Zeta functions connecting multiple zeta values and poly-Bernoulli numbers

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Abstract.

We first review our previous works of Arakawa and the authors on two, closely related single-variable zeta functions. Their special values at positive and negative integer arguments are respectively multiple zeta values and poly-Bernoulli numbers. We then introduce, as a generalization of Sasaki's work, level 2 analogue of one of the two zeta functions and prove results analogous to those by Arakawa and the first named author.

§1. Introduction

In this (half expository) paper, we discuss some properties of two single-variable functions $\xi_k(s)$ and $\eta_k(s)$, which are closely related with each other, and their generalizations. We are interested in these functions because multiple zeta values and poly-Bernoulli numbers appear as special values, respectively at positive and negative integer arguments.

The multiple zeta value (MZV) and its variant multiple zeta-star value (MZSV), a vast amount of researches on which from various points of view has been carried out in recent years, are defined by

$$\zeta(k_1,\ldots,k_r)=\sum_{1\le m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

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and

$$\zeta^{\star}(k_1,\dots,k_r)=\sum_{1\le m_1\le\cdots\le m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

for $k_1,\dots,k_r\in\mathbb{Z}_{\ge1}$ with $k_r>1$ (for convergence), respectively. MZVs appear as special values of $\xi_k(s)$ and MZSV as those of $\eta_k(s)$ (Theorem 2.2).

Poly-Bernoulli numbers, having also two versions $B_n^{(k)}$ and $C_n^{(k)}$, were defined by the first named author in [12] and in Arakawa-Kaneko [2] by using generating series: For an integer $k\in\mathbb{Z}$, define sequences of rational numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!} \tag{1.1}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{t}-1}=\sum_{n=0}^{\infty}C_n^{(k)}\frac{t^n}{n!}, \tag{1.2}$$

where $\mathrm{Li}_k(z)$ is the polylogarithm function (or rational function when $k\le0$) defined by

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}\quad(|z|<1). \tag{1.3}$$

Since $\mathrm{Li}_1(z)=-\log(1-z)$, the generating functions on the left-hand sides respectively become

$$\frac{te^t}{e^t-1}\quad\text{and}\quad\frac{t}{e^t-1}$$

when $k=1$, and hence $B_n^{(1)}$ and $C_n^{(1)}$ are usual Bernoulli numbers, the only difference being $B_n^{(1)}=1/2$ and $C_n^{(1)}=-1/2$. When $k\ne1$, $B_n^{(k)}$'s and $C_n^{(k)}$'s are totally different numbers. We mention in passing that $B_n^{(-k)}$ $(n,k\ge0)$ coincides with the number of acyclic orientations of the complete bipartite graph $K_{n,k}$ (see [5]), and is also equal to the number of 'lonesum' matrices of size $n \times k$ (see [3]).

In [16] and [2], we showed that poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ appear as special values at nonpositive integers of $\eta_k(s)$ and $\xi_k(s)$ respectively. Multi-indexed version of these results were established in [16] and will be reviewed in §2 ((2.6) and (2.8)).

In §3, we give formulas obtained in [16] relating ξ and η (Proposition 3.2) and also an expression of ξ in terms of multiple zeta functions (Theorems 3.6).

In §4, we focus on the duality properties of $B_n^{(k)}$ and $C_n^{(k)}$, namely

$$B_n^{(-k)} = B_k^{(-n)}, \tag{1.4}$$

$$C_n^{(-k-1)} = C_k^{(-n-1)} \tag{1.5}$$

for $k, n \in \mathbb{Z}_{\geq 0}$ (see [12, Theorems 1 and 2] and [13, § 2]). We can interpret (1.4) and (1.5) as the identities

$$\eta_{-k}(-n) = \eta_{-n}(-k) \quad \text{and} \quad \widetilde{\xi}_{-k-1}(-n) = \widetilde{\xi}_{-n-1}(-k)$$

for $k, n \in \mathbb{Z}_{\geq 0}$, respectively, where $\widetilde{\xi}_{-k}(s)$ is another type of function interpolating $C_n^{(k)}$ (see (4.8)). These relations even hold if we extend k and n to complex variables, as shown by Yamamoto [26] and Komori-Tsumura [19] (see (4.17) and (4.20)).

In §5, we generalize Sasaki's zeta function (see [22]) from the viewpoint that it gives a level 2-version of $\xi(k_1, \ldots, k_r; s)$. Our previous methods work well in this case and we obtain several formulas related to multiple zeta values of level 2. This section is substantially new.

§2. Multi-poly-Bernoulli numbers and related zeta functions

Imatomi, Takeda, and the first named author [10] introduced multi-index generalizations of poly-Bernoulli numbers ("multi-poly-Bernoulli numbers") as follows.

Definition 1. *For $k_1, \ldots, k_r \in \mathbb{Z}$, define two types of multiple poly-Bernoulli numbers by*

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!} \tag{2.1}$$

and

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}, \tag{2.2}$$

where

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{1\le m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2}\cdots m_r^{k_r}} \tag{2.3}$$

is the multiple polylogarithm.

Remark 2.1. In [10], the following relation between $C_{p-2}^{(k_1,\ldots,k_r)}$ and the 'finite multiple zeta value' was proved:

$$\sum_{1\le m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \equiv -C_{p-2}^{(k_1,\ldots,k_{r-1},k_r-1)} \mod p \tag{2.4}$$

for any prime number p.

In connection with these numbers, we consider the following two types of zeta functions. The first one, $\xi(k_1,\ldots,k_r;s)$, was defined in [2] as follows.

Definition 2. *For $r \in \mathbb{Z}_{\ge 1}$, $k_1,\ldots,k_r \in \mathbb{Z}_{\ge 1}$ and $\Re s > 0$,*

$$\xi(k_1,\ldots,k_r;s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t-1}\, dt, \tag{2.5}$$

where $\Gamma(s)$ is the gamma function. In the case $r=1$, denote $\xi(k;s)$ by $\xi_k(s)$. Note that $\xi_1(s) = s\zeta(s+1)$.

This can be analytically continued to an entire function for $s \in \mathbb{C}$, and satisfies the following (see [2, Remark 2.4]):

$$\xi(k_1,\ldots,k_r;-m) = (-1)^m C_m^{(k_1,\ldots,k_r)} \quad (m \in \mathbb{Z}_{\ge 0}) \tag{2.6}$$

for $(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$. This can be regarded as a poly-analogue of the classical evaluation

$$\xi_1(-m)=(-m)\zeta(1-m)=(-1)^m C_m.$$

The second, $\eta(k_1,\dots,k_r;s)$, is defined as follows (see [16]).

Definition 3. *For $r\in\mathbb{Z}_{\geq 1}$, $k_1,\dots,k_r\in\mathbb{Z}_{\geq 1}$ and $\Re s>1-r$,*

$$\eta(k_1,\dots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^t)}{1-e^t}dt \tag{2.7}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$, In the case $r=1$, denote $\eta(k;s)$ by $\eta_k(s)$. Note that $\eta_1(s)=s\zeta(s+1)$.

Similar to $\xi(k_1,\dots,k_r;s)$, we see that $\eta(k_1,\dots,k_r;s)$ can be analytically continued to an entire function for $s\in\mathbb{C}$, and satisfies the following (see [16, Theorem 2.3]):

$$\eta(k_1,\dots,k_r;-m)=B_m^{(k_1,\dots,k_r)}\quad (m\in\mathbb{Z}_{\geq 0}) \tag{2.8}$$

for positive integers $k_1,\dots,k_r\in\mathbb{Z}_{\geq 1}$. This can be regarded as a poly-analogue of

$$\eta_1(-m)=(-m)\zeta(1-m)=B_m.$$

As for their values at positive integers, we can obtain explicit expressions in terms of multiple zeta/zeta-star values as follows. We prepare several notations. For an index set $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$, put $\mathbf{k}_+=(k_1,\dots,k_{r-1},k_r+1)$. The usual dual index of an admissible index $\mathbf{k}$ is denoted by $\mathbf{k}^*$. For $\mathbf{j}=(j_1,\dots,j_r)\in\mathbb{Z}_{\geq 0}^r$, we set $|\mathbf{j}|=j_1+\cdots+j_r$ and call it the weight of $\mathbf{j}$, and $d(\mathbf{j})=r$, the depth of $\mathbf{j}$. For two such indices $\mathbf{k}$ and $\mathbf{j}$ of the same depth, we denote by $\mathbf{k}+\mathbf{j}$ the index obtained by the component-wise addition, $\mathbf{k}+\mathbf{j}=(k_1+j_1,\dots,k_r+j_r)$, and by $b(\mathbf{k};\mathbf{j})$ the quantity given by

$$b(\mathbf{k};\mathbf{j}):=\prod_{i=1}^r\binom{k_i+j_i-1}{j_i}.$$

Theorem 2.2 ([16] Theorem 2.5)**.** *For any index set $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$ and any $m\in\mathbb{Z}_{\geq 1}$, we have*

$$\xi(k_1,\dots,k_r;m)=\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=n} b((\mathbf{k}_+)^*;\mathbf{j})\,\zeta((\mathbf{k}_+)^*+\mathbf{j}) \tag{2.9}$$

and

$$\eta(k_1,\dots,k_r;m) = (-1)^{r-1} \sum_{|\mathbf{j}|=m-1,\, d(\mathbf{j})=n} b((\mathbf{k}_+)^*;\mathbf{j}) \times \zeta^\star((\mathbf{k}_+)^*+\mathbf{j}), \tag{2.10}$$

where both sums run over all $\mathbf{j}\in\mathbb{Z}^r_{\geq 0}$ *of weight* $m-1$ *and depth* $n := d(\mathbf{k}_+^*)$ $(= |\mathbf{k}|+1-d(\mathbf{k}))$.

In particular, we have

$$\xi(k_1,\dots,k_r;1) = \zeta(\mathbf{k}_+)$$

and

$$\eta(k_1,\dots,k_r;1) = (-1)^{r-1}\zeta^\star((\mathbf{k}_+)^*).$$

Here we have used the duality $\zeta((\mathbf{k}_+)^*) = \zeta(\mathbf{k}_+)$.

Remark 2.3. In [2, Theorem 9 (i)], we proved (2.9) in the case when $(k_1,\dots,k_r) = (1,\dots,1,k)$. The above formulas generalize this and give its η-version. In fact, these can be proved by the same method as in [2], i.e., by considering the integral expressions

$$\begin{aligned}
\zeta(k_1,\dots,k_r) &= \frac{1}{\prod_{j=1}^r \Gamma(k_j)} \int_0^\infty\cdots\int_0^\infty \frac{x_1^{k_1-1}\cdots x_r^{k_r-1}}{e^{x_1+\cdots+x_r}-1} \\
&\quad\times \frac{1}{e^{x_2+\cdots+x_r}-1}\cdots\frac{1}{e^{x_r}-1}dx_1\cdots dx_r, \\
\zeta^\star(k_1,\dots,k_r) &= \frac{1}{\prod_{j=1}^r \Gamma(k_j)} \int_0^\infty\cdots\int_0^\infty \frac{x_1^{k_1-1}\cdots x_r^{k_r-1}}{e^{x_1+\cdots+x_r}-1} \\
&\quad\times \frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r}-1}\cdots\frac{e^{x_r}}{e^{x_r}-1}dx_1\cdots dx_r
\end{aligned}$$

for $k_1,\dots,k_r\in\mathbb{Z}_{\geq 1}$ with $k_r\geq 2$.

We emphasize that the formulas (2.9) and (2.10) have remarkable similarity in that one obtains (2.10) just by replacing multiple zeta values in (2.9) with multiple zeta-star values.

Noting the duality $(k+1)^* = (\underbrace{1,\dots,1}_{k-1},2)$, we can obtain the following two identities. The former is a special case of [2, Theorem 9 (i)] and the latter is [16, Corollary 2.8].

Corollary 2.4. *For $k, m \geq 1$, we have*

$$\xi_k(m) = \sum_{\substack{j_1,\dots,j_{k-1}\geq 1, j_k\geq 2\\ j_1+\cdots+j_k=k+m}} (j_k-1)\zeta(j_1,\dots,j_{k-1},j_k), \tag{2.11}$$

$$\eta_k(m) = \sum_{\substack{j_1,\dots,j_{k-1}\geq 1, j_k\geq 2\\ j_1+\cdots+j_k=k+m}} (j_k-1)\zeta^{\star}(j_1,\dots,j_{k-1},j_k). \tag{2.12}$$

§3. Relations among ξ, η and multiple zeta functions

In this section, we give formulas describing relations among ξ, η and multiple zeta functions by employing two types of connection formulas for the multiple polylogarithm.

First we show that each of the functions η and ξ can be written as a linear combination of the other in exactly the same way, using the so-called Landen-type connection formula for the multiple polylogarithm $\mathrm{Li}_{k_1,\dots,k_r}(z)$.

For two indices $\mathbf{k}$ and $\mathbf{k}'$ of the same weight, we say $\mathbf{k}'$ refines $\mathbf{k}$, denoted $\mathbf{k} \preceq \mathbf{k}'$, if $\mathbf{k}$ is obtained from $\mathbf{k}'$ by replacing some commas by $+$'s. For example,

$$(3) = (1+1+1) \preceq (1,1,1),\ (2,3) = (2,2+1) \preceq (2,2,1)$$

etc. Using this notation, the Landen connection formula for the multiple polylogarithm is as follows.

Lemma 3.1 (Okuda-Ueno [21] Proposition 9)**.** *For any index $\mathbf{k}$ of depth r, we have*

$$\mathrm{Li}_{\mathbf{k}}\left(\frac{z}{z-1}\right) = (-1)^r \sum_{\mathbf{k}\preceq\mathbf{k}'} \mathrm{Li}_{\mathbf{k}'}(z). \tag{3.1}$$

Using (3.1) for the case $z = 1-e^{-t}$ (resp. $1-e^{t}$), namely $z/(z-1) = 1-e^{t}$ (resp. $1-e^{-t}$), we can prove the following.

Proposition 3.2 ([16] Proposition 3.2)**.** *Let $\mathbf{k}$ be any index set and r its depth. We have the relations*

$$\eta(\mathbf{k};s) = (-1)^{r-1} \sum_{\mathbf{k}\preceq\mathbf{k}'} \xi(\mathbf{k}';s) \tag{3.2}$$

and

$$\xi(\mathbf{k};s)=(-1)^{r-1}\sum_{\mathbf{k}\preceq\mathbf{k}'}\eta(\mathbf{k}';s). \tag{3.3}$$

The reason of the symmetry is that the transformation $z\to z/(z-1)$ is involutive.

Here we recall a certain formula between ξ and the single-variable multiple zeta function

$$\zeta(k_1,\ldots,k_r;s)=\sum_{1\le m_1<\cdots<m_r<m}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}m^s} \tag{3.4}$$

defined for integers $k_1,\ldots,k_r$ as follows.

Theorem 3.3 ([2] Theorem 8)**.** *For $r,k\in\mathbb{Z}_{\ge1}$,*

(3.5)

$$\begin{aligned}&\xi(\underbrace{1,\ldots,1}_{r-1},k;s)\\ &=(-1)^{k-1}\sum_{\substack{a_1+\cdots+a_k=r\\ \forall a_j\ge0}}\binom{s+a_k-1}{a_k}\zeta(a_1+1,\ldots,a_{k-1}+1;a_k+s)\\ &\quad+\sum_{j=0}^{k-2}(-1)^j\zeta(\underbrace{1,\ldots,1}_{r-1},k-j)\zeta(\underbrace{1,\ldots,1}_{j};s).\end{aligned}$$

Concerning a generalization of this result, Arakawa and the first named author posed the following question.

Problem 3.4 ([2] §8, Problem (i))**.** *For a general index set $(k_1,\ldots,k_r)$, is the function $\xi(k_1,\ldots,k_r;s)$ also expressed by multiple zeta functions as in Theorem 3.3 stated above?*

An affirmative answer was given in [16]. To describe it, we consider an Euler-type connection formula for the multiple polylogarithm.

Lemma 3.5 ([16] Lemma 3.5)**.** *Let $\mathbf{k}$ be any index. Then we have*

$$\mathrm{Li}_{\mathbf{k}}(1-z)=\sum_{\mathbf{k}',\,j\ge0}c_{\mathbf{k}}(\mathbf{k}';j)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\mathrm{Li}_{\mathbf{k}'}(z), \tag{3.6}$$

where the sum on the right-hand side runs over indices $\mathbf{k}'$ *and integers* $j \geq 0$ *that satisfy* $|\mathbf{k}'|+j \leq |\mathbf{k}|$*, and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}$*-linear combination of multiple zeta values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$*. We understand* $\mathrm{Li}_{\emptyset}(z)=1$ *and* $|\emptyset|=0$ *for the empty index* $\emptyset$*, and the constant* 1 *is interpreted as a multiple zeta value of weight* 0*.*

From this, we can obtain formulas expressing $\xi(k_1,\ldots,k_r;s)$ in terms of multiple zeta functions, which can be regarded as a general answer to the above problem. However, we should note that there are no closed formulas for the coefficients $c_{\mathbf{k}}(\mathbf{k}';j)$, and we can only compute them inductively from low weights.

Theorem 3.6 ([16] Theorem 3.6)**.** *Let* $\mathbf{k}$ *be any index set. The function* $\xi(\mathbf{k};s)$ *can be written in terms of multiple zeta functions as*

$$\xi(\mathbf{k};s)=\sum_{\mathbf{k}',\,j\geq 0} c_{\mathbf{k}}(\mathbf{k}';j)\binom{s+j-1}{j}\zeta(\mathbf{k}';s+j). \tag{3.7}$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j \geq 0$ *satisfying* $|\mathbf{k}'|+j \leq |\mathbf{k}|$*, and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}$*-linear combination of multiple zeta values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$*. The index* $\mathbf{k}'$ *may be* $\emptyset$ *and for this we set* $\zeta(\emptyset;s+j)=\zeta(s+j)$*.*

As an example, we used the identity

$$\mathrm{Li}_{2,1}(1-z)=2\mathrm{Li}_3(z)-\log z\cdot\mathrm{Li}_2(z)-\zeta(2)\log z-2\zeta(3), \tag{3.8}$$

obtained by integrating the well-known

$$\mathrm{Li}_2(1-z)+\mathrm{Li}_2(z)=\zeta(2)-\log z\log(1-z). \tag{3.9}$$

Applying (3.8) to the definition of ξ in (2.5), we obtained

$$\xi(2,1;s)=2\zeta(3;s)+s\zeta(2;s+1)+\zeta(2)s\zeta(s+1)-2\zeta(3)\zeta(s). \tag{3.10}$$

Lemma 3.5 (and its proof in [16]) gives an inductive way to compute the functional equation under $z\mapsto 1-z$. Here we give a further example which implies a multiple version of (3.10). The following identity is an example of Lemma 3.5 because $(\log z)^n = (-1)^n n!\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{n}}(1-z)$ (see e.g. [2, Lemma 1]).

Lemma 3.7. *For $r \in \mathbb{Z}_{\geq 0}$ and $0 < z < 1$,*

$$(3.11) \qquad (-1)^r \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{r}}(1-z) = -(r+1)\mathrm{Li}_{r+2}(z) + (\log z)\mathrm{Li}_{r+1}(z) + \sum_{j=0}^{r} \frac{r-j+1}{j!}\zeta(r-j+2)(\log z)^j.$$

Proof. We proceed by induction on r. When $r=0$, (3.11) is nothing but (3.9). For the case $r \geq 1$, if we differentiate the right-hand side of (3.11), the result is equal to

$$(-1)^r \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{r-1}}(1-z)\frac{1}{z} = (-1)^r \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{r-1},0}(1-z),$$

by the induction hypothesis for the case of $r-1$. Integrating it again, we obtain the assertion for the case of r. Thus we complete the proof. Q.E.D.

Applying (3.11) with $z = e^{-t}$ $(t > 0)$ to (2.5), we obtain the following generalization of (3.10).

Theorem 3.8. *For $r \in \mathbb{Z}_{\geq 1}$,*

$$(3.12) \qquad (-1)^r \xi(2, \underbrace{1,\ldots,1}_{r}; s) = -(r+1)\zeta(r+2; s) - s\zeta(r+1; s+1) + \sum_{j=0}^{r} (-1)^j (r-j+1)\zeta(r-j+2)\binom{s+j-1}{j}\zeta(s+j).$$

Example 3.9. The case $r=1$ is (3.10) and the case $r=2$ is

$$\xi(2,1,1;s) = -3\zeta(4;s) + 3\zeta(4)\zeta(s) - s\zeta(3;s+1) - 2s\zeta(3)\zeta(s+1) + \frac{s(s+1)}{2}\zeta(2)\zeta(s+2).$$

These coincide with the formula in [16, Example 3.8].

§4. The function $\eta(k_1,\ldots,k_r;s)$ for nonpositive indices and related topics

In this section, we consider multi-polylogarithms with nonpositive indices.

Lemma 4.1 ([16] Lemma 4.1). *For $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $P(x;k_1,\ldots,k_r) \in \mathbb{Z}[x]$ such that*

$$\mathrm{Li}_{-k_1,\ldots,-k_r}(z) = \frac{P(z;k_1,\ldots,k_r)}{(1-z)^{k_1+\cdots+k_r+r}}, \tag{4.1}$$

$$\deg P(x;k_1,\ldots,k_r) = \begin{cases} r & (k_1=\cdots=k_r=0) \\ k_1+\cdots+k_r+r-1 & (\text{otherwise}), \end{cases} \tag{4.2}$$

$$x^r \mid P(x;k_1,\ldots,k_r). \tag{4.3}$$

Specifically, $P(x;\underbrace{0,0,\ldots,0}_{r}) = x^r$.

The case of $r=1$ is well-known (see, for example, Shimura [23, Equations (2.17), (4.2) and (4.6)]). For example,

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}.$$

However, even if we apply this definition to (2.5) as well as in the case of positive indices, we cannot define the function ξ with nonpositive indices. In fact, if we set, for example,

$$\xi_0(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_0(1-e^{-t})}{e^t-1}dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt,$$
$$\xi_{-1}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-1}(1-e^{-t})}{e^t-1}dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t dt,$$

we see that these integrals are divergent for any $s \in \mathbb{C}$.

On the other hand, we can define the function η with nonpositive indices as follows.

Definition 4. *For $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$, define*

$$\eta(-k_1,\ldots,-k_r;s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)}{1-e^t}dt \tag{4.4}$$

for $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - r$. *In the case* $r = 1$, *denote* $\eta(-k; s)$ *by* $\eta_{-k}(s)$.

We can easily check that the integral on the right-hand side of (4.4) is absolutely convergent for $\mathrm{Re}(s) > 1 - r$. Similar to the case with positive indices, we can see that $\eta(-k_1, \ldots, -k_r; s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies

$$\eta(-k_1, \ldots, -k_r; -m) = B_m^{(-k_1,\ldots,-k_r)} \quad (m \in \mathbb{Z}_{\geq 0}) \tag{4.5}$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$. In particular when $r = 1$, we have

$$\eta_{-k}(-m) = B_m^{(-k)} \quad (k \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}_{\geq 0}). \tag{4.6}$$

Furthermore, we modify the definition (2.5) as follows.

Definition 5. *For* $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ *with* $(k_1, \ldots, k_r) \neq (0, \ldots, 0)$, *define*

$$\widetilde{\xi}(-k_1, \ldots, -k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1 - e^t)}{e^{-t} - 1} dt \tag{4.7}$$

for $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - r$. *In the case* $r = 1$, *denote* $\widetilde{\xi}(-k; s)$ *by* $\widetilde{\xi}_{-k}(s)$ *for* $k \geq 1$.

We see that $\widetilde{\xi}(-k_1, \ldots, -k_r; s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies

$$\widetilde{\xi}(-k_1, \ldots, -k_r; -m) = C_m^{(-k_1,\ldots,-k_r)} \quad (m \in \mathbb{Z}_{\geq 0}) \tag{4.8}$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, \ldots, k_r) \neq (0, \ldots, 0)$. In particular, $\widetilde{\xi}_{-k}(-m) = C_m^{(-k)}$ $(k \in \mathbb{Z}_{\geq 1},\ m \in \mathbb{Z}_{\geq 0})$.

Remark 4.2. Note that we cannot define $\widetilde{\xi}(k_1, \ldots, k_r; s)$ by replacing $(-k_1, \ldots, -k_r)$ with $(k_1, \ldots, k_r)$ in (4.7). In fact, if we set, for example,

$$\widetilde{\xi}_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_1(1 - e^t)}{e^{-t} - 1} dt = s\zeta(s+1) + \frac{1}{\Gamma(s)} \int_0^\infty t^s dt,$$

which is not convergent for any $s \in \mathbb{C}$.

Here we extend definitions of poly-Bernoulli numbers (1.1) and (1.2) as follows. For $s \in \mathbb{C}$, we define

$$\frac{\mathrm{Li}_s(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(s)} \frac{t^n}{n!}, \tag{4.9}$$

$$\frac{\mathrm{Li}_s(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(s)} \frac{t^n}{n!}, \tag{4.10}$$

where

$$\mathrm{Li}_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \quad (|z|<1). \tag{4.11}$$

Using

(4.12)

$$\prod_{j=1}^{r} \frac{e^{\sum_{\nu=j}^{r} x_\nu}(1-e^t)}{1-e^{\sum_{\nu=j}^{r} x_\nu}(1-e^t)} = \sum_{k_1,\ldots,k_r \geq 0} \mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t) \frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!},$$

we have the following.

Theorem 4.3 ([16] Theorem 4.7)**.** *For $k \in \mathbb{Z}_{\geq 0}$,*

$$\eta(-k;s) = B_k^{(s)}. \tag{4.13}$$

Setting $s=-n \in \mathbb{Z}_{\leq 0}$ in (4.13) in the case $r=1$ and using (4.6), we obtain the duality relation $B_n^{(-k)} = B_k^{(-n)}$ in (1.4), which can be written as

$$\eta_{-k}(-n) = \eta_{-n}(-k). \tag{4.14}$$

Similarly, we can prove that

$$\widetilde{\xi}_{-k-1}(-n) = \widetilde{\xi}_{-n-1}(-k) \quad (n,k \in \mathbb{Z}_{\geq 0}), \tag{4.15}$$

namely the duality relation $C_n^{(-k-1)} = C_k^{(-n-1)}$ in (1.5).

On the other hand, for $n,k \in \mathbb{Z}_{\geq 1}$, we found experimentally the identities ([16, Eq. (36)])

$$\eta_k(n) = \eta_n(k), \tag{4.16}$$

which was soon proved and generalized by Yamamoto [26]. In particular when $r=1$, he showed

$$\eta_u(s) = \eta_s(u) \tag{4.17}$$

for $s, u \in \mathbb{C}$, where

$$\eta_u(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_u(1-e^t)}{1-e^t} dt \ (s, u \in \mathbb{C};\ \Re(s) > 1), \tag{4.18}$$

which can be analytically continued to $(s,u) \in \mathbb{C}^2$. More recently Kawasaki and Ohno gave an alternative proof of (4.16) in [18].

Inspired by Yamamoto's result, Komori and the second named author [19] consider a more general type of zeta function denoted by $\xi_D(u,s;y,w;g)$ $(u,s,y,w \in \mathbb{C}; g \in GL(2,\mathbb{C}))$ which satisfies

$$\xi_D(u,s;y,w-1;g) = -\frac{1}{\det g} \xi_D(s,u;w,y-1;g^{-1}). \tag{4.19}$$

When $g = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $\xi_D(u,s;1,0;g) = \eta_u(s)$. Hence (4.19) in this case implies (4.17). Also, we have $\xi_D(u,s;1,-1;g) = \widetilde{\xi}_u(s)$ which is defined by replacing $-k$ with u in the definition of $\widetilde{\xi}_{-k}(s)$ (see Definition 5). Hence (4.19) in this case implies

$$\widetilde{\xi}_{u-1}(s) = \widetilde{\xi}_{s-1}(u), \tag{4.20}$$

which includes (4.15).

Furthermore, Yamamoto proved the identity ([26, §1])

$$\eta_k(n) = \sum_{0<a_1 \le \cdots \le a_k = b_n \ge \cdots \ge b_1 > 0} \frac{1}{a_1 \cdots a_k b_1 \cdots b_n} \quad (k, n \in \mathbb{Z}_{\ge 1}),$$

which directly reveals the symmetry (4.16). Similar expression for $\xi_k(n)$ is

$$\xi_k(n) = \sum_{0<a_1 = \cdots = a_k = b_n \ge \cdots \ge b_1 > 0} \frac{1}{a_1 \cdots a_k b_1 \cdots b_n} \quad (k, n \in \mathbb{Z}_{\ge 1}),$$

which unfortunately is not symmetric. We do not know if any duality property holds for $\xi_k(s)$.

In addition, recall that we mention at the end of §3 in [16] the identity

(4.21)
$$\eta_k(m) = \binom{m+k}{k}\zeta(m+k) - \sum_{\substack{2\le r\le k+1\\ j_1+\cdots+j_r=m+k-r-1}} \binom{j_1+\cdots+j_{r-1}}{k-r+1}\cdot\zeta(j_1+1,\cdots,j_{r-1}+1,j_r+2),$$

without proof. Recently Shingu proved

$$\eta_k(m) = \sum_{\substack{k_1+\cdots+k_r=k+n\\ 1\le r\le k,\ k_r\ge 2}} \sum_{i=1}^{k_r-1} \binom{k+n-r-i}{n-i}\zeta(k_1,\ldots,k_r) \tag{4.22}$$

in his master's thesis [24] by using Yamamoto's multiple integrals introduced in [25]. It is easy to derive (4.21) from (4.22).

At the end of this section, we consider an application of the duality relation $\eta(k;n)=\eta(n;k)$ in (4.16). By combining Proposition 3.2 and Theorem 3.6, we obtain, for $k,n\in\mathbb{Z}_{\ge 1}$,

$$\begin{aligned}\eta(k;n) &= \sum_{(k)\preceq\mathbf{k}'} \xi(\mathbf{k}';n)\\ &= \sum_{(k)\preceq\mathbf{k}'} \sum_{\mathbf{k}'',\,j\ge 0} c_{\mathbf{k}'}(\mathbf{k}'';j)\binom{n+j-1}{j}\zeta(\mathbf{k}'';n+j),\end{aligned}$$

where the sum is over indices $\mathbf{k}''$ and integers $j\ge 0$ satisfying $|\mathbf{k}''|+j\le|\mathbf{k}'|$, and $c_{\mathbf{k}'}(\mathbf{k}'';j)$ is a $\mathbb{Q}$-linear combination of multiple zeta values of weight $|\mathbf{k}'|-|\mathbf{k}''|-j$ determined by (3.7).

We see that Proposition 3.2 and Theorem 3.6 were given by the connection formulas of Euler type and Landen type, respectively. From (4.16), we obtain the following.

Theorem 4.4. *With the above notation, for $k,n\in\mathbb{Z}_{\ge 1}$,*

$$\sum_{(k)\preceq\mathbf{k}'} \sum_{\mathbf{k}'',\,j\ge 0} c_{\mathbf{k}'}(\mathbf{k}'';j)\binom{n+j-1}{j}\zeta(\mathbf{k}'';n+j) \tag{4.23}$$

$$= \sum_{(n)\preceq \mathbf{n}'} \sum_{\mathbf{n}'',\, j\geq 0} c_{\mathbf{n}'}(\mathbf{n}''; j)\binom{k+j-1}{j}\zeta(\mathbf{n}''; k+j).$$

Example 4.5. For example, set $(k,n)=(3,2)$ in (4.23). Then, by [16, Example 3.8], we have

$$\begin{aligned}&\zeta(1,2,2)+\zeta(2,1,2)+2\zeta(1,1,3)-\zeta(2)\zeta(1,2)+\zeta(3,2)-3\zeta(1,4)\\ &+2\zeta(2)\zeta(3)+4\zeta(5)=6\zeta(5)-3\zeta(1,4)-\zeta(2,3)+\zeta(2)\zeta(3).\end{aligned}$$

This can of course be checked by known identities, for example, double shuffle relations. We do not pursue here connections between identities of MZVs obtained by $\eta(k;n)=\eta(n;k)$ as above and known sets of identities. Are there some interesting aspects?

§5. Zeta functions interpolating multiple zeta values of level 2

In this section, we define a certain level 2-version of the function $\xi(k_1,\ldots,k_r;s)$ which interpolates multiple zeta values of level 2 at positive integers. Here, we mean by MZVs of level 2 the quantities essentially equivalent to those often referred to as the Euler sums. But we only look at a special subclass of them. Specifically, we look at the quantity

$$\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \bmod 2}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}},$$

i.e., the sum is restricted to $m_1, m_2, m_3, \ldots$ with odd, even, odd, $\ldots$ in alternating manner. These numbers in depth 2 were considered in [15] in connection to modular forms of level 2, establishing a generalization of the work by Gangle-Kaneko-Zagier [6].

In [22, Section 4], Sasaki considered the polylogarithm of level 2 defined by

$$\mathrm{Ath}_k(z)=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)^k}=\mathrm{Li}_k(z)-\frac{1}{2^k}\mathrm{Li}_k(z^2)$$

for $k\in\mathbb{Z}$. When $k=1$, this becomes the well-known

$$\mathrm{Ath}_1(z)=\tanh^{-1} z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}=\mathrm{Li}_1(z)-\frac{1}{2}\mathrm{Li}_1(z^2).$$

We generalize this to a multiple version. For $k_1, \ldots, k_r \in \mathbb{Z}$, define

$$\begin{aligned}\mathrm{Ath}(k_1, \ldots, k_r; z) &= \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{z^{\sum_{\nu=1}^{r}(2n_\nu+1)}}{\prod_{j=1}^{r}\left(\sum_{\nu=1}^{j}(2n_\nu+1)\right)^{k_j}}.\end{aligned} \tag{5.1}$$

Note that since $\mathrm{Ath}(1; z) = \tanh^{-1} z$, we have

$$\mathrm{Ath}(1; \tanh t) = t. \tag{5.2}$$

Similar to [2, Lemma 1], we can easily obtain the following.

Lemma 5.1. (i) *For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}&\frac{d}{dz}\mathrm{Ath}(k_1, \ldots, k_r; z) \\ &= \begin{cases} \frac{1}{z}\mathrm{Ath}(k_1, \ldots, k_{r-1}, k_r - 1; z) & (k_r \geq 2), \\ \frac{1}{1-z^2}\mathrm{Ath}(k_1, \ldots, k_{r-1}; z) & (k_r = 1). \end{cases}\end{aligned}$$

(ii) $\mathrm{Ath}(\underbrace{1, \ldots, 1}_{r}; z) = \dfrac{1}{r!}(\mathrm{Ath}(1; z))^r.$

We define a kind of multiple zeta function of level 2 as follows.

Definition 6. *For $k_1, \ldots, k_{r-1} \in \mathbb{Z}_{\geq 1}$ and $\Re s > 1$, let*

$$\begin{aligned}\mathscr{T}(k_1, \ldots, k_{r-1}, s) &= \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^s} \\ &= \sum_{n_1,\ldots,n_r \geq 0} \prod_{j=1}^{r-1}\left(\sum_{\nu=1}^{j}(2n_\nu+1)\right)^{-k_j} \times \left(\sum_{\nu=1}^{r}(2n_\nu+1)\right)^{-s}.\end{aligned} \tag{5.3}$$

Furthermore, as its normalized version, let

$$\widehat{\mathscr{T}}(k_1, \ldots, k_{r-1}, s) = 2^r \mathscr{T}(k_1, \ldots, k_{r-1}, s). \tag{5.4}$$

When $k_r > 1$, we see that

$$\mathrm{Ath}(k_1,\ldots,k_r;1) = \mathscr{T}(k_1\ldots,k_r).$$

Corresponding to these functions, we define a level 2-version of $\xi(k_1,\ldots,k_r;s)$.

Definition 7. *For $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 1}$, let*

$$\begin{aligned}(5.5)\quad &\psi(k_1,\ldots,k_r;s)\\ &= \frac{2^r}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Ath}(k_1,\ldots,k_r;\tanh(t/2))}{\sinh(t)}\,dt \quad (\Re s > 0).\end{aligned}$$

Remark 5.2. In [22, Section 4], Sasaki essentially considered (5.3), and also $\psi(k_1;s)$. In fact, Sasaki considered a little more general function $\psi_k(s,a)$ $(0 < a < 1)$, and our $\psi(k;s)$ coincides with his $2^{s+2}\psi_k(s,1/2)$.

Similar to [2, Theorem 6], we can see that $\psi(k_1,\ldots,k_r;s)$ can be continued to $\mathbb{C}$ as an entire function. Further we can prove the following theorem which is exactly a level 2-analogue of [2, Theorem 8]. Note that this theorem for the case $r = 1$ was essentially proved by Sasaki (see [22, Theorem 7]).

Theorem 5.3. *For $r, k \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}&\psi(\underbrace{1,\ldots,1}_{r-1},k;s)\\ &= (-1)^{k-1}\sum_{\substack{a_1,\ldots,a_k\geq 0\\ a_1+\cdots+a_k=r}}\binom{s+a_k-1}{a_k}\cdot\widehat{\mathscr{T}}(a_1+1,\ldots,a_{k-1}+1,a_k+s)\\ &\quad+\sum_{j=0}^{k-2}(-1)^j\,\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{r-1},k-j)\cdot\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{j},s).\end{aligned}$$

In order to prove this theorem, we prepare the following lemma which is a level 2-version of [2, Theorem 3 (i)]. The proof is completely similar and is omitted.

Lemma 5.4. *For $l_1,\ldots,l_{m-1} \in \mathbb{Z}_{\geq 1}$ and $\Re s > 1$,*

$$\widehat{\mathscr{T}}(l_1,\ldots,l_{m-1},s)$$

$$= \frac{1}{\Gamma(l_1)\cdots\Gamma(l_{m-1})\Gamma(s)} \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1}\cdots x_{m-1}^{l_{m-1}-1} x_m^{s-1} \times \prod_{j=1}^{m} \frac{1}{\sinh(x_j+\cdots+x_m)} dx_1\cdots dx_m.$$

Proof of Theorem 5.3. The method of the proof is similar to that in [2, Theorem 8] (see also [22, Theorem 7]). Given $r,k\geq 1$, introduce the following integrals

$$I_\nu^{(r,k)}(s) = \frac{2^r}{\Gamma(s)}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{Ath}(\overbrace{1,\ldots,1}^{r-1},\nu;\tanh((x_\nu+\cdots+x_k)/2))}{\prod_{l=\nu}^k \sinh(x_l+\cdots+x_k)} \times x_k^{s-1}\,dx_\nu\cdots dx_k.$$

We compute $I_1^{(r,k)}(s)$ in two different ways. First, since

$$\mathrm{Ath}(\underbrace{1,\ldots,1}_{r};\tanh((x_1+\cdots+x_k)/2)) = \frac{1}{r!}\left(\frac{x_1+\cdots+x_k}{2}\right)^r$$

by Lemma 5.1 (ii) and (5.2), we have

$$\begin{aligned}
&I_1^{(r,k)}(s)\\
&= \frac{1}{\Gamma(s)\,r!}\int_0^\infty\cdots\int_0^\infty \frac{(x_1+\cdots+x_k)^r x_k^{s-1}}{\prod_{l=1}^k \sinh(x_l+\cdots+x_k)}\,dx_1\cdots dx_k\\
&= \frac{1}{\Gamma(s)}\sum_{a_1+\cdots+a_k=r}\frac{1}{a_1!\cdots a_k!}\int_0^\infty\cdots\int_0^\infty x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}x_k^{s+a_k-1}\\
&\qquad\times\frac{1}{\prod_{l=1}^k \sinh(x_l+\cdots+x_k)}\,dx_1\cdots dx_k\\
&= \sum_{a_1+\cdots+a_k=r}\frac{\Gamma(s+a_k)}{\Gamma(s)a_k!}\times\frac{1}{\Gamma(a_1+1)\cdots\Gamma(a_{k-1}+1)\Gamma(s+a_k)}\\
&\qquad\times\int_0^\infty\cdots\int_0^\infty\frac{x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}x_k^{s+a_k-1}}{\prod_{l=1}^k \sinh(x_l+\cdots+x_k)}\,dx_1\cdots dx_k.
\end{aligned}$$

Using Lemma 5.4 for the last integral, we obtain

$$I_1^{(r,k)}(s) = \sum_{a_1+\cdots+a_k=r}\binom{s+a_k-1}{a_k}\times \tag{5.6}$$

$$\widehat{\mathscr{T}}(a_1+1,\ldots,a_{k-1}+1,s+a_k).$$

Secondly, by using

$$\frac{\partial}{\partial x_\nu}\mathrm{Ath}(\underbrace{1,\ldots,1}_{r-1},\nu+1;\tanh((x_\nu+\cdots+x_k)/2)) = \frac{\mathrm{Ath}(\overbrace{1,\ldots,1}^{r-1},\nu;\tanh((x_\nu+\cdots+x_k)/2))}{\sinh(x_\nu+\cdots+x_k)} \tag{5.7}$$

(see Lemma 5.1) and Lemma 5.4, we compute

$$\begin{aligned}
&I_\nu^{(r,k)}(s)\\
&=\frac{2^r}{\Gamma(s)}\int_0^\infty\cdots\int_0^\infty\left[\mathrm{Ath}(\underbrace{1,\ldots,1}_{r-1},\nu+1;\tanh((x_\nu+\cdots+x_k)/2))\right]_{x_\nu=0}^{\infty}\\
&\qquad\times\frac{1}{\prod_{l=\nu+1}^k\sinh(x_l+\cdots+x_k)}\,x_k^{s-1}\,dx_{\nu+1}\cdots dx_k\\
&=2^r\,\mathscr{T}(\underbrace{1,\ldots,1}_{r-1},\nu+1)\cdot\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{k-\nu-1},s)-I_{\nu+1}^{(r,k)}\\
&=\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{r-1},\nu+1)\cdot\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{k-\nu-1},s)-I_{\nu+1}^{(r,k)}.
\end{aligned}$$

Therefore, using this relation repeatedly, we obtain

$$\begin{aligned}
&I_1^{(r,k)}(s)\\
&=\sum_{\nu=1}^{k-1}(-1)^{\nu-1}\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{r-1},\nu+1)\cdot\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{k-\nu-1},s)+(-1)^{k-1}I_k^{(r,k)}\\
&=\sum_{j=0}^{k-2}(-1)^{k-j}\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{r-1},k-j)\cdot\widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{j},s)+(-1)^{k-1}I_k^{(r,k)}.
\end{aligned}$$

By definition, we have

$$I_k^{(r,k)}(s)=\psi(\underbrace{1,\ldots,1}_{r-1},k;s),$$

and thus

$$I_1^{(r,k)}(s) = \sum_{j=0}^{k-2} (-1)^{k-j} \widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{r-1}, k-j) \cdot \widehat{\mathscr{T}}(\underbrace{1,\ldots,1}_{j}, s) + (-1)^{k-1} \psi(\underbrace{1,\ldots,1}_{r-1}, k; s). \tag{5.8}$$

Comparing (5.6) and (5.8), we obtain the assertion. Q.E.D.

Next, we show a level 2-version of [2, Theorem 9 (i)].

Theorem 5.5. *For $r, k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$,*

(5.9)

$$\psi(\underbrace{1,\ldots,1}_{r-1}, k; m+1) = \sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+\cdots+a_k=m}} \binom{a_k+r}{r} \cdot \widehat{\mathscr{T}}(a_1+1, \ldots, a_{k-1}+1, a_k+r+1).$$

Proof. By (5.7), we have

$$\begin{aligned}
&\psi(1,\ldots,1,k;m+1) \\
&= \frac{2^r}{m!} \int_0^\infty \frac{t_k^m}{\sinh t_k} \int_0^{t_k} \frac{\mathrm{Ath}(\overbrace{1,\ldots,1}^{r-1}, k-1; \tanh(t_{k-1}/2))}{\sinh t_{k-1}} \, dt_{k-1} dt_k \\
&= \frac{2^r}{m!} \int_0^\infty \frac{t_k^m}{\sinh t_k} \int_0^{t_k} \frac{1}{\sinh t_{k-1}} \int_0^{t_{k-1}} \\
&\qquad \frac{\mathrm{Ath}(\overbrace{1,\ldots,1}^{r-1}, k-2; \tanh(t_{k-2}/2))}{\sinh t_{k-2}} \, dt_{k-2} dt_{k-1} dt_k \\
&= \cdots \\
&= \frac{2^r}{m!} \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} \frac{t_k^m \, \mathrm{Ath}(\overbrace{1,\ldots,1}^{r}; \tanh(t_1/2))}{\sinh(t_k) \cdots \sinh(t_1)} \, dt_1 \cdots dt_k \\
&= \frac{1}{m!r!} \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} \frac{t_k^m \, t_1^r}{\sinh(t_k) \cdots \sinh(t_1)} \, dt_1 \cdots dt_k.
\end{aligned}$$

By the change of variables

$$t_1 = x_k, t_2 = x_{k-1} + x_k, \ldots, t_k = x_1 + \cdots + x_k,$$

we obtain

$$\begin{aligned}
&\psi(1, \ldots, 1, k; m+1) \\
&= \frac{1}{m!r!} \int_0^\infty \int_0^\infty \frac{(x_1 + \cdots + x_k)^m \, x_k^r}{\prod_{l=1}^k \sinh(x_l + \cdots + x_k)} \, dt_1 \cdots dt_k \\
&= \sum_{a_1+\cdots+a_k=m} \binom{a_k + r}{r} \cdot \widehat{\mathscr{T}}(a_1 + 1, \ldots, a_{k-1} + 1, a_k + r + 1).
\end{aligned}$$

Q.E.D.

Corollary 5.6. *For $r, k \geq 1$, we have the "height one" duality*

$$\widehat{\mathscr{T}}(\underbrace{1, \ldots, 1}_{r-1}, k+1) = \widehat{\mathscr{T}}(\underbrace{1, \ldots, 1}_{k-1}, r+1). \tag{5.10}$$

Proof. If we set $m = 0$ in (5.9), we have

$$\psi(\underbrace{1, \ldots, 1}_{r-1}, k; 1) = \widehat{\mathscr{T}}(\underbrace{1, \ldots, 1}_{k-1}, r+1). \tag{5.11}$$

On the other hand, from the definition we have in general

$$\begin{aligned}
\psi(k_1, \ldots, k_r; 1) &= 2^r \int_0^\infty \frac{\mathrm{Ath}(k_1, \ldots, k_r; \tanh(t/2))}{\sinh t} \, dt \\
&= 2^r \int_0^\infty \frac{d}{dt} \mathrm{Ath}(k_1, \ldots, k_{r-1}, k_r + 1; \tanh(t/2)) \, dt \\
&= \widehat{\mathscr{T}}(k_1, \ldots, k_{r-1}, k_r + 1)
\end{aligned}$$

and in particular

$$\psi(\underbrace{1, \ldots, 1}_{r-1}, k; 1) = \widehat{\mathscr{T}}(\underbrace{1, \ldots, 1}_{r-1}, k+1). \tag{5.12}$$

Thus from (5.11) and (5.12) we obtain (5.10). Q.E.D.

We remark that, by computing $\xi(\underbrace{1,...,1}_{r-1},k;1)$ in two ways as above, we obtain an alternative proof of the usual height one duality $\zeta(\underbrace{1,...,1}_{r-1},k+1)=\zeta(\underbrace{1,...,1}_{k-1},r+1)$.

In the forthcoming paper [17], we extend the duality (5.10) in full generality.

By setting $s=m+1$ in Theorem 5.3 and comparing with Theorem 5.5, we obtain a level 2-version of [2, Corollary 11] as follows.

Theorem 5.7. *For $m,r\geq 1$ and $k\geq 2$,*

$$\begin{aligned}&\sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+a_k=m}}\binom{a_k+r}{r}\cdot\widehat{\mathscr{T}}(a_1+1,\dots,a_{k-1}+1,a_k+r+1)\\ &+(-1)^k\sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+a_k=r}}\binom{a_k+m}{m}\cdot\widehat{\mathscr{T}}(a_1+1,\dots,a_{k-1}+1,a_k+m+1)\\ &=\sum_{j=0}^{k-2}(-1)^j\widehat{\mathscr{T}}(\underbrace{1,\dots,1}_{r-1},k-j)\cdot\widehat{\mathscr{T}}(\underbrace{1,\dots,1}_{j},m+1).\end{aligned}$$

If we use the duality $\widehat{\mathscr{T}}(\underbrace{1,\dots,1}_{j},m+1)=\widehat{\mathscr{T}}(\underbrace{1,\dots,1}_{m-1},j+2)$, the right-hand side becomes the exact analogue of the one in [2, Corollary 11].

Example 5.8. We recall

$$\begin{aligned}\zeta^{o}(s)\,(=\mathscr{T}(s))&=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}=\left(1-2^{-s}\right)\zeta(s),\\ \zeta^{oe}(k,s)\,(=\mathscr{T}(k,s))&=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m+1)^k(2m+2n)^s}\end{aligned}$$

(see Kaneko-Tasaka [15]). Since

$$\begin{aligned}\widehat{\mathscr{T}}(s)&=2\mathscr{T}(s)=2\zeta^{o}(s),\\ \widehat{\mathscr{T}}(k,s)&=2^2\mathscr{T}(k,s)=2^2\zeta^{oe}(k,s),\end{aligned}$$

Theorem 5.7 for the case $k = 2$ and $r = 1$ gives

$$\sum_{a=0}^{a}(a+1)\zeta^{oe}(a-a+1, a+2)$$
$$+\zeta^{oe}(2, a+1) + (a+1)\zeta^{oe}(1, a+2) = \zeta^{o}(2)\zeta^{o}(a+1).$$

Remark 5.9. We have introduced the function $\psi(k_1,\ldots,k_r;s)$ as a level 2-version of $\xi(k_1,\ldots,k_r;s)$, and proved results corresponding to those in [2]. In the forthcoming paper [17], we will further discuss level 2-versions of poly-Bernoulli numbers and multiple zeta values in connection to $\psi(k_1,\ldots,k_r;s)$, and hopefully, a version corresponding to $\eta(k_1,\ldots,k_r;s)$

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References

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, Bernoulli Numbers and Zeta Functions, Springer, Tokyo, 2014.

[2] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J., **153** (1999), 189–209.

[3] C. Brewbaker, A combinatorial interpretation of the Poly-Bernoulli numbers and two Fermat analogues, Integers, **8** (2008), ♯A02.

[4] M. -A. Coppo and B. Candelpergher, The Arakawa-Kaneko zeta function, Ramanujan J., **22** (2010), 153–162.

[5] P. J. Cameron, C. A. Glass and R. U. Schumacher, Acyclic orientations and poly-Bernoulli numbers, preprint, arXiv:1412.3685.

[6] H. Gangle, M. Kaneko, D. Zagier, Double zeta values and modular forms, in 'Automorphic forms and Zeta functions', Proceedings of the conference in memory of Tsuneo Arakawa, World Scientific, (2006), 71–106.

[7] A. Granville, A decomposition of Riemann's zeta-function, in London Math. Soc. Lecture Note Ser. **247**, Cambridge, 1997, pp. 95–101.

[8] M. Hoffman, Multiple harmonic series, Pacific J. Math., **152** (1992), 275–290.

[9] K. Imatomi, Multi-poly-Bernoulli-star numbers and finite multiple zeta-star values, Integers, **14** (2014), A51.

[10] K. Imatomi, M. Kaneko and E. Takeda, Multi-poly-Bernoulli numbers and finite multiple zeta values, J. Integer Sequences, **17** (2014), Article 14.4.5.

[11] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math., **142** (2006), 307–338.

[12] M. Kaneko, Poly-Bernoulli numbers, J. Théor. Nombres Bordeaux, **9** (1997), 199–206.

[13] M. Kaneko, Poly-Bernoulli numbers and related zeta functions, Algebraic and Analytic Aspects of Zeta Functions and L-functions, MSJ Mem., **21**, pp. 73–85, Math. Soc. Japan, Tokyo, 2010.

[14] M. Kaneko and M. Sakata, On multiple zeta values of extremal height, Bull. Aust. Math. Soc., **93** (2016), 186–193.

[15] M. Kaneko and K. Tasaka, Double zeta values, double Eisenstein series, and modular forms of level 2, Math. Ann. **357** (2013), 1091–1118.

[16] M. Kaneko and H. Tsumura, Multi-poly-Bernoulli numbers and related zeta functions, Nagoya Math. J., published online (2017): doi.org/10.1017/nmj.2017.16.

[17] M. Kaneko and H. Tsumura, Poly-Bernoulli numbers of level 2 and related zeta functions, in preparation.

[18] N. Kawasaki and Y. Ohno, Combinatirial proofs of identities for special values of Arakawa-Kaneko multiple zeta functions, Kyushu J. Math. **72** (2018), 215–222.

[19] Y. Komori and H. Tsumura, On Arakawa-Kaneko zeta-functions associated with $GL_2(\mathbf{C})$ and their functional relations, J. Math. Soc. Japan **70** (2018), 179-213.

[20] S. Oi, Gauss hypergeometric functions, multiple polylogarithms, and multiple zeta values, Publ. Res. Inst. Mat. Sci. **45** (2009), 981–1009.

[21] J. Okuda and K. Ueno, Relations for multiple zeta values and Mellin transforms of multiple polylogarithms, Publ. Res. Inst. Math. Sci., **40** (2004), 537–564.

[22] Y. Sasaki, On generalized poly-Bernoulli numbers and related L-functions, J. Number Theory, **132** (2012), 156–170.

[23] G. Shimura, Elementary Dirichlet Series and Modular Forms, Springer Monographs in Mathematics, Springer, New York, 2007.

[24] K. Shingu, The Kaneko-Tsumura zeta function and its related area (in Japanese), Master's thesis, Nagoya University (2018).

[25] S. Yamamoto, Multiple zeta-star values and multiple integrals, RIMS Kokyuroku Bessatsu B68, 2017, pp. 3–14.

[26] S. Yamamoto, Multiple zeta functions of Kaneko-Tsumura type and their values at positive integers, preprint, arXiv: 1607.01978.

[27] D. Zagier, Values of zeta functions and their applications, in ECM volume, Progress in Math., **120** (1994), 497–512.

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