ON POLY-COSECANT NUMBERS

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Abstract. We introduce and study a "level two" generalization of the poly-Bernoulli numbers, which may also be regarded as a generalization of the cosecant numbers. We prove a recurrence relation, two exact formulas, and a duality relation for negative upper-index numbers.

1. Introduction

Poly-Bernoulli numbers were first introduced in [6] and later a slightly modified version was studied in [2]. They are, denoted $B_n^{(k)}$ and $C_n^{(k)}$ respectively, defined by using generating series, as follows. For an integer $k \in \mathbb{Z}$, let $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ be the sequences of rational numbers given respectively by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \tag{1.1}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}, \tag{1.2}$$

where $\mathrm{Li}_k(z)$ is the polylogarithm function (or rational function when $k \leq 0$) defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z|<1). \tag{1.3}$$

In the sequel, we regard this or any other series only as a formal power series.

Since $\mathrm{Li}_1(z) = -\log(1-z)$, the generating functions on the left-hand sides of (1.1) and (1.2) when $k=1$ become

$$\frac{te^t}{e^t-1} \quad \text{and} \quad \frac{t}{e^t-1}$$

respectively, and hence $B_n^{(1)}$ and $C_n^{(1)}$ are usual Bernoulli numbers, the only difference being $B_1^{(1)} = 1/2$ and $C_1^{(1)} = -1/2$ and otherwise $B_n^{(1)} = C_n^{(1)}$.

Various properties of poly-Bernoulli numbers, including combinatorial applications, are known. Among them we mention the explicit formulas

$$B_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n \atop i} \right\}}{(i+1)^k}, \quad C_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n+1 \atop i+1} \right\}}{(i+1)^k}$$

for $k \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ using the Stirling numbers of the second kind, and the dualities

$$B_n^{(-k)} = B_k^{(-n)}, \tag{1.4}$$

$$C_n^{(-k-1)} = C_k^{(-n-1)} \tag{1.5}$$

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for $k, n \in \mathbb{Z}_{\geq 0}$ (see [6, Theorems 1 and 2] and [7, § 2]). For combinatorial applications, see [3].

In this paper, we study the following "level 2" analog of poly-Bernoulli numbers, denoted $D_n^{(k)}$, which we also call the poly-cosecant numbers. For each $k \in \mathbb{Z}$, define $D_n^{(k)}$ by

$$\frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!}, \tag{1.6}$$

where $\mathrm{A}_k(z)$ is the series

$$\mathrm{A}_k(z) = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k} \tag{1.7}$$

and $\tanh(z)$ and $\sinh(z)$ are the usual hyperbolic tangent and sine functions respectively. Since $\mathrm{A}_k(z)$, $\tanh(z)$ and $\sinh(z)$ are all odd functions, we immediately see that $D_{2n+1}^{(k)} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Note that $\mathrm{A}_1(z) = 2\tanh^{-1}(z)$, and thus

$$\sum_{n=0}^{\infty} D_n^{(1)} \frac{t^n}{n!} = \frac{t}{\sinh t} = \frac{it}{\sin(it)} \quad (i = \sqrt{-1}).$$

Hence, up to sign, $D_n^{(1)}$ is the cosecant number D_n (see Nörlund [10, p. 458]).

We should mention that our $D_n^{(k)}$ is (if slightly modified) a special case of a generalization of the poly-Bernoulli number introduced by Y. Sasaki in [11, Definition 5].

2. Recurrence and explicit formulas for poly-cosecant numbers

In this section, we obtain a recurrence and explicit formulas for poly-cosecant numbers.

We first give a recurrence. Note that $D_0^{(0)} = 1$ and $D_n^{(0)} = 0$ for all $n \geq 1$ because $\mathrm{A}_0(\tanh(t/2)) = \sinh(t)$. Starting from this, the following formula gives a way to compute $D_n^{(k)}$ recursively for any integer k.

Proposition 2.1. *For any integer k and $n \geq 0$, it holds*

$$D_n^{(k-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)}.$$

Proof. We differentiate the defining relation

$$\mathrm{A}_k(\tanh(t/2)) = \sinh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!}$$

to obtain

$$\frac{\mathrm{A}_{k-1}(\tanh(t/2))}{\sinh t} = \cosh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sinh t \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!}.$$

From this we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k-1)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_{n-2m}^{(k)} \frac{t^n}{(2m)!(n-2m)!} + \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_{n-2m}^{(k)} \frac{t^n}{(2m+1)!(n-2m-1)!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} D_{n-2m}^{(k)} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!}.$$

By equating the coefficients of $t^n/n!$ on both sides, we obtain the desired result. □

When $k > 0$, we may want to write this as

$$(n+1)D_n^{(k)} = D_n^{(k-1)} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)} \quad (n > 0).$$

Note that $D_0^{(k)} = 1$ for all $k \in \mathbb{Z}$.

We proceed to give two explicit formulas for $D_n^{(k)}$. Recall that $\left[{n \atop m} \right]$ and $\left\{ {n \atop m} \right\}$ are Stirling numbers of the first and the second kinds, respectively, and $B_n = B_n^{(1)}$ is the Bernoulli number. See [1, Chapter 2] for the precise definition and formulas we use in the proof. In [11], Sasaki gave a different formula, but one needs to define yet another sequences to describe the formula.

Theorem 2.2. *For any $k \in \mathbb{Z}$ and $n \geq 0$, we have*

1)

$$D_n^{(k)} = 4 \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=1}^{2m+1} \sum_{q=0}^{n-2m} (2^{p+q+1} - 1) \binom{n}{q} \left[{2m+1 \atop p} \right] \left\{ {n-q \atop 2m} \right\} \frac{B_{p+q+1}}{p+q+1},$$

and

2)

$$D_n^{(k)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p (p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\}.$$

Proof. To prove 1), we need the following lemma. We may prove this in the same manner as in [1, Proposition 2.6 (4)] and we omit the proof here.

Lemma 2.3. *For $n \geq 1$ we have,*

$$x^n \left(\frac{d}{dx} \right)^n = \sum_{m=1}^{n} (-1)^{n-m} \left[{n \atop m} \right] \left(x \frac{d}{dx} \right)^m.$$

We write

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} \\ &= 2 \sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^k} \frac{1}{\sinh t} \\ &= 4 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} \frac{e^t (e^t - 1)^{2m}}{(e^t + 1)^{2m+2}}. \end{aligned} \tag{2.1}$$

Since

$$\frac{1}{(x+1)^{n+1}} = \frac{(-1)^n}{n!} \left(\frac{d}{dx} \right)^n \frac{1}{x+1}, \tag{2.2}$$

we see by setting $x = e^t$ and using Lemma 2.3 that

$$\frac{e^{nt}}{(e^t+1)^{n+1}} = \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \left(\frac{d}{dt}\right)^p \frac{1}{e^t+1}. \tag{2.3}$$

From

$$\frac{t}{e^t-1} = \sum_{q=0}^{\infty} B_q \frac{t^q}{q!}$$

and

$$\frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1},$$

we have

$$\frac{1}{e^t+1} = \sum_{q=0}^{\infty}(1-2^q)B_q\frac{t^{q-1}}{q!}.$$

By taking the p-th derivative of both sides, we get

$$\left(\frac{d}{dt}\right)^p\left(\frac{1}{e^t+1}\right) = \sum_{q=p+1}^{\infty}(1-2^q)\frac{B_q}{q}\frac{t^{q-p-1}}{(q-p-1)!} = \sum_{q=p+1}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}$$

and we substitute this in (2.3) to obtain

$$\begin{aligned}\frac{e^{nt}}{(e^t+1)^{n+1}} &= \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!} \\ &= \frac{1}{n!}\sum_{q=0}^{\infty}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} (1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

From this, we have

$$\begin{aligned}\frac{e^t}{(e^t+1)^{2m+2}} &= \frac{e^{-(2m+1)t}}{(e^{-t}+1)^{2m+2}} \\ &= \frac{1}{(2m+1)!}\sum_{q=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q} \begin{bmatrix} 2m+1 \\ p \end{bmatrix} (1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

Together with the well-known generating series ([1, Proposition 2.6 (7)], note that $\begin{Bmatrix} s \\ 2m \end{Bmatrix} = 0$ if $s < 2m$)

$$(e^t-1)^{2m} = (2m)!\sum_{s=0}^{\infty}\begin{Bmatrix} s \\ 2m \end{Bmatrix}\frac{t^s}{s!},$$

we obtain

$$\begin{aligned}&\frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}} \\ &= \frac{1}{2m+1}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\begin{bmatrix} 2m+1 \\ p \end{bmatrix}\begin{Bmatrix} s \\ 2m \end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^{q+s}}{q!s!} \\ &= \frac{1}{2m+1}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix} 2m+1 \\ p \end{bmatrix}\begin{Bmatrix} n-q \\ 2m \end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}.\end{aligned}$$

Substituting this into (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} \\
&= 4 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=0}^{\infty} \sum_{q=0}^{n} \sum_{p=1}^{2m+1} (-1)^{p+q}(1-2^{p+q+1}) \binom{n}{q} \left[{2m+1 \atop p} \right] \left\{ {n-q \atop 2m} \right\} \frac{B_{p+q+1}}{p+q+1} \frac{t^n}{n!} \\
&= 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=1}^{2m+1} \sum_{q=0}^{n-2m} (2^{p+q+1}-1) \binom{n}{q} \left[{2m+1 \atop p} \right] \left\{ {n-q \atop 2m} \right\} \frac{B_{p+q+1}}{p+q+1} \frac{t^n}{n!}.
\end{aligned}$$

(We have used the facts that $B_{p+q+1} = 0$ if $p+q \geq 1$ is even and $\left\{ {n-q \atop 2m} \right\} = 0$ if $n-q < 2m$.) By equating the coefficients of $t^n/n!$ on both sides, we obtain the desired result.

To prove 2), we employ the following formula ([4, Proposition 9]) for the numbers $T_{n,m}$ ("higher order tangent numbers") defined by

$$\frac{\tan^m t}{m!} = \sum_{n=m}^{\infty} T_{n,m} \frac{t^n}{n!}, \tag{2.4}$$

namely

$$T_{n,m} = \frac{i^{n-m}}{m!} \sum_{p=m}^{n} (-2)^{n-p} p! \binom{p-1}{m-1} \left\{ {n \atop p} \right\}. \tag{2.5}$$

From the definition we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \frac{d}{dt} \mathrm{A}_{k+1}(\tanh(t/2)) \\
&= 2 \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^{k+1}}.
\end{aligned} \tag{2.6}$$

By using $\tanh t = -i \tan(it)$ and equations (2.4) and (2.5), we can write

$$\begin{aligned}
(\tanh(t/2))^m &= (-i)^m m! \sum_{n=m}^{\infty} T_{n,m} \frac{i^n}{2^n} \frac{t^n}{n!} \\
&= (-i)^m (-1)^{\frac{n-m}{2}} \sum_{n=m}^{\infty} \sum_{p=m}^{n} (-2)^{n-p} p! \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{i^n}{2^n} \frac{t^n}{n!} \\
&= (-1)^m \sum_{n=m}^{\infty} \sum_{p=m}^{n} (-1)^p \frac{p!}{2^p} \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{t^n}{n!}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m+1}^{\infty} \sum_{p=2m+1}^{n} (-1)^{p+1} \frac{p!}{2^{p-1}} \binom{p-1}{2m} \left\{ {n \atop p} \right\} \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m}^{\infty} \sum_{p=2m}^{n} (-1)^p \frac{(p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p (p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!}.
\end{aligned}$$

By equating the coefficients of $t^n/n!$, we complete the proof of the theorem. □

3. Duality

We now prove the duality property of $D_n^{(k)}$ similar to (1.4) and (1.5).

Theorem 3.1. *For $n, k \in \mathbb{Z}_{\geq 0}$, it holds*

$$D_{2n}^{(-2k-1)} = D_{2k}^{(-2n-1)}. \tag{3.1}$$

We give two proofs using a generating function. The first proof gives a closed, symmetric formula for the generating function, whereas the second is more indirect and a little involved. We however think the second way may be of independent interest and decided to include it here.

Consider the following generating function of $D_{2n}^{(-2k-1)}$:

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n}}{(2n)!}\frac{y^{2k}}{(2k)!}.$$

We establish the closed formula of $F(x,y)$ as follows. The theorem follows immediately from the symmetry of the formula.

Proposition 3.2. *Set*

$$G(x,y) = \frac{e^{x+y}}{(1+e^x+e^y-e^{x+y})^2}.$$

Then we have

$$F(x,y) = G(x,y) + G(x,-y) + G(-x,y) + G(-x,-y).$$

In other words, $F(x,y)$ is the sub-series of $4G(x,y)$ which is even both in x and y.

Proof. We first compute the generating function of all $D_n^{(-k)}$,

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!}. \tag{3.2}$$

Proposition 3.3. *We have*

$$f(x,y) = \frac{e^x(e^y-1)}{1+e^x+e^y-e^{x+y}} + \frac{e^{-x}(e^y-1)}{1+e^{-x}+e^y-e^{-x+y}}. \tag{3.3}$$

Proof. By definition

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{\infty} \frac{\mathrm{A}_{-k}(\tanh(x/2))}{\sinh x}\frac{y^k}{k!} \\ &= \frac{2}{\sinh x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)^k(\tanh(x/2))^{2n+1}\frac{y^k}{k!}. \end{aligned}$$

We note that

$$2\sum_{n=0}^{\infty}(2n+1)^k t^{2n+1} = 2\left(t\frac{d}{dt}\right)^k \frac{t}{1-t^2} = \left(t\frac{d}{dt}\right)^k \left(\frac{1}{1-t} - \frac{1}{1+t}\right),$$

and by using the standard formula (*cf.*, *e.g.*, [1, Proposition 2.6 (4)])

$$\left(t\frac{d}{dt}\right)^k = \sum_{m=1}^{k} \left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m,$$

we see the right-hand side is equal to

$$\sum_{m=1}^{k} \left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m \left(\frac{1}{1-t} - \frac{1}{1+t}\right)$$

$$= \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}} \right).$$

Hence, by setting $t = \tanh(x/2)$ and noting $t/(1-t) = (e^x - 1)/2$, $-t/(1+t) = (e^{-x} - 1)/2$, $(\sinh x)(1-t) = e^{-x}(e^x - 1)$, $(\sinh x)(1+t) = e^x - 1$, we have

$$\begin{aligned} f(x,y) &= \frac{1}{\sinh x} \sum_{k=0}^{\infty} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}} \right) \frac{y^k}{k!} \qquad (t = \tanh(x/2)) \\ &= \sum_{k=0}^{\infty} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left\{ \frac{e^x}{e^x - 1} \left(\frac{e^x - 1}{2} \right)^m - \frac{1}{e^x - 1} \left(\frac{e^{-x} - 1}{2} \right)^m \right\} \frac{y^k}{k!} \\ &= \sum_{m=1}^{\infty} (e^y - 1)^m \left\{ \frac{e^x}{e^x - 1} \left(\frac{e^x - 1}{2} \right)^m - \frac{1}{e^x - 1} \left(\frac{e^{-x} - 1}{2} \right)^m \right\} \\ &= \frac{e^x}{e^x - 1} \cdot \frac{(e^y - 1)(e^x - 1)}{2 - (e^y - 1)(e^x - 1)} - \frac{1}{e^x - 1} \cdot \frac{(e^y - 1)(e^{-x} - 1)}{2 - (e^y - 1)(e^{-x} - 1)} \\ &= \frac{e^x(e^y - 1)}{1 + e^x + e^y - e^{x+y}} + \frac{e^{-x}(e^y - 1)}{1 + e^{-x} + e^y - e^{-x+y}}. \end{aligned}$$

□

From (3.3) we see that $f(x,y)$ is even in x, and so we have

$$\frac{f(x,y) - f(x,-y)}{2} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n}}{(2n)!} \frac{y^{2k+1}}{(2k+1)!}.$$

Our generating function $F(x,y)$ is the derivative of this with respect to y, and Proposition 3.2 follows from a straightforward calculation. Theorem 3.1 is thus proved.

□

Remark 3.4. We recall that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_n^{(-k-1)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{(e^x + e^y - e^{x+y})^2}$$

(see [7, Section 2]), which is remarkably similar to $G(x,y)$. The general coefficients of $4G(x,y)$ not necessarily even either in x or y may worth studying. The first several terms are given as

$$\begin{aligned} 4G(x,y) = 1 &+ \frac{x}{1!} + \frac{y}{1!} + \frac{x^2}{2!} + 2\frac{x}{1!}\frac{y}{1!} + \frac{y^2}{2!} + \frac{x^3}{3!} + 4\frac{x^2}{2!}\frac{y}{1!} + 4\frac{x}{1!}\frac{y^2}{2!} + \frac{y^3}{3!} \\ &+ \frac{x^4}{4!} + 8\frac{x^3}{3!}\frac{y}{1!} + 13\frac{x^2}{2!}\frac{y^2}{2!} + 8\frac{x}{1!}\frac{y^3}{3!} + \frac{y^4}{4!} + \cdots. \end{aligned}$$

For the second proof of Theorem 3.1, we need several lemmas.

Lemma 3.5.

$$F(x,y) = 2\sum_{n=0}^{\infty} \frac{\partial}{\partial x} \left(\tanh^{2n+1}(x/2) \right) \cosh((2n+1)y).$$

Proof. By (1.6), we have

$$\begin{aligned} F(x,y) &= 2\sum_{k=0}^{\infty} \frac{\mathrm{A}_{-2k-1}(\tanh(x/2))}{\sinh(x)} \frac{y^{2k}}{(2k)!} \\ &= \frac{2}{\sinh(x)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (2n+1)^{2k+1} \tanh^{2n+1}(x/2) \frac{y^{2k}}{(2k)!} \end{aligned}$$

$$= \frac{2}{\sinh(x)} \sum_{n=0}^{\infty} (2n+1) \tanh^{2n+1}(x/2) \cosh((2n+1)y)$$
$$= \frac{1}{\sinh(x/2)\cosh(x/2)} \sum_{n=0}^{\infty} (2n+1) \tanh^{2n}(x/2) \frac{\sinh(x/2)}{\cosh(x/2)} \cosh((2n+1)y)$$
$$= 2 \sum_{n=0}^{\infty} \frac{\partial}{\partial x} \left(\tanh^{2n+1}(x/2)\right) \cosh((2n+1)y).$$

Thus we have the assertion. □

We write

$$F(x,y) = \sum_{m=0}^{\infty} g_m(x) \frac{y^{2m}}{(2m)!} = \sum_{m=0}^{\infty} h_m(y) \frac{x^{2m}}{(2m)!}.$$

Then if we could prove $g_m(x) = h_m(x)$ for any $m \geq 0$, we are done.

First, we look at $g_m(x)$. Using Lemma 3.5, we have

$$g_m(x) = \left(\frac{\partial}{\partial y}\right)^{2m} F(x,y) \bigg|_{y=0} = 2 \frac{d}{dx} \sum_{n=0}^{\infty} (2n+1)^{2m} \tanh^{2n+1}(x/2).$$

Here we note that

$$\sum_{n=0}^{\infty} (2n+1)^{2m} t^{2n+1} = \left(t \frac{d}{dt}\right)^{2m} \sum_{n=0}^{\infty} t^{2n+1} = \left(t \frac{d}{dt}\right)^{2m} \frac{t}{1-t^2}. \tag{3.4}$$

Setting $t = \tanh(x/2)$ and noting

$$dt = \frac{1}{2} \frac{1}{\cosh^2(x/2)} dx, \qquad \frac{t}{1-t^2} = \frac{\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{1}{2} \sinh x,$$

we have

$$t \frac{d}{dt} = \tanh(x/2) \cdot 2\cosh^2(x/2) \frac{d}{dx} = \sinh x \frac{d}{dx}.$$

Therefore we obtain

$$g_m(x) = \frac{d}{dx} \left(\sinh x \frac{d}{dx}\right)^{2m} \sinh x. \tag{3.5}$$

We can explicitly write down the right-hand side by using the following lemma.

For $m \in \mathbb{Z}_{\geq 0}$, we define sequences $\{a_i^{(m)}\}_{0 \leq i \leq m} \subset \mathbb{Q}$ inductively by

$$\begin{aligned} & a_0^{(0)} = 1, \\ & a_i^{(m)} = \frac{1}{2} \left\{ i(2i-1) a_{i-1}^{(m-1)} - (2i+1)^2 a_i^{(m-1)} + (i+1)(2i+3) a_{i+1}^{(m-1)} \right\} \quad (m \geq 1), \end{aligned} \tag{3.6}$$

where we formally interpret $a_i^{(m)} = 0$ for $i < 0$ or $i > m$.

Lemma 3.6. *For $m \in \mathbb{Z}_{\geq 0}$,*

$$\left(\sinh x \frac{d}{dx}\right)^{2m} \sinh x = \sum_{i=0}^{m} a_i^{(m)} \sinh((2i+1)x). \tag{3.7}$$

Proof. We give the proof by induction on m. For $m = 0$, the identity trivially holds. We assume

$$\left(\sinh x \frac{d}{dx}\right)^{2(m-1)} \sinh x = \sum_{i=0}^{m-1} a_i^{(m-1)} \sinh((2i+1)x).$$

Using

$$\cosh(kx)\sinh(x) = \frac{1}{2} \left(\sinh((k+1)x) - \sinh((k-1)x)\right),$$

we have

$$\left(\sinh x\,\frac{d}{dx}\right)^{2m-1}\sinh x=\frac{1}{2}\sum_{i=0}^{m-1}(2i+1)a_i^{(m-1)}\left(\sinh((2i+2)x)-\sinh(2ix)\right),$$

and

$$\begin{aligned}
&\left(\sinh x\,\frac{d}{dx}\right)^{2m}\sinh x\\
&=\sum_{i=0}^{m-1}(2i+1)a_i^{(m-1)}\left\{\frac{i+1}{2}\left(\sinh((2i+3)x)-\sinh((2i+1)x)\right)\right.\\
&\qquad\left.-\frac{i}{2}\left(\sinh((2i+1)x)-\sinh((2i-1)x)\right)\right\}\\
&=\frac{1}{2}\sum_{i=1}^{m}i(2i-1)a_{i-1}^{(m-1)}\sinh((2i+1)x)\\
&\quad-\frac{1}{2}\sum_{i=0}^{m-1}(2i+1)^2a_i^{(m-1)}\sinh((2i+1)x)\\
&\quad+\frac{1}{2}\sum_{i=0}^{m-2}(i+1)(2i+3)a_{i+1}^{(m-1)}\sinh((2i+1)x).
\end{aligned}$$

Hence, using (3.6), we complete the proof by induction. □

Using this lemma, we obtain

$$g_m(x)=\sum_{i=0}^{m}(2i+1)a_i^{(m)}\cosh((2i+1)x). \tag{3.8}$$

Secondly, we compute $h_m(y)$. Again by using Lemma 3.5, we have

$$\begin{aligned}
h_m(y)&=\left(\frac{\partial}{\partial x}\right)^{2m}F(x,y)\Big|_{x=0}\\
&=2\sum_{n=0}^{\infty}\left(\frac{d}{dx}\right)^{2m+1}\left(\tanh^{2n+1}(x/2)\right)\cosh((2n+1)y)\Big|_{x=0}\\
&=2\sum_{n=0}^{m}\left(\frac{d}{dx}\right)^{2m+1}\tanh^{2n+1}(x/2)\Big|_{x=0}\cdot\cosh((2n+1)y)
\end{aligned}\tag{3.9}$$

because

$$\tanh^{2n+1}(x/2)=\frac{x^{2n+1}}{2^{2n+1}}+O(x^{2n+2})\quad(x\to 0).$$

We write down the right-hand side of (3.9) by using the following lemma.

Lemma 3.7. *For $n,l\in\mathbb{Z}_{\geq 0}$, there exist sequences $\{b_j^{(n,l)}\}_{0\leq j\leq l}\subset\mathbb{Q}$ such that*

$$\left(\frac{d}{dx}\right)^{l}\tanh^{2n+1}(x/2)=\sum_{j=0}^{l}b_j^{(n,l)}\tanh^{2n+1-l+2j}(x/2), \tag{3.10}$$

where $b_j^{(n,l)}=0$ if $2n+1-l+2j<0$. In particular,

$$\left(\frac{d}{dx}\right)^{2m+1}\tanh^{2n+1}(x/2)\Big|_{x=0}=b_{m-n}^{(n,2m+1)}. \tag{3.11}$$

Proof. For each n, we can immediately obtain the form (3.10) by induction on l, using the relation

$$\frac{d}{dx}\tanh^{2n+1}(x/2)=\frac{2n+1}{2}\left(\tanh^{2n}(x/2)-\tanh^{2n+2}(x/2)\right).$$

□

Combining this lemma and (3.9), we obtain

$$h_m(y)=2\sum_{n=0}^{m} b_{m-n}^{(n,2m+1)}\cosh((2n+1)y). \tag{3.12}$$

Now we are going to show $2b_{m-n}^{(n,2m+1)}=(2i+1)a_i^{(m)}$, which implies $g_m(x)=h_m(x)$. For $m,n\in\mathbb{Z}_{\geq 0}$ with $n\leq m$, set $\widetilde{b}_n^{(m)}=2b_{m-n}^{(n,2m+1)}$. Then, by (3.11), we have $\widetilde{b}_0^{(0)}=1$. Furthermore the following lemma holds.

Lemma 3.8. *For $m\in\mathbb{Z}_{\geq 1}$, we have the recursion*

$$\widetilde{b}_n^{(m)}=\frac{2n+1}{2}\left\{n\widetilde{b}_{n-1}^{(m-1)}-(2n+1)\widetilde{b}_n^{(m-1)}+(n+1)\widetilde{b}_{n+1}^{(m-1)}\right\}\quad (n\leq m), \tag{3.13}$$

where we interpret $b_i^{(k)}=0$ for $i<0$ or $i>k$.

Proof. It follows from (3.10) that

$$\left(\frac{d}{dx}\right)^{2m+1}\tanh^{2n+1}(x/2)=\sum_{j=0}^{2m+1} b_j^{(n,2m+1)}\tanh^{2n-2m+2j}(x/2). \tag{3.14}$$

Differentiating twice and using (3.10), we see that the left-hand side is equal to

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^{2m}\left(\frac{2n+1}{2}\tanh^{2n}(x/2)-\tanh^{2n+2}(x/2)\right)\\
&=\frac{2n+1}{2}\left(\frac{d}{dx}\right)^{2m-1}\left\{n\tanh^{2n-1}(x/2)-(2n+1)\tanh^{2n+1}(x/2)+(n+1)\tanh^{2n+3}(x/2)\right\}\\
&=\frac{2n+1}{2}\Bigg\{n\sum_{j=0}^{2m-1} b_j^{(n-1,2m-1)}\tanh^{2n-2m+2j}(x/2)\\
&\qquad-(2n+1)\sum_{j=0}^{2m-1} b_j^{(n,2m-1)}\tanh^{2n-2m+2+2j}(x/2)\\
&\qquad+(n+1)\sum_{j=0}^{2m-1} b_j^{(n+1,2m-1)}\tanh^{2n-2m+4+2j}(x/2)\Bigg\}.
\end{aligned}$$

If we let $x\to 0$, this goes to

$$\begin{aligned}
&\frac{2n+1}{2}\left\{nb_{m-n}^{(n-1,2m-1)}-(2n+1)b_{m-n-1}^{(n,2m-1)}+(n+1)b_{m-n-2}^{(n+1,2m-1)}\right\}\\
&=\frac{2n+1}{4}\left\{n\widetilde{b}_{n-1}^{(m-1)}-(2n+1)\widetilde{b}_n^{(m-1)}+(n+1)\widetilde{b}_{n+1}^{(m-1)}\right\}.
\end{aligned}$$

On the other-hand, the right-hand side of equation (3.14) tends to $b_{m-n}^{(n,2m+1)}=\widetilde{b}_n^{(m)}/2$ as $x\to 0$. Thus we obtain (3.13). □

Proof of Theorem 3.1. For $\{a_i^{(m)}\}$ defined by (3.6), set $\widetilde{a}_i^{(m)}=(2i+1)a_i^{(m)}$. Then (3.6) can be written as $\widetilde{a}_0^{(0)}=1$ and

$$\widetilde{a}_i^{(m)}=\frac{2i+1}{2}\left\{i\widetilde{a}_{i-1}^{(m-1)}-(2i+1)^2\widetilde{a}_i^{(m-1)}+(i+1)\widetilde{a}_{i+1}^{(m-1)}\right\}$$

which has exactly the same form as (3.13) for $\widetilde{b}_n^{(m)}$, namely $\widetilde{a}_n^{(m)} = \widetilde{b}_n^{(m)}$. Comparing (3.8) and (3.12), we obtain $g_m(x) = h_m(x)$. Thus we complete our second proof of Theorem 3.1. □

4. Multi-index case

We may define the multi-poly-cosecant numbers $D_n^{(k_1,\dots,k_r)}$ by

$$\frac{\mathrm{A}(k_1,\dots,k_r;\tanh(t/2))}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k_1,\dots,k_r)} \frac{t^n}{n!},$$

where the function

$$\mathrm{A}(k_1,\dots,k_r;z) = 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

for $k_1,\dots,k_r \in \mathbb{Z}$ is 2^r times $\mathrm{Ath}(k_1,\dots,k_r;z)$ which was introduced in [9, §5]. (Our $\mathrm{A}_k(z)$ is $\mathrm{A}(k;z)$.) We can regard $D_n^{(k_1,\dots,k_r)}$ as a level 2-version of the multi-poly-Bernoulli numbers $B_n^{(k_1,\dots,k_r)}$ and $C_n^{(k_1,\dots,k_r)}$ defined in [5].

In [9], we introduced the function

$$\psi(k_1,\dots,k_r;s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{A}(k_1,\dots,k_r;\tanh(t/2))}{\sinh(t)}\, dt \quad (\Re s > 0),$$

which can be analytically continued to $\mathbb{C}$ as an entire function. In the same manner as in the "level 1" case (ξ- and η-functions reviewed in the same paper), we see that the numbers $D_n^{(k_1,\dots,k_r)}$ appear as special values of $\psi(k_1,\dots,k_r;s)$ at non-positive integer arguments:

$$\psi(k_1,\dots,k_r;-n) = (-1)^n D_n^{(k_1,\dots,k_r)} \quad (n = 0,1,2,\dots).$$

Also, we can obtain a similar recurrence relation for multi-poly-cosecant numbers as

$$D_n^{(k_1,\dots,k_{r-1},k_r-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k_1,\dots,k_r)}$$

for any $r \geq 1, k_i \in \mathbb{Z}$ and $n \geq 0$.

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