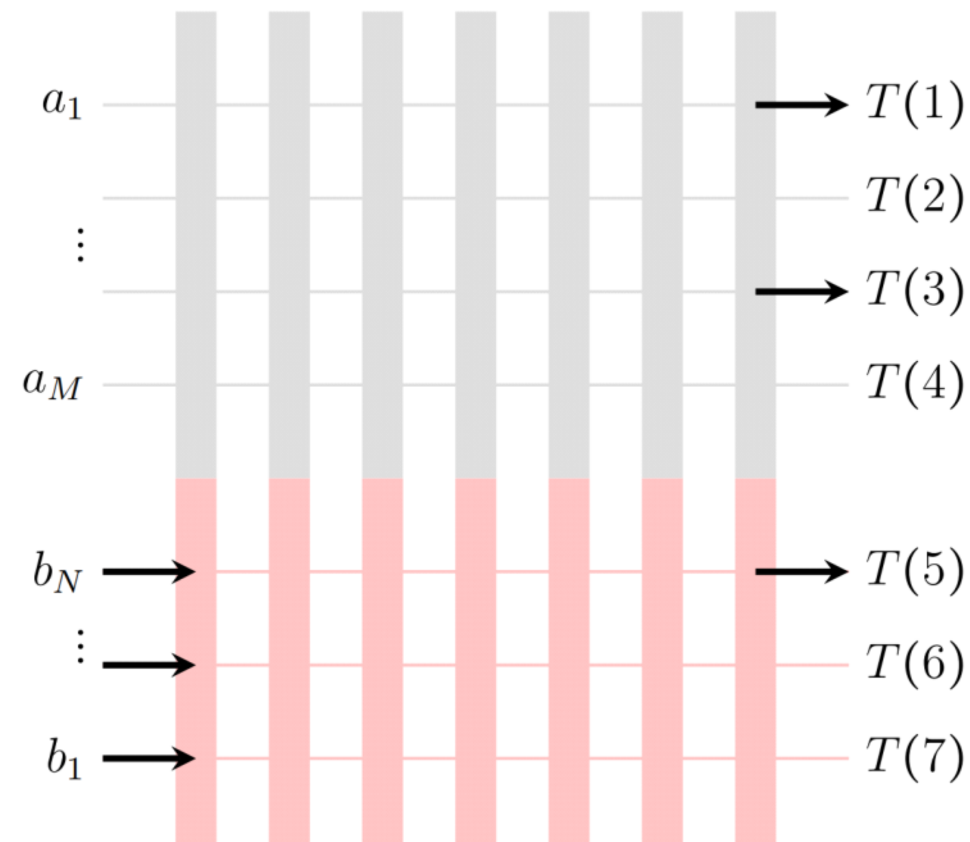
?'s relate to change in length of partitions.



Hall-Littlewood process lengths

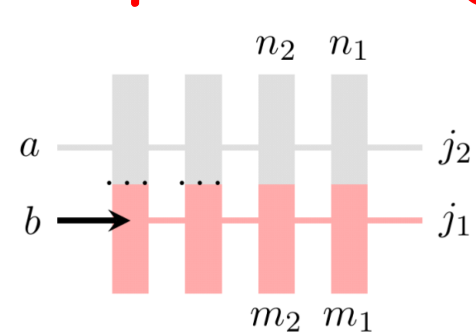
Consider the probability of seeing lengths T(1),…,T(M+N) under the Hall-Littlewood process. This equals the sum of weights of all configurations like

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\left(\frac{1-a_ib_j}{1-ta_ib_j}\right)\times$$



Now, use the YBE to sequentially swap b and a rows:

$$\left(\frac{1-ab}{1-tab}\right)$$



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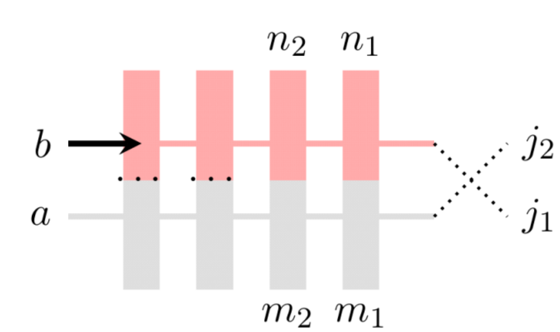
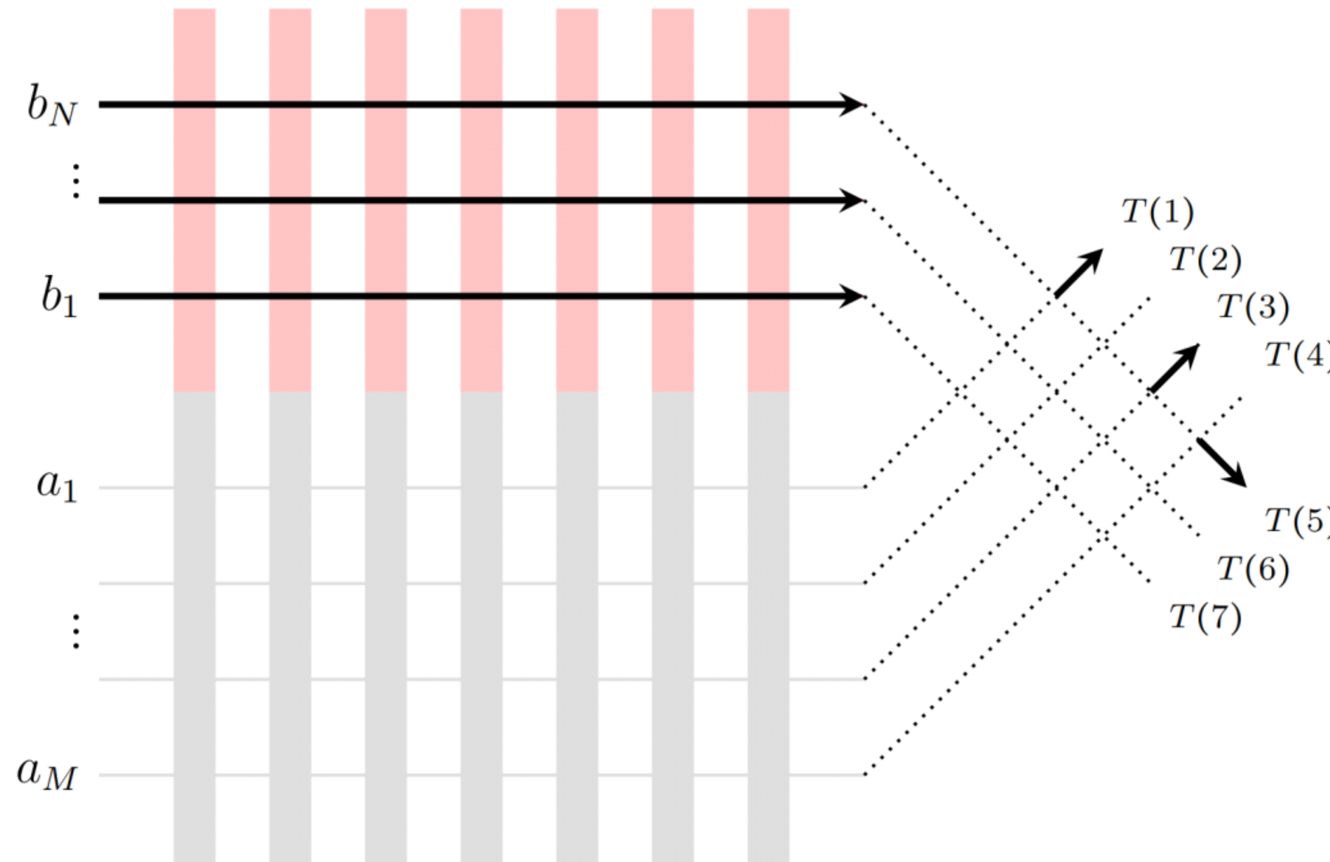


Enter the S6V height function

After swapping the order of all a's and b's, the normalizing constant has been fully absorbed and we are left equality to the weight of



The weight of the left side of the picture is 1 and the right side weight is precisely the probability of seeing the given sequence of output lines for the step initial data S6V model.

We have proved identity 2

Recall identity 2: $\mathbb{E}_{\mathrm{S6V}}\left[\frac{1}{(-ut^{H(x,y)};t)_\infty}\right] \overset{\mathbf{2}}{=\!=} \mathbb{E}_{\mathrm{HL}}\left[\frac{1}{(-ut^{\ell(\lambda)};t)_\infty}\right]$

- We used the Yang-Baxter equation to match the distribution of the Hall-Littlewood process lengths to the distribution of the output lines for the S6V model. Identity 2 follows readily from this.
- The marginal of the Hall-Littlewood measure on a single intermediate partition is the Hall-Littlewood measure on a partition

$$\mathbb{P}(\lambda) = \frac{P_\lambda(a_1,\ldots,a_M)Q_\lambda(b_1,\ldots,b_N)}{Z(a;b)}$$

with Hall-Littlewood polynomials P and Q, and same Z as before.

Identity 3: Enter the Macdonald measure

Replacing P and Q by Macdonald symmetric polynomials depending on q and t in [0,1), [Borodin-C '11] define the Macdonald measure:

$$\mathbb{P}_{q,t}(\lambda) = \frac{P_\lambda(a_1,\ldots,a_M;q,t)Q_\lambda(b_1,\ldots,b_N;q,t)}{Z(a;b)}$$

The normalization is now given by the Cauchy-Littlewood identity:

$$\sum_\lambda P_\lambda(a_1,\ldots,a_M)Q_\lambda(b_1,\ldots,b_N) = Z_{q,t}(a,b) = \prod_{i=1}^{M}\prod_{j=1}^{N}\frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty}$$

Macdonald polynomials have many degenerations, including

- Hall-Littlewood polynomials when q=0,
- Schur polynomials when q=t.

q-independence

Define the Macdonald difference operator which act the a-variables

$$D_M^u = \sum_{I\subset\{1,\ldots,M\}} t^{|I|(|I|-1)/2} \prod_{i\in I, j\notin I} \frac{ta_i - a_j}{a_i - a_j} \prod_{i\in I} T_{q,i} \quad .$$

Here, u is arbitrary and $T_{q,i}$ shifts a_i to qa_i. The eigenrelation

$$D_M^u P_\lambda(a;q,t) = e_\lambda(u,q,t) P_\lambda(a;q,t) \text{ with } \quad e_\lambda(u,q,t) = \prod_{i=1}^{M} (1+uq^{\lambda_i} t^{M-i})$$

not only defines the polynomials, but also enables us to calculate

$$\mathbb{E}_{q,t}\left[e_\lambda(u,q,t)\right] = \frac{D_M^u Z(a,b;q,t)}{Z(a,b;q,t)} \quad .$$

It is easy to see that the right side is, in fact, q-independent!

Equating q=0 (Hall-Littlewood) and q=t (Schur) yields identity 3.

Recap

$$\mathbb{E}_{\mathrm{S6V}}\left[\frac{1}{(-ut^{H(x,y)};t)_\infty}\right] \overset{\mathbf{2}}{=\!=} \mathbb{E}_{\mathrm{HL}}\left[\frac{1}{(-ut^{\ell(\lambda)};t)_\infty}\right] \overset{\mathbf{3}}{=\!=} \mathbb{E}_{\mathrm{Schur}}\left[\prod_{k\notin\{\lambda_i-i\}_{i\geq 1}}\frac{1}{1+ut^{k+x}}\right]$$

$\mathbf{1}\ \downarrow$ Limit as $t\nearrow 1$ and $x,y\nearrow\infty$ $\downarrow\ \mathbf{4}$

$$\mathbb{E}_{\mathrm{SHE}}\left[e^{-u\mathcal{Z}(2\tau,0)e^{\tau/12}}\right] \longleftrightarrow \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty}\frac{1}{1+ue^{\tau^{1/3}a_k}}\right]$$

The identity!

To prove an identity between SHE and Airy, we found an identity between two discrete regularizations, S6V and Schur measure.

- Identity 2 relates S6V to the Hall-Littlewood process using the Yang-Baxter equation for t-Bosons.
- Identity 3 relate the Hall-Littlewood and Schur measure using a further lifting to Macdonald measures.

Some extensions

- Hall-Littlewood RSK:
  - [Bufetov-Matveev '17] provide a Markov chain on interlacing partitions which preserves that class of Hall-Littlewood processes and whose marginal on lengths is the S6V model.
- Yang-Baxter fields and bijectivization:
  - [Bufetov-Petrov '17] and [Bufetov-Mucciconi-Petrov '19] use Yang-Baxter equation to construct other Markov chains like above, also for higher-spin models.
- Half-space:
  - [Barraquand-Borodin-C-Wheeler '17] provide relation between half-space versions of the S6V and Hall-Littlewood process (special case of half-space Macdonald process [Barraquand-Borodin-C '18]) and prove half-space identity and asymptotics.
- Gibbsian line ensembles
  - [C-Dimitrov '17] interpret S6V and Hall-Littlewood relationship via Gibbsian line ensembles [C-Hammond '11] and prove predicted KPZ 2/3 transversal exponent.
- And much more...

Summary of three lectures

- What did I do
  - Lecture 1: Conjured and 'solved' the Beta RWRE out of thin air.
  - Lecture 2: 'Solved' S6V via Bethe ansatz diagonalization and Markov duality.
  - Lecture 3: Revealed a key source of solvability, the Yang-Baxter equation, and connected vertex models to symmetric function measures on partitions.
- What didn't I do?
  - Lots! For example, how does the Beta RWRE arises from S6v? From where does duality come? How does one do actually perform asymptotics?...
- What will other people do at this program?
  - Imamura: Higher-spin models and another other routes to get Fredholm determinant formulas for measures in the Macdonald hierarchy.
  - Tsai: How to use the identity we discussed in this lecture to prove KPZ equation tail and large deviation results.
  - Basu: How inputs from integrable probability inform geometric problems in LPP.