# Exceptional points of two-dimensional random walks at multiples of the cover time

Yoshihiro Abe (Chiba University)

The 12th MSJ-SI
Stochastic Analysis, Random Fields and Integrable Probability
August 1, 2019

Joint work with Marek Biskup (UCLA)

# Abstract

We have studied the statistics of
exceptional points for 2D SRW such as

- Avoided points (i.e. points not visited at all, late points)
- Thick points (i.e. heavily visited sites)
- Thin points (i.e. lightly visited sites)
- Light points (i.e. points where the local time is $O(1)$)

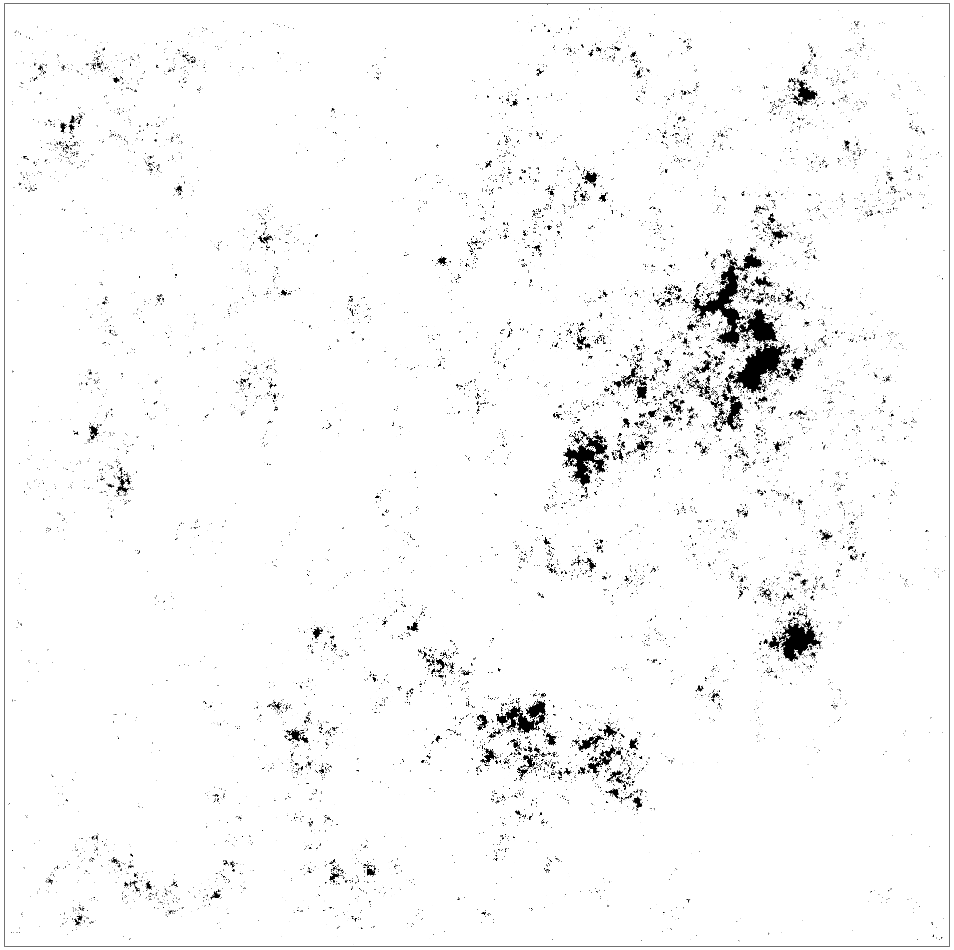
In this talk, we will focus on avoided points.

cf. Okada's talk (tomorrow)

# Figure: Avoided points (Simulation by Marek Biskup)

$2000 \times 2000$ square, run-time $= 0.3\times$ (cover time)

Note: Cover time is the first time at which the SRW visits every vertex.



# SRW on $D_N$ with wired boundary condition

$\boldsymbol{D} \subset \mathbb{R}^{\mathbf{2}}$ : "good" bounded open set

$\boldsymbol{D_N} \subset \mathbb{Z}^{\mathbf{2}}$ : "good" lattice approximation of $D$

$\boldsymbol{x} \in \boldsymbol{D_N} \Rightarrow \frac{\boldsymbol{x}}{\boldsymbol{N}} \in \boldsymbol{D}$

$(\boldsymbol{X_t})_{\boldsymbol{t \geq 0}}$: Continuous-time SRW on $\boldsymbol{D_N}$ with
Exp$(1)$-holding times

Technical Assumption: When $\boldsymbol{X}$ exits $\boldsymbol{D_N}$, it re-enters $\boldsymbol{D_N}$ through a uniformly-chosen boundary edge.

$\rightsquigarrow$ Regard $\boldsymbol{\partial D_N}$ as a single point $\boldsymbol{\rho}$

We assume this to relate our local times to DGFF with zero boundary conditions via the 2nd Ray-Knight theorem.

## Local time $L_t^{D_N}$

Recall: $(X_t)_{t\geq 0} =$ SRW on $D_N$, $\rho =$ the boundary vertex

Local time:

$$L_t^{D_N}(x) := \int_0^{\tau_\rho(t)} 1_{\{X_s=x\}} ds \frac{1}{\deg(x)},$$

where

$$\tau_\rho(t) := \inf\left\{ s \geq 0 : \int_0^s 1_{\{X_r=\rho\}} dr \frac{1}{\deg(\rho)} > t \right\}.$$

Let $t_N$ be a sequence with $\frac{t_N}{\frac{1}{\pi}(\log N)^2} \xrightarrow{N\to\infty} \theta \in (0,1)$.

$\rightsquigarrow \tau_\rho(t_N) \approx \theta \times$ (cover time of $D_N$)

$\rightsquigarrow L_{t_N}^{D_N} \approx$ local time at $\theta \times$ (cover time of $D_N$)

## Main Result

Recall: $L^{D_N}_{t_N} \approx$ local time at $\theta \times$ (cover time of $D_N$)

$$\frac{t_N}{\frac{1}{\pi}(\log N)^2} \xrightarrow{N\to\infty} \theta \in (0,1),\ W_N := N^2 e^{-\frac{2t_N}{\frac{1}{\pi}\log N}} = N^{2-2\theta+o(1)}$$

$$\kappa^D_N := \frac{1}{W_N} \sum_{x\in D_N} 1_{\{L^{D_N}_{t_N}(x)=0\}} \delta_{\frac{x}{N}} \otimes \delta_{\{L^{D_N}_{t_N}(x+z)\,:\,z\in\mathbb{Z}^2\}}$$

Main Theorem. (A.-Biskup)

$$\kappa^D_N \underset{N\to\infty}{\xrightarrow{\text{law}}} c_\theta\ Z^D_{\sqrt{\theta}}(dx) \otimes \nu^{\mathrm{RI}}_\theta(d\phi),$$

- $Z^D_\lambda(dx)$ "$=$" $r^D(x)^{\frac{(2\lambda)^2}{2}} e^{2\lambda\varphi^D_x - \frac{1}{2}\mathrm{Var}(2\lambda\varphi^D_x)} dx, \quad \lambda \in (0,1)$

  a Liouville Quantum Gravity measure on $D$

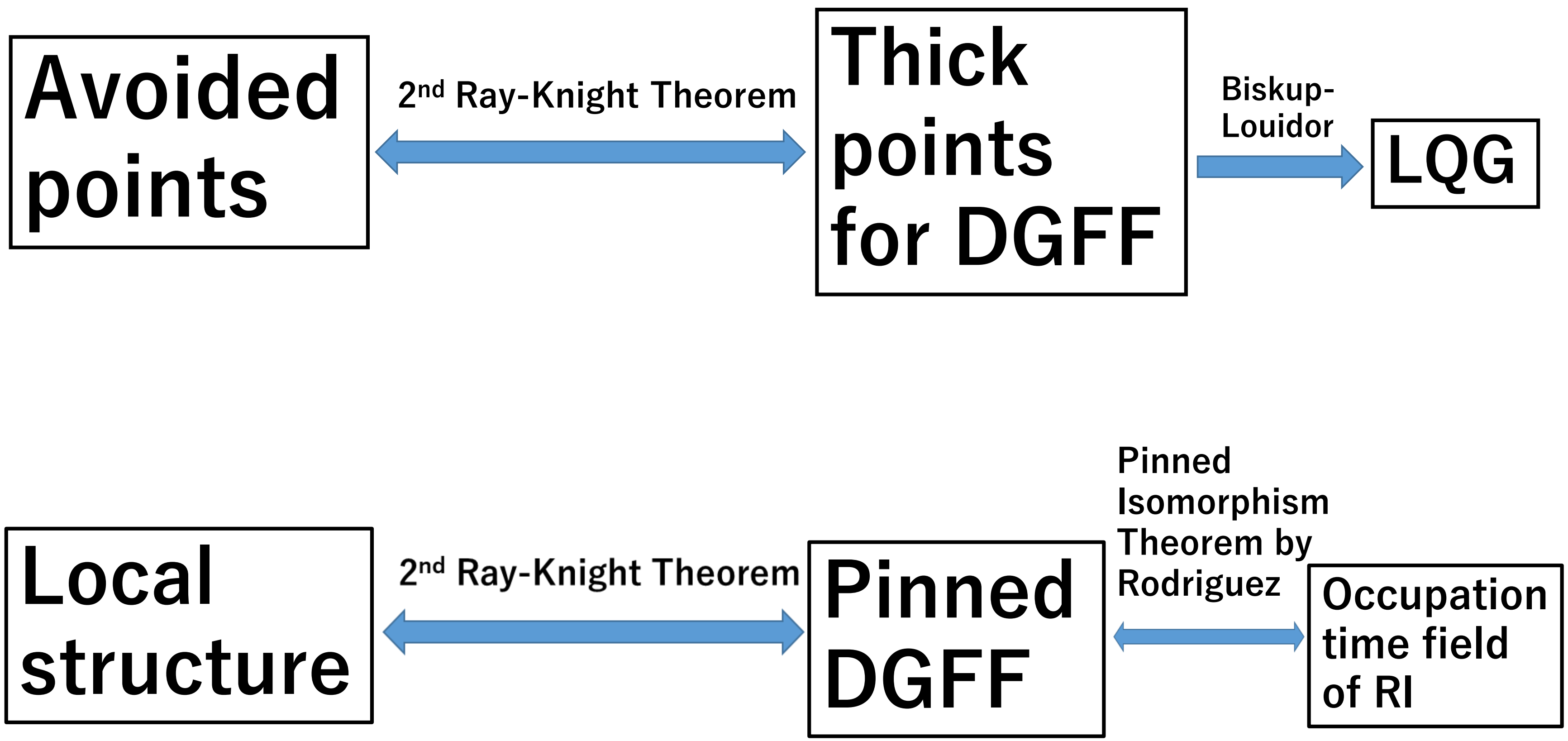
- $\nu^{\mathrm{RI}}_\theta$ is the law of occupation time field of

  the two-dimensional random interlacement at level $\theta$.

# Idea of proof.

Recall: $\kappa_N^D \xrightarrow[N\to\infty]{\text{law}} c_\theta \, Z^D_{\sqrt{\theta}}(dx) \otimes \nu_\theta^{\mathrm{RI}}(d\phi).$

Recall: $\kappa_N^D := \frac{1}{W_N} \sum_{x\in D_N} 1_{\{L^{D_N}_{t_N}(x)=0\}} \delta_{\frac{x}{N}} \otimes \delta_{\{L^{D_N}_{t_N}(x+z)\,:\, z\in\mathbb{Z}^2\}}$



**Heuristics of**

**Avoided points**

$\leftrightarrow$ **Thick points for DGFF**

# Discrete Gaussian Free Field (DGFF) and the maximum

**Definition.** $h^{D_N} = (h_x^{D_N})_{x \in D_N}$ is DGFF on $D_N$

$\overset{\text{def}}{\Leftrightarrow}$ $h^{D_N}$ is centered Gaussian with

$$\mathbb{E}\left[h_x^{D_N} h_y^{D_N}\right] = G^{D_N}(x,y) \quad := E^x\left[\int_0^{H_{\partial D_N}} 1_{\{X_s = y\}} ds\right] \frac{1}{\deg(y)}.$$

Theorem. (Bolthausen-Deuschel-Giacomin (2001))

$$\frac{\max_{x \in D_N} h_x^{D_N}}{\log N} \xrightarrow{N \to \infty} \sqrt{\frac{2}{\pi}} \quad \text{in probab.}$$

Note: $\mathrm{Var}(h_x^{D_N}) = \frac{1}{2\pi} \log N + O(1)$

Remark. 2nd order: Bramson-Zeitouni (2011)

Convergence in law: Bramson-Ding-Zeitouni (2016)

# Convergence of $\lambda$-thick points $\lambda \in (0,1)$

Recall: $\frac{\max_{x \in D_N} h_x^{D_N}}{\log N} \xrightarrow{N\to\infty} \sqrt{\frac{2}{\pi}}$ in probab.

$$\eta_N^D := \frac{1}{K_N} \sum_{x \in D_N} \delta_{\frac{x}{N}} \otimes \delta_{h_x^{D_N} - a_N} \otimes \delta_{\{h_x^{D_N} - h_{x+z}^{D_N} \,:\, z \in \mathbb{Z}^2\}}$$

$$\frac{a_N}{\log N} \overset{N\to\infty}{\to} \lambda \sqrt{\frac{2}{\pi}}, \; K_N := \frac{N^2}{\sqrt{\log N}} e^{-\frac{a_N^2}{\frac{1}{\pi}\log N}} = N^{2-2\lambda^2+o(1)}.$$

Theorem. (Biskup-Louidor (2016))

$$\eta_N^D \xrightarrow[N\to\infty]{\text{law}} c(\lambda) Z_\lambda^D(dx) \otimes e^{-2\sqrt{2\pi}\lambda h} dh \otimes \nu_\lambda,$$

where $Z_\lambda^D(dx)$ "$=$" $r^D(x)^{\frac{(2\lambda)^2}{2}} e^{2\lambda\varphi_x^D - \frac{1}{2}\mathrm{Var}(2\lambda\varphi_x^D)} dx$ LQG on $D$,

$\nu_\lambda$ is the law of $\phi + 2\sqrt{2\pi}\,\lambda\mathfrak{a}$, $\phi$ is DGFF on $\mathbb{Z}^2$ pinned to zero at the origin.

# Heuristics of avoided points $\leftrightarrow$ thick points

Recall: $t_N \approx \theta \frac{1}{\pi}(\log N)^2$, $x$ is a $\lambda$-thick point $\Leftrightarrow h_x^{D_N} \approx \lambda \sqrt{\frac{2}{\pi}} \log N$

Key: **2nd Ray-Knight Theorem (Eisenbaum-Kaspi-Marcus-Rosen-Shi (2000))**

$$\left\{ L_{t_N}^{D_N}(x) + \frac{1}{2}(h_x^{D_N})^2 : x \in D_N \right\} \text{ under } P^\rho \otimes \mathbb{P}$$

$$\overset{\text{law}}{=} \left\{ \frac{1}{2}\left( h_x^{D_N} + \sqrt{2t_N} \right)^2 : x \in D_N \right\}.$$

Thus,

$$Z^D_{\sqrt{\theta}} \leftrightarrow x \text{ is } \sqrt{\theta}\text{-thick point}$$

$$\leftrightarrow h_x^{D_N} + \sqrt{2t_N} \approx 0$$

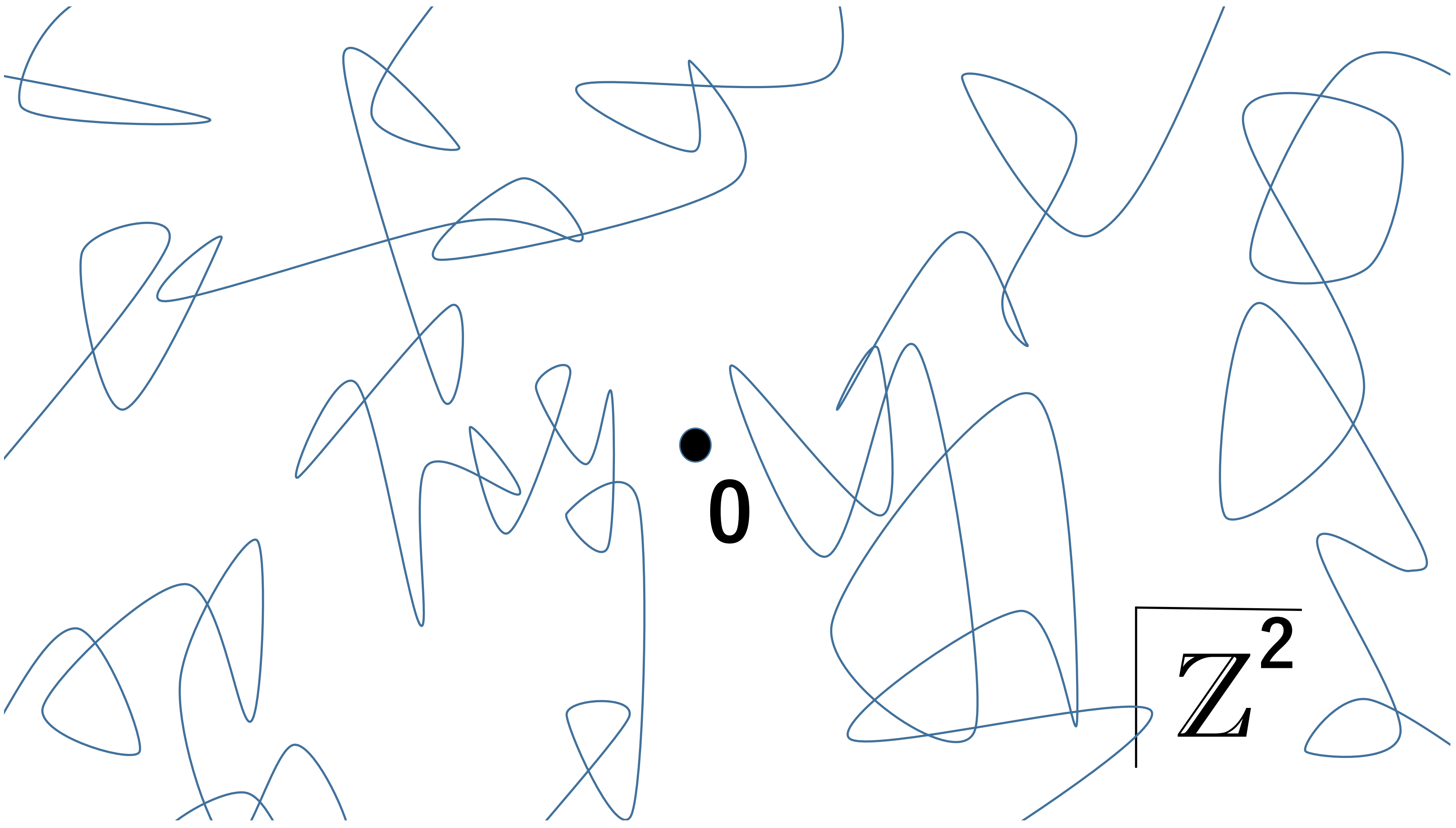
$$\leftrightarrow L_{t_N}^{D_N}(x) = 0.$$ i.e. $x$ is an avoided point.

**Heuristics of**

**Local structure of avoided points**
**$\leftrightarrow$ 2D random interlacements**

# Two-dimensional random interlacements

Poissonian soup of trajectories of SRWs conditioned on never hitting the origin.



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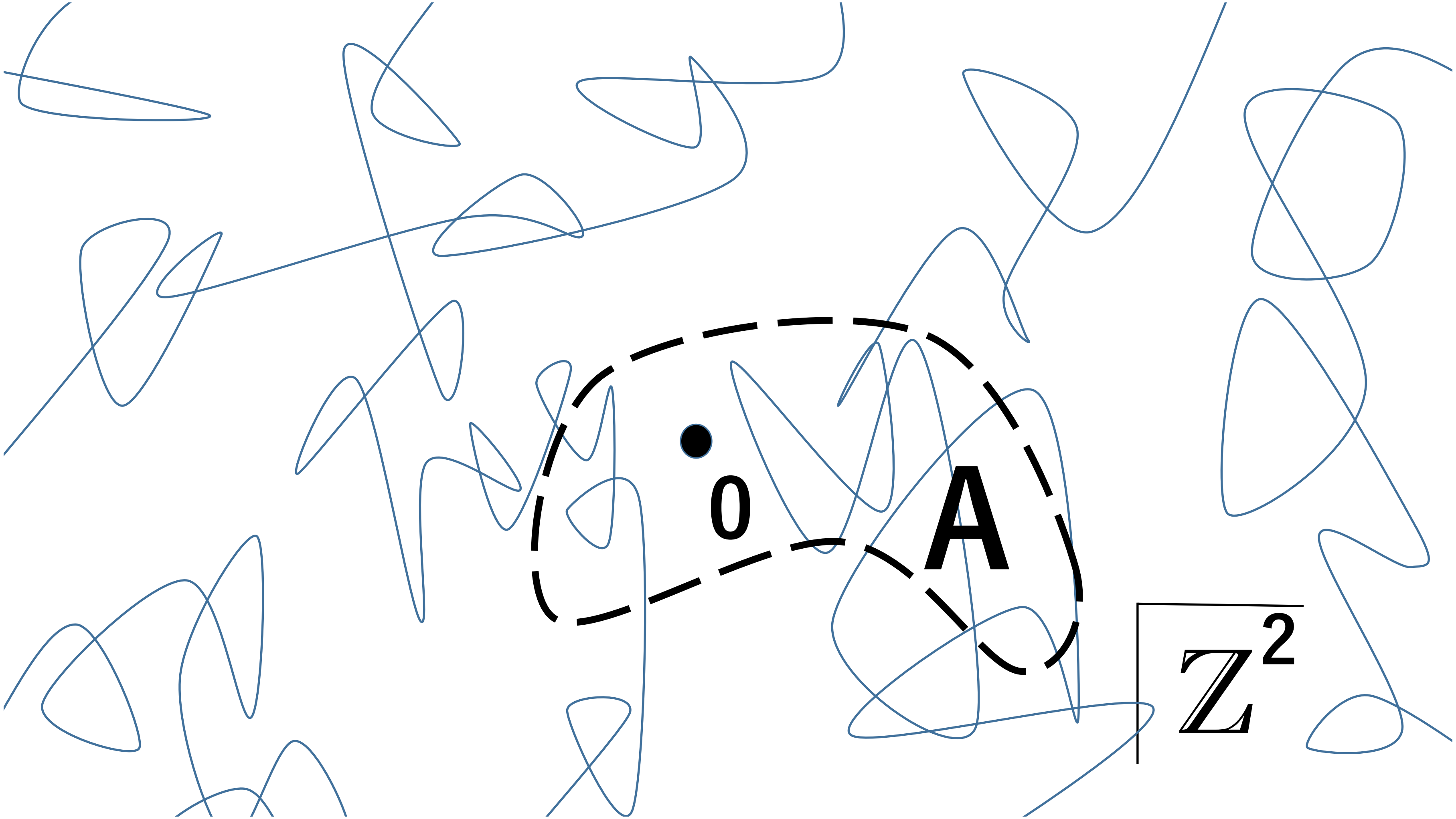


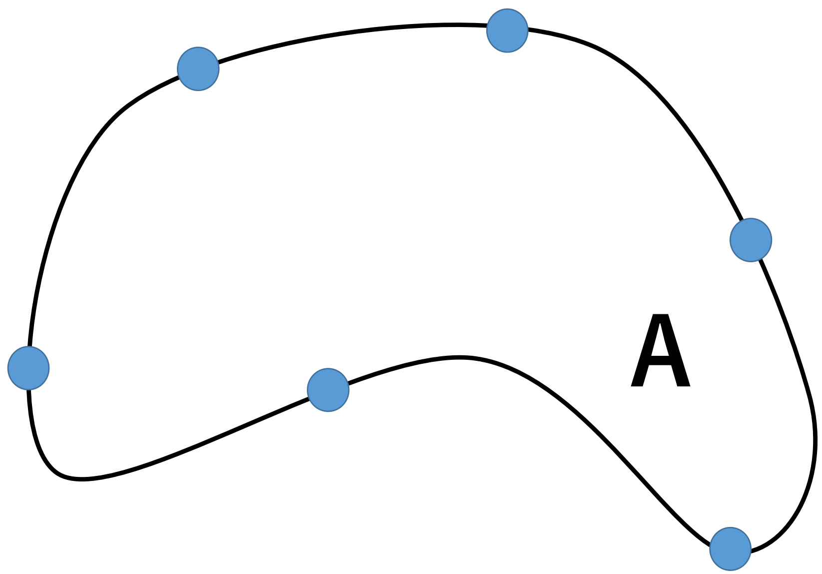
# What 2D RI at level $\theta$ looks like $\quad 0 \in A \subset \mathbb{Z}^2$ finite

Take i.i.d. Poi $(\pi\theta\text{cap}(A))$ samples from the law $\frac{e_A(\,\cdot\,)}{\text{cap}(A)}$,

where $e_A$ is the equilibrium measure and cap is the capacity:

$e_A(x) := 4\mathfrak{a}(x)\text{hm}_A(x) := 4\mathfrak{a}(x) \lim_{|y|\to\infty} P^y[X_{H_A} = x],\ x \in A,$

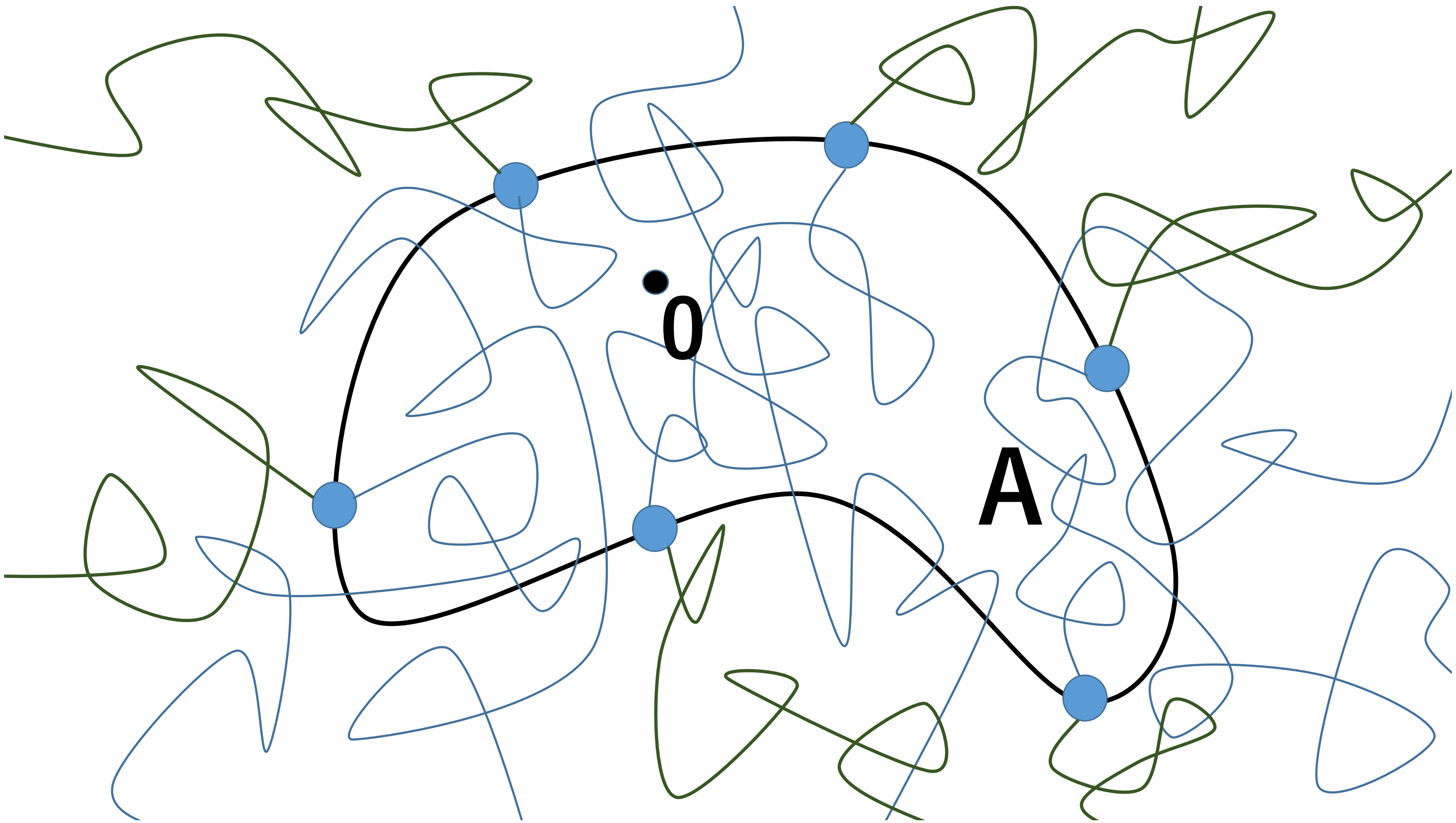
$\text{cap}(A) := \sum_{x\in A} e_A(x).$



# <u>What 2D RI at level $\theta$ looks like</u> $\mathbf{0} \in \boldsymbol{A} \subset \mathbb{Z}^{\mathbf{2}}$ finite

From each point, start indep two walks (blue and green);

Blue paths avoid $\mathbf{0}$ and green paths never return to $\boldsymbol{A}$.



# Construction of 2D RI

Two-dimensional random interlacements was constructed by Comets-Popov-Vachkovskaia (2016) and Rodriguez (2019).

One of the main motivations is to study the local structure of the uncovered set by SRW on 2D torus $(\mathbb{Z}/N\mathbb{Z})^2$.

cf. Sznitman ('10) : $\mathbb{Z}^d, d \geq 3$,

Teixeira ('09) : general transient weighted graphs

# Occupation time field for 2D RI at level $\theta$

Let $(w_i)_{i\in\mathbb{N}}$ be the doubly-infinite trajectories in the two-dimensional random interlacements at level $\theta$.

The occupation time field is defined by

$$\ell_\theta^{\mathrm{RI}}(x) := \sum_{i\in\mathbb{N}} \frac{1}{4}\int_{-\infty}^{\infty} 1_{\{w_i(t)=x\}}dt, \quad x\in\mathbb{Z}^2.$$

Recall the main theorem: $\kappa_N^D \xrightarrow[N\to\infty]{\text{law}} c_\theta\ Z^D_{\sqrt{\theta}}(dx)\otimes \nu_\theta^{\mathrm{RI}}(d\phi)$

with $\kappa_N^D = \frac{1}{W_N}\sum_{x\in D_N} 1_{\{L_{t_N}^{D_N}(x)=0\}}\delta_{\frac{x}{N}}\otimes\delta_{\{L_{t_N}^{D_N}(x+z)\,:\,z\in\mathbb{Z}^2\}}$,

$t_N \approx \theta\frac{1}{\pi}(\log N)^2$, $L_{t_N}^{D_N} \approx$ local time at $\theta\times$ (cover time of $D_N$)

$$\nu_\theta^{\mathrm{RI}} = \text{ the law of } (\ell_\theta^{\mathrm{RI}}(x))_{x\in\mathbb{Z}^2}.$$

# Heuristics of local picture $\leftrightarrow$ RI

Recall: $t_N \approx \theta \frac{1}{\pi} (\log N)^2$, $(\ell_\theta^{\mathrm{RI}}(z))_{z\in\mathbb{Z}^2}$ : Occupation time field of RI

Key: **Pinned Isomorphism Theorem (Rodriguez (2019))**

$$\left\{ \ell_\theta^{\mathrm{RI}}(z) + \frac{1}{2}(\phi_z)^2 : z \in \mathbb{Z}^2 \right\} \overset{\text{law}}{=} \left\{ \frac{1}{2} \left( \phi_z + 2\sqrt{2\pi\theta}\ \mathfrak{a} \right)^2 : z \in \mathbb{Z}^2 \right\},$$

where $(\phi_z)_{z\in\mathbb{Z}^2}$ be DGFF on $\mathbb{Z}^2$ pinned to zero at the origin.

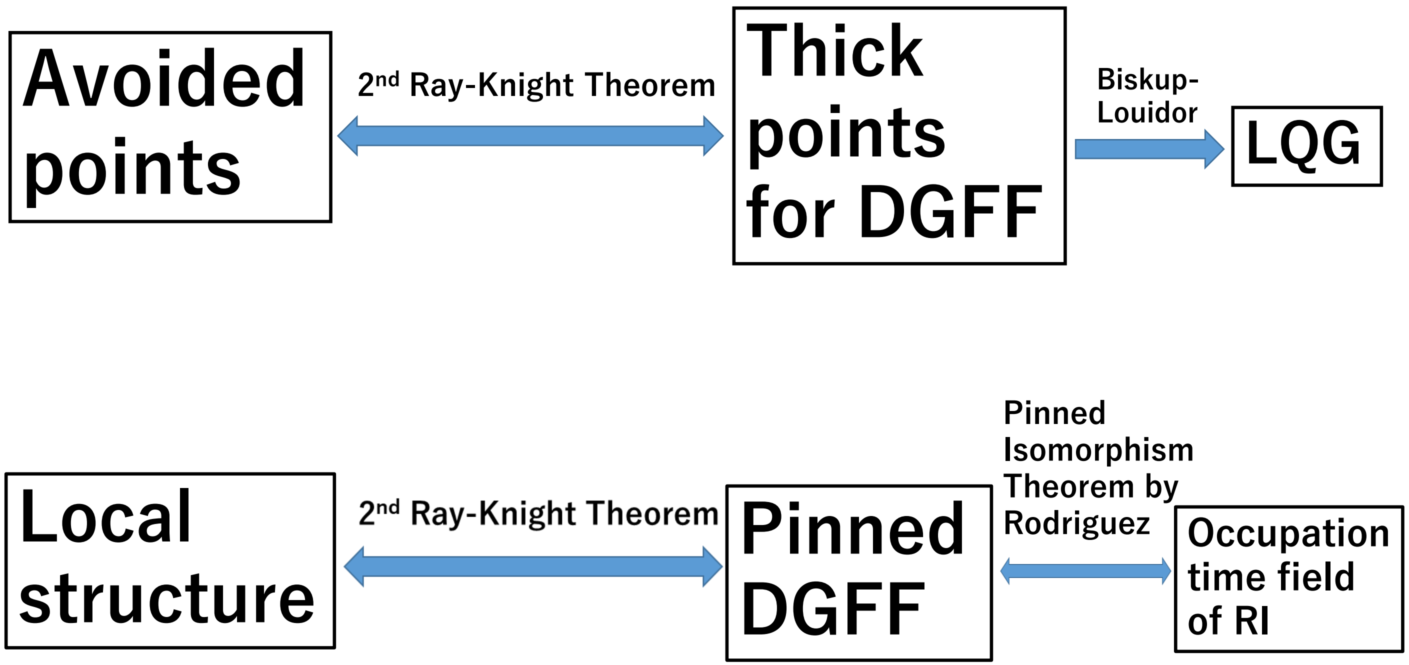
Recall: $L_{t_N}^{D_N} \approx$ local time at $\theta \times$ (cover time of $D_N$)

$$\left( (L_{t_N}^{D_N}(x+z))_z \mid L_{t_N}^{D_N}(x) = 0 \right) \leftrightarrow \left( (h_{x+z}^{D_N} - h_x^{D_N})_z \mid h_x^{D_N} \approx \sqrt{\theta} \times \text{max} \right)$$

$$\leftrightarrow (\phi_z + 2\sqrt{2\pi\theta}\ \mathfrak{a})_z .$$

$$\therefore \left( (L_{t_N}^{D_N}(x+z))_{z\in\mathbb{Z}^2} \mid L_{t_N}^{D_N}(x) = 0 \right) \overset{\text{law}}{\approx} (\ell_\theta^{\mathrm{RI}}(z))_{z\in\mathbb{Z}^2} .$$

# Thank you.



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