Three-Parametric Marcenko–Pastur Density

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Path from previous work

◆ Marcenko–Pastur law (parameter *r*)

$$\rho(x;r) = \frac{\sqrt{(x - x_{\mathrm{L}}(r))(x_{\mathrm{R}}(r) - x)}}{2\pi r x}\mathbf{1}_{(x_{\mathrm{L}}(r),x_{\mathrm{R}}(r))}(x) \tag{1.1}$$

$$x_{\mathrm{L}}(r) := (1-\sqrt{r})^2, \quad x_{\mathrm{R}}(r) := (1+\sqrt{r})^2.$$

◆ Dynamical extension (parameters *r*, *t*)(Blaizot, Nowak, Warchoł : PRE (2013))

$$\rho(x;r,t) := \rho_{\delta_0}(x;r,t) = \frac{\sqrt{(x - x_{\mathrm{L}}(r,t))(x_{\mathrm{R}}(r,t) - x)}}{2\pi r t x}\mathbf{1}_{(x_{\mathrm{L}}(r,t),x_{\mathrm{R}}(r,t))}(x) \tag{1.2}$$

$$x_{\mathrm{L}}(r,t) := (1-\sqrt{r})^2 t, \quad x_{\mathrm{R}}(r,t) := (1+\sqrt{r})^2 t, \quad t \in (0,\infty).$$

◆ Three-Parametric Marcenko–Pastur Density (parameters *r*, *t*, *a*) (Present results)

$$\rho(x;r,t,a) = \frac{\sqrt{(x - f_{\mathrm{L}}(x;r,t,a))(f_{\mathrm{R}}(x;r,t,a) - x)}}{2\pi r x t}\mathbf{1}_{(x_{\mathrm{L}}(r,t,a),x_{\mathrm{R}}(r,t,a))}(x) \tag{1.3}$$

Setting for Three-Parametric Marcenko–Pastur Density

◆ Random $M \times N$ matrix K

- the ratio of M and N is a constant r: $r = \lim_{N\to 0, M\to 0} \frac{N}{M} \in (0, 1]$.
- Each element is an independent random variable with variance t
- Wishart eigenvalue process started from a

$$L(t) = \frac{1}{M} K(t)^{\dagger} K(t)\text{'s eigenvalues } X_j(t), \text{ satisfying } \lim_{t\to 0} X_j(t) = a, \forall j \in \{1, 2, ..., N\}.$$

For example ,

$$\begin{aligned}
&\Re K_{kk} \sim N(\sqrt{Ma}, t/2), \quad k = 1, \ldots, N, \\
&\Re K_{jk} \sim N(0, t/2), \quad j = 1, \ldots, M, \quad k = 1, \ldots, N, \quad j \neq k, \qquad (2.1) \\
&\Im K_{jk} \sim N(0, t/2), \quad j = 1, \ldots, M, \quad k = 1, \ldots, N.
\end{aligned}$$

$X_j(t)$ follows our three-parametric MP law as $N \to \infty$ and $M \to \infty$.

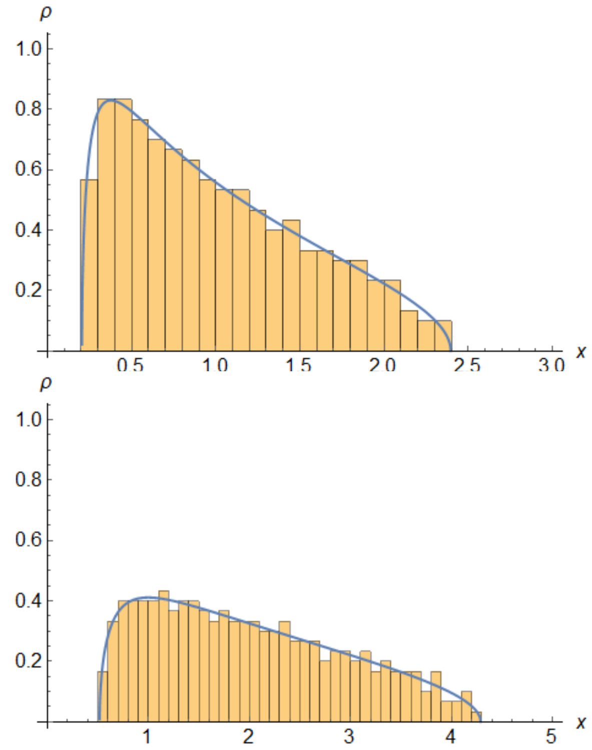
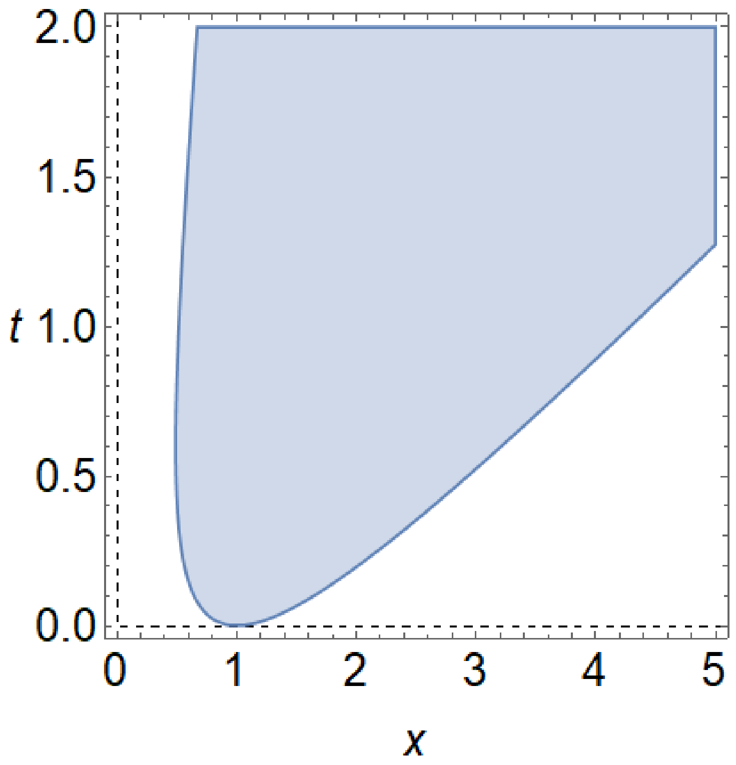
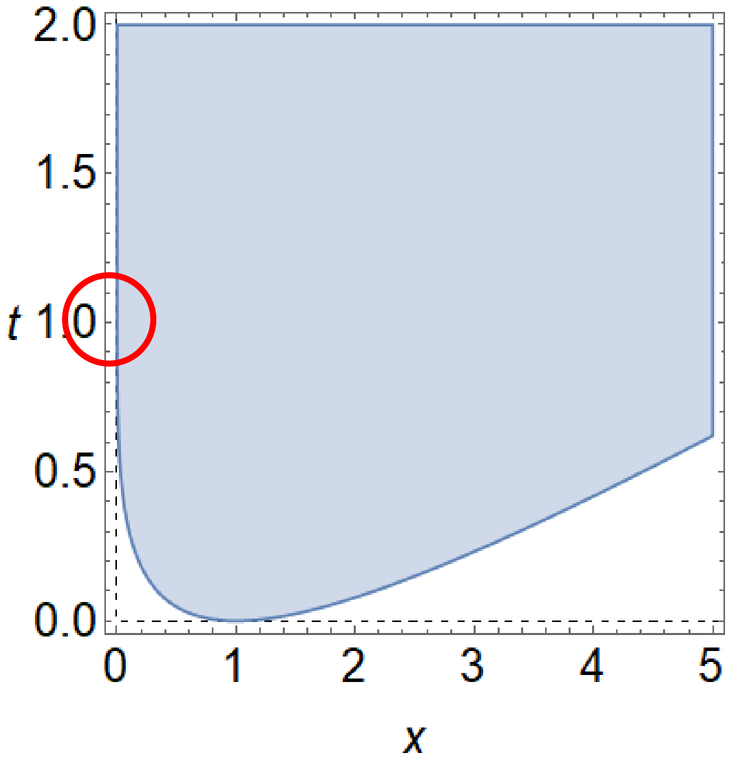


Figure1:The MP density, with parameter $(r, t, a)=(0.3,1,0)$, and,(0.3,1,1) . The histograms with 1000×300 matrixes whose elements are randomly generalized following the law (2.1).



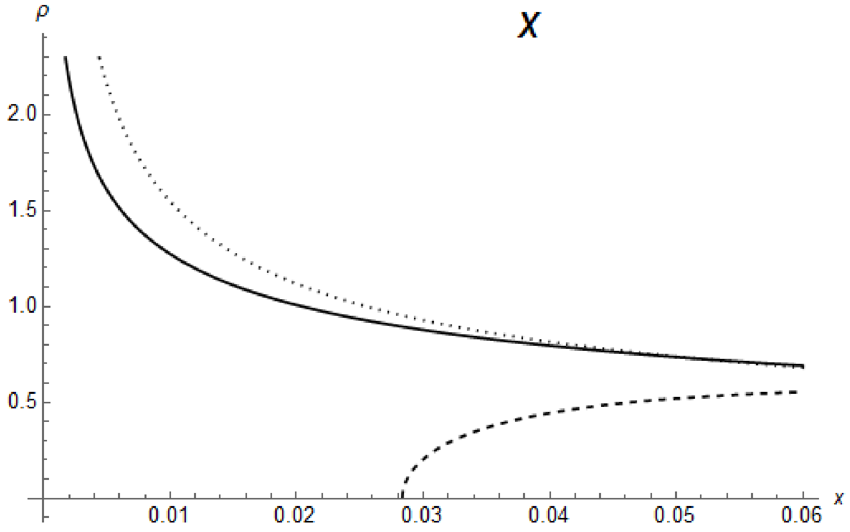
Time evolution of the support of the three-parametric Marcenko–Pastur density on the (x, t)-plane.

Left figure : r=0.3, a=1
Right figure : r=1, a=1



We study dynamic critical phenomena observed at the critical time $t_c(a) = a$ when r=1.

Right figure shows critical behavior of the three-parametric Marcenko–Pastur density.



Please see the preprint (**arXiv: math.PR/1907.07413**)