Annulus SLE partition functions and martingale-observables

Joint work with Nam-Gyu Kang, Hee-Joon Tak (arXiv:1806.03638)

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Overview

Goal

Analytic implementation of CFT constructed from GFF with Dirichlet/excursion reflected boundary conditions to the theory of annulus SLE.

- *To calculate correlation functions of conformal fields.*
- *To derive Eguchi-Ooguri, Ward and BPZ-Cardy type equations.*
- *To find explicit representations of SLE partition functions.*
- *To get a large class of SLE martingale-observables.*

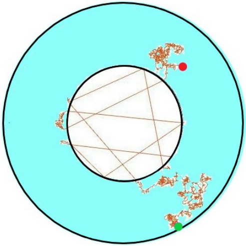

Figure: ERBM (Drenning, Lawler)

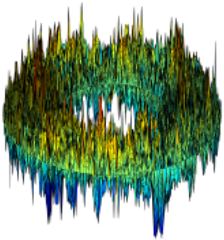

Figure: GFF with Dirichlet b.c.

Eguchi-Ooguri's type equation in a doubly connected domain

Theorem

For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ with a stress tensor $A_{\boldsymbol{\beta}}$,

$$\frac{1}{\pi}\oint_{\partial_{\text{inner}}} \mathbf{E}\left[A_{\boldsymbol{\beta}}(\zeta)\mathcal{X}\right] d\zeta = \partial_r \mathbf{E}\left[\mathcal{X}\right].$$

▷ If $\mathcal{X}$ is a tensor product of two GFF, then the above equation is equivalent to

$$-\frac{1}{\pi}\oint_{\partial_{\text{inner}}} \partial_\zeta G_r(\zeta, z_1)\, \partial_\zeta G_r(\zeta, z_2) d\zeta = \partial_r G_r(z_1, z_2),$$

where G_r is the Green's function for Laplace equation with Dirichlet/ER b.c.

▷ Set

$$G_r(z_1, z_2) := p(r)\, G_r^{Diri}(z_1, z_2) + (1 - p(r))\, G_r^{ER}(z_1, z_2).$$

Then Eguchi-Ooguri equation holds *if and only if* $p(r) = r/(r + \chi)$.

Cf. On a complex torus of genus one, similar form of Ward's equation holds.

- Eguchi-Ooguri: *Conformal and current algebras on a general Riemann surface*, Nuclear Phys. B, 282(2):308-328, 1987.
- Kang-Makarov: *Calculus of conformal fields on a compact Riemann surface*, arXiv:1708.07361, 86 pp.

Annulus SLE partition functions

▷ Null-vector equation for annulus SLE partition functions $Z(r,x)$: (Lawler, Zhan)

$$\partial_r Z = \frac{\kappa}{2} Z'' + H Z' + \frac{6-\kappa}{2\kappa} H' Z, \quad H(r,x) = 2(\log \Theta(r,x))'.$$

▷ Annulus CFT provides family of solutions to null-vector equation for each $\kappa > 0$:

$$Z(r,x) = \Theta(r,x)^{\frac{2}{\kappa}} \oint_\gamma \Theta(r,x-\zeta)^{-\frac{4}{\kappa}} \Theta(r,\zeta)^{-\frac{4}{\kappa}} \exp\left(-\frac{(\sqrt{\frac{2}{\kappa}}(x-2\zeta) - 2\beta\pi)^2}{4(r+\chi)} \right) d\zeta$$

(**Cf.** β: *back ground charge parameter*, χ: *boundary condition parameter*)

▷ Examples of $Z(r,x)$ associated to lattice models:

• *GFF with boundary value* β *on* ∂_{inner}

$$Z(r,x) = \Theta(r,x)^{-\frac{1}{2}} \times \exp\left(-\frac{(x - 2\sqrt{2}\beta\pi)^2}{8r} \right)$$

• *LERWs reflecting on* ∂_{inner} *with angle* β

$$Z(r,x) = \frac{\Theta(r, x+\pi-2\beta\pi)}{\Theta(r,x)} \times \Big(H(r,x) - H(r, x+\pi-2\beta\pi) \Big)$$