ALGEBRAIC INDEPENDENCE RESULTS FOR THE VALUES OF THE THETA-CONSTANTS AND SOME IDENTITIES

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Abstract

In the present work, we give algebraic independence results for the values of the classical theta-constants $\vartheta_2(\tau)$, $\vartheta_3(\tau)$, and $\vartheta_4(\tau)$. For example, the two values $\vartheta_\alpha(m\tau)$ and $\vartheta_\beta(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any τ in the upper half-plane when $e^{\pi i\tau}$ is an algebraic number, where $m, n \geq 1$ are integers and $\alpha, \beta \in \{2, 3, 4\}$ with $(m, \alpha) \neq (n, \beta)$. This algebraic independence result provides new examples of transcendental numbers through some identities found by S. Ramanujan. We additionally give some explicit identities among the three theta-constants in particular cases.

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1 Introduction and statement of the results

The Jacobi theta function is defined for two complex variables z and τ by

$$\vartheta(z \,|\, \tau) \;=\; \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau + 2\pi i\nu z}\,,$$

which converges for all complex numbers z, and τ in the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \Im(\tau) > 0\}$. Then the following three holomorphic functions defined in $\mathbb{H}$,

$$\vartheta_2(\tau) := e^{\pi i\tau/4} \cdot \vartheta\big(\tau/2 \,|\, \tau\big) = 2\sum_{\nu=0}^{\infty} e^{\pi i(\nu+1/2)^2\tau}, \qquad \vartheta_3(\tau) := \vartheta\big(0 \,|\, \tau\big) = 1 + 2\sum_{\nu=1}^{\infty} e^{\pi i\nu^2\tau},$$

$$\vartheta_4(\tau) := \vartheta\big(1/2 \,|\, \tau\big) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^\nu e^{\pi i\nu^2\tau},$$

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are known as theta-constants or Thetanullwerte, and the function $\vartheta_3(\tau)$ is called the Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\vartheta(z\,|\,\tau)$. As is well known, the theta-constants are never zero in $\mathbb{H}$ and have modular properties (cf. [13, Chapter 10]). In 1996, Yu. V. Nesterenko [8] found a new approach to the arithmetic nature of values of modular forms, proving the algebraic independence results for the values of the Ramanujan functions

$$P(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)z^n, \qquad Q(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)z^n, \qquad R(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)z^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$;

Theorem A ([8, Theorem 1]). *For each $q \in \mathbb{C}$ with $0 < |q| < 1$, at least three of the numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent over $\mathbb{Q}$.*

Theorem A has a number of remarkable consequences on algebraic independence (cf. [8, 9, 11]); for example, the two numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. D. Bertrand [3] translated Theorem A in terms of the theta-constants as follows. Let $D := \frac{1}{\pi i}\frac{d}{d\tau}$ be a differential operator.

Theorem B ([3, Theorem 4]). *Let $\alpha, \beta, \gamma \in \{2, 3, 4\}$ with $\alpha \neq \beta$. Then for any $\tau \in \mathbb{H}$, at least three of the numbers $e^{\pi i\tau}$, $\vartheta_\alpha(\tau)$, $\vartheta_\beta(\tau)$, $D\vartheta_\gamma(\tau)$ are algebraically independent over $\mathbb{Q}$.*

Note that we can derive from Theorem B that the sum $\sum_{n=1}^{\infty} q^{n^2}$ is transcendental for any algebraic number q with $0 < |q| < 1$ (cf. [4]). It is a natural question to ask whether Theorem B continues to hold if τ is replaced by $n\tau$ for a positive integer n. In this direction, the first author [5] has investigated the algebraic independence of the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ for special integers $n \geq 2$, namely, in the case when n is a power of two, and for $n = 3, 5, 6, 7, 9, 10, 11, 12$. As an application of the case $n = 5$, he obtained the transcendence of each of the infinite sums

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} \qquad \text{and} \qquad \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1+q^n},$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol and q is an algebraic number with $0 < |q| < 1$, by using the identities among the two functions $\vartheta_3(\tau)$ and $\vartheta_3(5\tau)$ due to Ramanujan (cf. [1, p. 249, (ii) and (iii) in Entry 8]). Recently, these results were generalized as follows;

Theorem C ([6, Theorem 1.2], [7, Theorem 1]). *Let m and n be distinct integers with $1 \leq m < n$ and $\gamma \in \{2, 3, 4\}$. Then for any $\tau \in \mathbb{H}$ at least three of the numbers $e^{\pi i\tau}$, $\vartheta_3(\tau)$, $\vartheta_3(n\tau)$, $D\vartheta_\gamma(\tau)$ are algebraically independent over $\mathbb{Q}$. Furthermore, at least two of the numbers $e^{\pi i\tau}$, $\vartheta_3(m\tau)$, $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$.*

The latter assertion in Theorem C implies that the two values of the theta-constant $\vartheta_3(\tau)$ at different points $\tau = m\tau_0, n\tau_0$ are algebraically independent over $\mathbb{Q}$ if the number $e^{\pi i\tau_0}$ is algebraic. The proof of Theorem C heavily depends on the constructive identities among the theta-constants, which are produced from the polynomials $P_m(X, Y)$ obtained by Yu. V. Nesterenko [10] (see Theorem D in Section 3). The first purpose of this paper is to extend a result of Theorem C to a more general form;

Theorem 1.1. *Let $m, n, \ell \geq 1$ be integers and $\alpha, \beta, \gamma \in \{2, 3, 4\}$ with $(m, \alpha) \neq (n, \beta)$. Then for any $\tau \in \mathbb{H}$, at least three of the numbers $e^{\pi i\tau}$, $\vartheta_\alpha(m\tau)$, $\vartheta_\beta(n\tau)$, $D\vartheta_\gamma(\ell\tau)$ are algebraically independent over $\mathbb{Q}$. In particular, the two numbers $\vartheta_\alpha(m\tau)$ and $\vartheta_\beta(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau \in \mathbb{H}$ when $e^{\pi i\tau}$ is an algebraic number.*

Note that Theorem 1.1 also generalizes Theorem B. The key of our improvement is the equality on the transcendence degrees

$$\mathrm{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}\big(\vartheta_\alpha(m\tau), \vartheta_\beta(n\tau)\big) = \mathrm{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}\big(\vartheta_2(\tau), \vartheta_3(\tau)\big) \tag{1.1}$$

for any $\tau \in \mathbb{H}$, provided that $(m, \alpha) \neq (n, \beta)$. The equality (1.1) will be confirmed through the theory of modular forms without the use of the specific identities among the theta-constants. This approach is completely different from those used in the previous papers [5], [6], and [7]. We give the proof of Theorem 1.1 in Section 2.

Example 1.1. *Let $m, n \geq 1$ be distinct integers and q be an algebraic number with $0 < |q| < 1$. Then, any two numbers among the six numbers*

$$\sum_{\nu=1}^{\infty} q^{m\nu(\nu-1)}, \quad \sum_{\nu=1}^{\infty} q^{n\nu(\nu-1)}, \quad \sum_{\nu=1}^{\infty} q^{m\nu^2}, \quad \sum_{\nu=1}^{\infty} q^{n\nu^2}, \quad \sum_{\nu=1}^{\infty} (-1)^\nu q^{m\nu^2}, \quad \sum_{\nu=1}^{\infty} (-1)^\nu q^{n\nu^2}$$

are algebraically independent over $\mathbb{Q}$, and any three numbers are not.

As an application of Theorem 1.1, we have the following corollary. Let $\left(\frac{n}{p}\right)$ denote the Legendre symbol.

Corollary 1.1. *Let q be an algebraic number with $0 < |q| < 1$. Then the infinite sums*

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n}, \qquad \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{3}\right) \frac{q^n}{1-q^n}, \qquad \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^{2n}} \tag{1.2}$$

are transcendental. The same holds for the infinite sums

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^{2n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1+q^n}. \tag{1.3}$$

Remark 1.1. It is well-known that the value of the elliptic modular j-function given by the formula

$$j(\tau) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is an algebraic number for any imaginary quadratic number $\tau \in \mathbb{H}$, where $\lambda := \lambda(\tau) = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$. Combining this fact and the equality (1.1), we find that the two numbers $\vartheta_\alpha(m\tau)$ and $\vartheta_\beta(n\tau)$ are algebraically dependent over $\mathbb{Q}$ if $\tau \in \mathbb{H}$ is an imaginary quadratic number. Indeed, the values of the theta-constants at $\tau = i, 2i \in \mathbb{H}$ are given by

$$\vartheta_2(i), \vartheta_4(i) = \frac{\pi^{1/4}}{2^{1/4}\Gamma(3/4)}, \qquad \vartheta_3(i) = \frac{\pi^{1/4}}{\Gamma(3/4)},$$

$$\vartheta_2(2i) = \frac{\sqrt{2-\sqrt{2}}}{2^{3/4}} \vartheta_2(i), \qquad \vartheta_3(2i) = \frac{\sqrt{2+\sqrt{2}}}{2} \vartheta_3(i), \qquad \vartheta_4(2i) = 2^{1/8} \vartheta_4(i)$$

(cf. [2, p. 325, Entry 1], see also [14]), where $\Gamma(z)$ is the gamma-function.

The second purpose of this paper is to give algebraic dependence relations over $\mathbb{Q}$ for the two rational functions of the theta-constants $\vartheta_j(n\tau)/\vartheta_3(\tau)$ and $\vartheta_4(\tau)/\vartheta_3(\tau)$, where $n \geq 2$ is an integer and $j \in \{2,3,4\}$. For an integer $n \geq 2$, we define the function $\psi(n)$ by

$$\psi(n) := n \prod_{\substack{p \mid n \\ p:\text{odd}}} \left(1 + \frac{1}{p}\right), \tag{1.4}$$

where the product on the right-hand side is taken over all odd prime numbers p with $p \mid n$.

Theorem 1.2. *Let $n \geq 2$ be an integer. For each $j \in \{2,3,4\}$, there exists a polynomial $Q_{j,n}(X,Y)$ with rational coefficients such that*

$$Q_{j,n}\left(\frac{\vartheta_j^4(n\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0 \tag{1.5}$$

holds for any $\tau \in \mathbb{H}$, where $Q_{j,n}(X,Y)$ has the form

$$Q_{j,n}(X,Y) = X^{\psi(n)} + \sum_{\nu=1}^{\psi(n)} R_{j,n,\nu}(Y) X^{\psi(n)-\nu} \tag{1.6}$$

with

$$\deg R_{j,n,\nu}(Y) \leq \nu, \qquad \nu = 1, 2, \ldots, \psi(n). \tag{1.7}$$

Theorem 1.2 generalizes a result of Yu. V. Nesterenko [10] (see Theorem D in Section 3). In Section 4, we derive from Theorem 1.2 a useful method to compute the explicit algebraic dependence relations among the theta-constants. For example, we compute polynomials for the three theta-constants $\vartheta_j(\tau)$, $\vartheta_j(2\tau)$, and $\vartheta_j(3\tau)$ for each $j \in \{2,3,4\}$ and list the first few polynomials $Q_{j,n}$ at the end of this paper.

2 Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1. We first observe the equality (1.1). Let $m, n, \ell \geq 1$ be integers and $\alpha, \beta, \gamma \in \{2,3,4\}$ with $(m,\alpha) \neq (n,\beta)$. Then the three theta-constants $\vartheta_\alpha^4(m\tau)$, $\vartheta_\beta^4(n\tau)$, and $\vartheta_\gamma^4(\ell\tau)$ are modular forms of weight 2 at least for the principal congruence subgroup of level $N := 2\ell mn$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

so that the two ratios

$$x := x(\tau) := \frac{\vartheta_\gamma^4(\ell\tau)}{\vartheta_\alpha^4(m\tau)} \qquad \text{and} \qquad y := y(\tau) := \frac{\vartheta_\beta^4(n\tau)}{\vartheta_\alpha^4(m\tau)}$$

are modular functions at least for $\Gamma(N)$. Let $\mathfrak{F}_N$ denote the field of all the modular functions for $\Gamma(N)$ whose Fourier expansions with respect to $e^{2\pi i\tau/N}$ have coefficients in $\mathbb{Q}(e^{2\pi i/N})$. Then the field $\mathfrak{F}_N$ is algebraic over the field $\mathbb{Q}(j(\tau))$ of weight zero modular functions for $SL_2(\mathbb{Z})$, where $j(\tau)$ is the elliptic modular j-function (cf. [12, Chapter 6, §6.2]). Hence, noting that $x, y \in \mathfrak{F}_N$, we find that the field $\mathbb{Q}(j(\tau), x, y)$ has transcendental degree one over $\mathbb{Q}$, and so the function x is algebraic over the field $\mathbb{Q}(y)$, since y is a non-constant function by the assumption $(m,\alpha) \neq (n,\beta)$. Thus, there exists a polynomial in two variables

$$g(X,Y) := b_0(Y)X^h + b_1(Y)X^{h-1} + \cdots + b_h(Y), \qquad b_0(Y) \not\equiv 0,$$

with $b_0(Y), \ldots, b_h(Y) \in \mathbb{Q}[Y]$, such that the function $g(\tau) := g(X,Y)|_{X=x,Y=y}$ is identically zero, where we may assume that the polynomials $b_0(Y), \ldots, b_h(Y)$ have no common factors in $\mathbb{Q}[Y]$.
Let $\tau_0 \in \mathbb{H}$ be a fixed complex number and put $y_0 := y(\tau_0) \in \mathbb{C}$. Suppose to the contrary that $b_\mu(y_0) = 0$ for all $\mu = 0, 1, \ldots, h$. Then y_0 is an algebraic number, since $b_0(Y)$ is a nonzero polynomial. Hence, all polynomials $b_\mu(Y)$ are divided by the minimal polynomial of y_0 over $\mathbb{Q}$, which is impossible. Thus, there exists a μ such that $b_\mu(y_0) \neq 0$, so that the polynomial $g(X, y_0)$ over $\mathbb{Q}(y_0)$ does not vanish. This implies that the number $x(\tau_0)$ is algebraic over $\mathbb{Q}(y_0)$, namely, the number $\vartheta_\gamma(\ell\tau_0)$ is algebraic over the field $\mathbb{Q}(\vartheta_\alpha(m\tau_0), \vartheta_\beta(n\tau_0))$. The above integers $m, n, \ell \geq 1$ and the subscripts $\alpha, \beta, \gamma \in \{2,3,4\}$ are chosen arbitrary, and therefore we obtain the equality

$$\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\left(\vartheta_\alpha(m\tau), \vartheta_\beta(n\tau)\right) = \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\left(\vartheta_2(\tau), \vartheta_3(\tau)\right)$$

for any $\tau \in \mathbb{H}$, which is (1.1) as desired. Theorem 1.1 follows from the equality (1.1), since

$$\begin{aligned}\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\left(e^{\pi i\tau}, \vartheta_\alpha(m\tau), \vartheta_\beta(n\tau), D\vartheta_\gamma(\ell\tau)\right) &= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\left(e^{\pi i\ell\tau}, \vartheta_2(\tau), \vartheta_3(\tau), D\vartheta_\gamma(\ell\tau)\right) \\ &= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\left(e^{\pi i\ell\tau}, \vartheta_2(\ell\tau), \vartheta_3(\ell\tau), D\vartheta_\gamma(\ell\tau)\right) \geq 3\end{aligned}$$

hold for any $\tau \in \mathbb{H}$, where we used Theorem B at the last inequality. The proof of Theorem 1.1 is completed. □

Proof of Corollary 1.1. Let q_0 be an algebraic number with $0 < |q_0| < 1$ and we choose $\tau_0 \in \mathbb{H}$ such that $q_0 = e^{2\pi i\tau_0}$. By Theorem 1.1 the numbers $\vartheta_2(\tau_0)$ and $\vartheta_2(3\tau_0)$ are algebraically independent over $\mathbb{Q}$, so that the number $\vartheta_2^3(3\tau_0)/\vartheta_2(\tau_0)$ is transcendental. On the other hand, the identity

$$\frac{\vartheta_2^3(3\tau)}{\vartheta_2(\tau)} = 4\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^{2n}}, \qquad q := e^{2\pi i\tau},$$

holds for any $\tau \in \mathbb{H}$ (cf. [2, p. 374, Entry 34]). Hence, substituting $\tau = \tau_0$, we obtain the transcendence of the infinite series on the right-hand side. Similarly, we can obtain the transcendence for other sums in (1.2) from the identities

$$\frac{\vartheta_3^3(3\tau)}{\vartheta_3(\tau)} = 1 - 2\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{3}\right) \frac{q^n}{1-q^n}, \qquad q := -e^{\pi i\tau},$$

and

$$\frac{\vartheta_3^3(\tau)}{\vartheta_3(3\tau)} + 3\frac{\vartheta_3^3(3\tau)}{\vartheta_3(\tau)} = 4\left(1 + 6\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n}\right), \qquad q := e^{2\pi i\tau},$$

which are given in [2, p. 375]. For the infinite sums in (1.3), see the identities [1, p. 249, (i) and (iv) in Entry 8]).

3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Let $\vartheta_j := \vartheta_j(\tau)$ $(j = 2,3,4)$ for brevity. It is well-known that the identities

$$\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4 \tag{3.1}$$

and

$$2\vartheta_2^2(2\tau) = \vartheta_3^2 - \vartheta_4^2, \qquad 2\vartheta_3^2(2\tau) = \vartheta_3^2 + \vartheta_4^2, \qquad \vartheta_4^2(2\tau) = \vartheta_3\vartheta_4 \tag{3.2}$$

hold for any $\tau \in \mathbb{H}$. We first show Theorem 1.2 in either case of $n = 2$ or an odd integer $n \geq 3$. Define the three polynomials as follows;

$$Q_{2,2}(X,Y),\ \ Q_{3,2}(X,Y) := X^2 - \frac{1}{2}(Y+1)X + \frac{1}{16}(Y-1)^2, \tag{3.3}$$
$$Q_{4,2}(X,Y) := X^2 - Y.$$

Lemma 3.1. *For each $j \in \{2,3,4\}$ the polynomial $Q_{j,2}(X,Y)$ satisfies*

$$Q_{j,2}\left(\frac{\vartheta_j^4(2\tau)}{\vartheta_3^4},\ \frac{\vartheta_4^4}{\vartheta_3^4}\right) = 0 \qquad (\tau \in \mathbb{H}). \tag{3.4}$$

Proof. By the first equality in (3.2) we have

$$\frac{\vartheta_2^8(2\tau)}{\vartheta_3^8} - \frac{1}{2}\left(\frac{\vartheta_4^4}{\vartheta_3^4}+1\right)\frac{\vartheta_2^4(2\tau)}{\vartheta_3^4} + \frac{1}{16}\left(\frac{\vartheta_4^4}{\vartheta_3^4}-1\right)^2 = 0,$$

so that the polynomial

$$Q_{2,2}(X,Y) = X^2 - \frac{1}{2}(Y+1)X + \frac{1}{16}(Y-1)^2$$

vanishes at $X = \vartheta_2^4(2\tau)/\vartheta_3^4$ and $Y = \vartheta_4^4/\vartheta_3^4$ for any $\tau \in \mathbb{H}$. Similarly we find that the polynomials $Q_{3,2}$ and $Q_{4,2}$ satisfy (3.4) from the second and the third equalities in (3.2), respectively. □

It is clear that the above polynomials $Q_{j,2}$ satisfy (1.6) and (1.7) in Theorem 1.2. Next we consider the case where $n = m \geq 3$ is an odd integer. We use the following result obtained by Yu. V. Nesterenko [10].

Theorem D ([10, Theorem 1, Corollaries 3, 4]). *For any odd integer $m \geq 3$ there exists an integer polynomial*

$$P_m(X,Y) = X^{\psi(m)} + \sum_{\nu=1}^{\psi(m)} R_\nu(Y)X^{\psi(m)-\nu} \tag{3.5}$$

with $\deg_Y R_\nu(Y) < \nu$ $(\nu = 1,2,\ldots,\psi(m))$, such that the identities

$$P_m\left(m^2\frac{\vartheta_2^4(m\tau)}{\vartheta_2^4(\tau)},\ -16\frac{\vartheta_4^4(\tau)}{\vartheta_2^4(\tau)}\right) = 0, \qquad P_m\left(m^2\frac{\vartheta_3^4(m\tau)}{\vartheta_3^4(\tau)},\ 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0, \tag{3.6}$$

and

$$P_m\left(m^2\frac{\vartheta_4^4(m\tau)}{\vartheta_4^4(\tau)},\ -16\frac{\vartheta_2^4(\tau)}{\vartheta_4^4(\tau)}\right) = 0 \tag{3.7}$$

hold for any $\tau \in \mathbb{H}$, where $\psi(m)$ is defined by (1.4).

Let $P_m(X,Y)$ be an integer polynomial in Theorem D. For example, the first two polynomials P_3 and P_5 are given in [10] by

$$P_3(X,Y) = X^4 - 12X^3 + 30X^2 - (Y^2 - 16Y + 28)X + 9, \tag{3.8}$$
$$P_5(X,Y) = X^6 - 30X^5 + 135X^4 - (20Y^2 - 320Y + 260)X^3 - (120Y^2 - 1920Y - 255)X^2$$
$$- (Y^4 - 32Y^3 + 308Y^2 - 832Y + 126)X + 25,$$

respectively. Define

$$Q_{2,m}(X,Y) := m^{-2\psi(m)}(1-Y)^{\psi(m)} \cdot P_m\left(m^2\frac{X}{1-Y}, -16\frac{Y}{1-Y}\right), \tag{3.9}$$

$$Q_{3,m}(X,Y) := m^{-2\psi(m)} \cdot P_m(m^2X, 16(1-Y)), \tag{3.10}$$

$$Q_{4,m}(X,Y) := m^{-2\psi(m)}Y^{\psi(m)} \cdot P_m\left(m^2\frac{X}{Y},\ -16\frac{1-Y}{Y}\right). \tag{3.11}$$

Lemma 3.2. *For each $j \in \{2,3,4\}$ the above $Q_{j,m}(X,Y)$ is a polynomial with rational coefficients, which satisfies*

$$Q_{j,m}\left(\frac{\vartheta_j^4(m\tau)}{\vartheta_3^4},\ \frac{\vartheta_4^4}{\vartheta_3^4}\right) = 0 \qquad (\tau \in \mathbb{H}),$$

and is of the form

$$Q_{j,m}(X,Y) = X^{\psi(m)} + \sum_{\nu=1}^{\psi(m)} R_{j,m,\nu}(Y)X^{\psi(m)-\nu},$$

where

$$\deg R_{j,m,\nu}(Y) \le \nu, \qquad \nu = 1,2,\ldots,\psi(m).$$

Proof. The identity

$$Q_{4,m}\left(\frac{\vartheta_4^4(m\tau)}{\vartheta_3^4},\ \frac{\vartheta_4^4}{\vartheta_3^4}\right) = 0 \qquad (\tau \in \mathbb{H})$$

follows from (3.7) together with (3.1). Furthermore by (3.5) and (3.11) we get the form

$$Q_{4,m}(X,Y) = X^{\psi(m)} + \sum_{\nu=1}^{\psi(m)} R_{4,m,\nu}(Y)X^{\psi(m)-\nu},$$

where

$$R_{4,m,\nu}(Y) := m^{-2\nu}Y^{\nu} \cdot R_\nu\left(-16\frac{1-Y}{Y}\right), \qquad \nu = 1,2,\ldots,\psi(m),$$

are polynomials in Y with

$$\deg R_{4,m,\nu}(Y) \le \nu, \qquad \nu = 1,2,\ldots,\psi(m),$$

since $R_\nu(X)$ are given by integer polynomials whose degrees are less than ν. Therefore Lemma 3.2 is true for $j = 4$. We can obtain the similar results for the polynomials $Q_{2,m}$ and $Q_{3,m}$ from the equalities (3.6). □

Finally we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix a subscript $j \in \{2,3,4\}$. The proof is by induction on n. We have just shown in Lemmas 3.1 and 3.2 that the assertion is true for $n = 2$ and an odd integer $n = m \ge 3$. Suppose that Theorem 1.2 is true for some fixed integer $n \ge 2$; namely there exists a polynomial

$$Q_{j,n}(X,Y) = X^{\psi(n)} + \sum_{\nu=1}^{\psi(n)} R_{j,n,\nu}(Y)X^{\psi(n)-\nu} \tag{3.12}$$

satisfying the properties (1.5), (1.6), and (1.7). In what follows, we show the existence of the polynomial $Q_{j,2n}(X,Y)$, which satisfies the properties (1.5), (1.6), and (1.7) with n replaced by $2n$. The identity (1.5) remains true when τ is replaced by 2τ, and the equalities

$$\frac{\vartheta_j^4(2n\tau)}{\vartheta_3^4(2\tau)} = 4\frac{\vartheta_j^4(2n\tau)}{\vartheta_3^4}\left(1+\frac{\vartheta_4^2}{\vartheta_3^2}\right)^{-2}, \qquad \frac{\vartheta_4^4(2\tau)}{\vartheta_3^4(2\tau)} = 4\frac{\vartheta_4^2}{\vartheta_3^2}\left(1+\frac{\vartheta_4^2}{\vartheta_3^2}\right)^{-2}$$

follow from (3.2). Hence by (3.12)

$$\begin{aligned} A_{j,n}(X,Y) &:= 4^{-\psi(n)}(1+Y)^{2\psi(n)}\cdot Q_{j,n}\left(\frac{4X}{(1+Y)^2},\ \frac{4Y}{(1+Y)^2}\right) \\ &= X^{\psi(n)} + \sum_{\nu=1}^{\psi(n)} 4^{-\nu}(1+Y)^{2\nu}\cdot R_{j,n,\nu}\left(4Y\,(1+Y)^{-2}\right)X^{\psi(n)-\nu} \\ &=: \sum_{\nu=0}^{\psi(n)} S_{j,n,\nu}(Y)X^{\psi(n)-\nu} \end{aligned}$$

vanishes at $X=\vartheta_j^4(2n\tau)/\vartheta_3^4$ and $Y=\vartheta_4^2/\vartheta_3^2$ for any $\tau\in\mathbb{H}$, where we denote $S_{j,n,0}(Y):=1$ and

$$S_{j,n,\nu}(Y) := 4^{-\nu}(1+Y)^{2\nu}\cdot R_{j,n,\nu}\left(4Y\,(1+Y)^{-2}\right), \qquad \nu=1,2,\ldots,\psi(n).$$

By the induction hypothesis (1.7), the above $S_{n,\nu}(Y)$ are polynomials with

$$\deg S_{j,n,\nu}(Y) \le 2\nu, \qquad \nu=0,1,2,\ldots,\psi(n). \tag{3.13}$$

Define

$$\begin{aligned} B_{j,n}(X,Y) &:= A_{j,n}(X,Y)A_{j,n}(X,-Y) \\ &= X^{2\psi(n)} + \sum_{\nu=1}^{2\psi(n)} T_{j,n,\nu}(Y)X^{2\psi(n)-\nu}, \end{aligned}$$

where $T_{j,n,\nu}(Y)$ are polynomials in Y given by

$$T_{j,n,\nu}(Y) := \sum_{\substack{0\le\nu_1,\nu_2\le\psi(n)\\ \nu_1+\nu_2=\nu}} S_{j,n,\nu_1}(Y)S_{j,n,\nu_2}(-Y). \tag{3.14}$$

Clearly the polynomials $T_{j,n,\nu}(Y)$ are even with respect to the variable Y; namely there exist polynomials $R_{j,2n,\nu}(Y)$ with rational coefficients such that

$$R_{j,2n,\nu}(Y^2) := T_{j,n,\nu}(Y), \qquad \nu=1,2,\ldots,2\psi(n). \tag{3.15}$$

Now we check that the polynomial

$$\begin{aligned} Q_{j,2n}(X,Y) &:= X^{2\psi(n)} + \sum_{\nu=1}^{2\psi(n)} R_{j,2n,\nu}(Y)X^{2\psi(n)-\nu} \\ &= X^{\psi(2n)} + \sum_{\nu=1}^{\psi(2n)} R_{j,2n,\nu}(Y)X^{\psi(2n)-\nu} \end{aligned} \tag{3.16}$$

fulfills the properties (1.5), (1.6), and (1.7) for n replaced by $2n$. The property (1.5) follows from the relation

$$Q_{j,2n}\left(X,Y^2\right) = B_{j,n}(X,Y) = A_{j,n}\left(X,Y\right) A_{j,n}\left(X,-Y\right)$$

and the fact that the polynomial $A_{j,n}(X,Y)$ vanishes at $X = \vartheta_j^4(2n\tau)/\vartheta_3^4$ and $Y = \vartheta_4^2/\vartheta_3^2$ for any $\tau \in \mathbb{H}$. The form (1.6) is given by (3.16). Moreover, for $\nu = 1, 2, \ldots, \psi(2n)$ we have by (3.13), (3.14), and (3.15)

$$\begin{aligned} 2\deg R_{j,2n,\nu}(Y) &= \deg T_{j,n,\nu}(Y) \\ &\leq \max_{\substack{0\leq\nu_1,\nu_2\leq\psi(n)\\ \nu_1+\nu_2=\nu}} \left(\deg S_{j,n,\nu_1}(Y) + \deg S_{j,n,\nu_2}(-Y)\right) \\ &\leq \max_{\substack{0\leq\nu_1,\nu_2\leq\psi(n)\\ \nu_1+\nu_2=\nu}} \left(2\nu_1 + 2\nu_2\right) \\ &= 2\nu, \end{aligned}$$

so that

$$\deg R_{j,2n,\nu}(Y) \leq \nu, \qquad \nu = 1, 2, \ldots, \psi(2n),$$

which is (1.7). The proof of Theorem 1.2 is completed. □

Remark 3.1. We have $Q_{2,2^\ell} = Q_{3,2^\ell}$ for any integer $\ell \geq 1$, since these polynomials are inductively constructed from the same initial polynomial (3.3). Moreover, by definitions (3.9), (3.10), and (3.11), we have

$$\begin{aligned} Q_{2,m}(X,Y) &= (1-Y)^{\psi(m)} \cdot Q_{3,m}\left(\frac{X}{1-Y}, \frac{1}{1-Y}\right), \\ Q_{4,m}(X,Y) &= Y^{\psi(m)} \cdot Q_{3,m}\left(\frac{X}{Y}, \frac{1}{Y}\right) \end{aligned}$$

for any odd integer $m \geq 3$.

Remark 3.2. Let $n \geq 2$ be an even integer. Then for each $j \in \{2,3,4\}$ we have

$$Q_{j,n}\left(\frac{\vartheta_j^4(n\tau)}{\vartheta_4^4}, \frac{\vartheta_3^4}{\vartheta_4^4}\right) = 0 \qquad (\tau \in \mathbb{H}), \tag{3.17}$$

which follows immediately from the transformation $\tau \mapsto \tau + 1$ in (1.5) and the equalities

$$\vartheta_j^4(\tau+2) = \vartheta_j^4(\tau) \quad (j = 2,3,4),$$

$$\vartheta_3^4(\tau+1) = \vartheta_4^4(\tau), \qquad \vartheta_4^4(\tau+1) = \vartheta_3^4(\tau).$$

4 Identities for the theta-constants

4.1 An application of Theorem 1.2

By the argument in the proof of Theorem 1.1, any three theta-constants $\vartheta_i(\ell\tau)$, $\vartheta_j(m\tau)$, and $\vartheta_k(n\tau)$ are algebraically dependent over $\mathbb{Q}$, but it is not easy to find the explicit algebraic dependence relations for given three theta-constants. In this section, as an application of Theorem 1.2, we give the explicit algebraic dependence relations among the three theta-constants $\vartheta_j(\tau)$, $\vartheta_j(2\tau)$, and $\vartheta_j(3\tau)$ for each fixed $j \in \{2,3,4\}$. Let $\tau \in \mathbb{H}$. Then by (3.6) and (3.17) the two polynomials

$$\begin{aligned} f(W) &:= W^2 \cdot Q_{2,2}\left(\frac{\vartheta_2^4(2\tau)}{\vartheta_2^4} \cdot \frac{1}{W},\ 1+\frac{1}{W}\right), \\ g(W) &:= P_3\left(9\frac{\vartheta_2^4(3\tau)}{\vartheta_2^4},\ -16W\right), \end{aligned}$$

have a common root at $W = \vartheta_4^4/\vartheta_2^4$, and hence the resultant of $f(W)$ and $g(W)$ is equal to zero. Thus, we find that the polynomial

$$R_2(X,Y,Z) := X^5Z - X^4Y^2 - 4X^3Y^2Z - 270X^2Y^2Z^2 + 256XY^4Z + 972XY^2Z^3 - 729Y^2Z^4$$

vanishes identically at $X = \vartheta_2^4(\tau)$, $Y = \vartheta_2^4(2\tau)$, and $Z = \vartheta_2^4(3\tau)$, where we used the forms (3.3) and (3.8). Similarly by considering the resultants

$$\mathrm{Res}_W\left(W^2 \cdot Q_{3,2}\left(\frac{\vartheta_3^4(2\tau)}{\vartheta_3^4} \cdot \frac{1}{W},\ \frac{1}{W}\right),\ P_3\left(9\frac{\vartheta_3^4(3\tau)}{\vartheta_3^4},\ 16(1-W)\right)\right)$$

and

$$\mathrm{Res}_W\left(Q_{4,2}\left(\frac{\vartheta_4^4(2\tau)}{\vartheta_4^4},\ 1+W\right),\ P_3\left(9\frac{\vartheta_4^4(3\tau)}{\vartheta_4^4},\ -16W\right)\right),$$

respectively, we can obtain integer polynomials

$$\begin{aligned} R_3(X,Y,Z) &:= X^8 - 56X^7Z - 10240X^6YZ + 1324X^6Z^2 - 8192X^5Y^2Z - 761856X^5YZ^2 \\ &\quad - 17064X^5Z^3 + 9666560X^4Y^2Z^2 - 2764800X^4YZ^3 + 128790X^4Z^4 \\ &\quad - 25165824X^3Y^3Z^2 - 2211840X^3Y^2Z^3 + 9953280X^3YZ^4 - 565704X^3Z^5 \\ &\quad + 16777216X^2Y^4Z^2 + 7962624X^2Y^2Z^4 - 7464960X^2YZ^5 \\ &\quad + 1338444X^2Z^6 - 5971968XY^2Z^5 - 1417176XZ^7 + 531441Z^8\,, \\ R_4(X,Y,Z) &:= X^5 - 28X^4Z + 270X^3Z^2 + 256X^2Y^2Z - 972X^2Z^3 + 729XZ^4 - 256Y^4Z\,, \end{aligned}$$

where $R_j(X,Y,Z)$ vanishes identically at $X = \vartheta_j^4(\tau)$, $Y = \vartheta_j^4(2\tau)$, and $Z = \vartheta_j^4(3\tau)$ for each $j = 3,4$.

4.2 Appendix

$$Q_{2,2} = Q_{3,2} = X^2 - \frac{1}{2}(Y+1)X + \frac{1}{2^4}(Y-1)^2,$$

$$Q_{2,3} = X^4 + \frac{4}{3}(Y-1)X^3 + \frac{10}{3^3}(Y-1)^2X^2 + \frac{4}{3^6}(Y-1)(Y+7)(7Y+1)X + \frac{1}{3^6}(Y-1)^4,$$

$$Q_{2,4} = Q_{3,4} = X^4 - \frac{1}{4}(Y+1)X^3 + \frac{1}{2^7}(3Y^2 - 62Y + 3)X^2 - \frac{1}{2^{10}}(Y+1)(Y^2+30Y+1)X + \frac{1}{2^{16}}(Y-1)^4,$$

$$\begin{aligned} Q_{2,5} = X^6 &+ \frac{6}{5}(Y-1)X^5 + \frac{27}{5^3}(Y-1)^2X^4 + \frac{4}{5^5}(Y-1)(13Y^2+230Y+13)X^3 \\ &+ \frac{3}{5^7}(Y-1)^2(17Y^2 - 2082Y + 17)X^2 \\ &+ \frac{2}{5^{10}}(Y-1)(63Y^4 + 6404Y^3 + 19834Y^2 + 6404Y + 63)X + \frac{1}{5^{10}}(Y-1)^6, \end{aligned}$$

$$Q_{3,3} = X^4 - \frac{4}{3}X^3 + \frac{10}{3^3}X^2 - \frac{4}{3^6}(8Y-1)(8Y-7)X + \frac{1}{3^6},$$

$$\begin{aligned} Q_{3,5} = X^6 &- \frac{6}{5}X^5 + \frac{27}{5^3}X^4 - \frac{4}{5^5}(256Y^2 - 256Y + 13)X^3 - \frac{3}{5^7}(2048Y^2 - 2048Y - 17)X^2 \\ &- \frac{2}{5^{10}}(32768Y^4 - 65536Y^3 + 39424Y^2 - 6656Y + 63)X + \frac{1}{5^{10}}, \end{aligned}$$

$$Q_{4,2} = X^2 - Y,$$

$$Q_{4,3} = X^4 - \frac{4}{3}YX^3 + \frac{10}{3^3}Y^2X^2 - \frac{4}{3^6}Y(Y-8)(7Y-8)X + \frac{1}{3^6}Y^4,$$

$$Q_{4,4} = X^4 - YX^2 - \frac{1}{2^4}Y(Y-1)^2,$$

$$\begin{aligned} Q_{4,5} = X^6 &- \frac{6}{5}YX^5 + \frac{27}{5^3}Y^2X^4 - \frac{4}{5^5}Y(13Y^2 - 256Y + 256)X^3 + \frac{3}{5^7}Y^2(17Y^2 + 2048Y - 2048)X^2 \\ &- \frac{2}{5^{10}}Y(63Y^4 - 6656Y^3 + 39424Y^2 - 65536Y + 32768)X + \frac{1}{5^{10}}Y^6. \end{aligned}$$

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