Analogues of the Aoki-Ohno and Le-Murakami relations for finite multiple zeta values

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Abstract

We establish finite analogues of the identities known as the Aoki-Ohno relation and the Le-Murakami relation in the theory of multiple zeta values. We use an explicit form of a generating series given by Aoki and Ohno.

1 Introduction and statement of the results

For an index set of positive integers $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_1 > 1$, the multiple zeta value $\zeta(\mathbf{k})$ and the multiple zeta-star value $\zeta^\star(\mathbf{k})$ are defined respectively by the nested series

$$\zeta(\mathbf{k}) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

and

$$\zeta^\star(\mathbf{k}) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

We refer to the sum $k_1 + \cdots + k_r$, the length r and the number of components k_i with $k_i > 1$ as the weight, depth, and height of the index $\mathbf{k}$ respectively.

For given k and s, let $I_0(k,s)$ be the set of indices $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_1 > 1$ of weight k and height s. We naturally have $k \geq 2s$ and $s \geq 1$; otherwise $I_0(k,s)$ is empty.

Aoki and Ohno proved in [1] the identity

$$\sum_{\mathbf{k} \in I_0(k,s)} \zeta^\star(\mathbf{k}) = 2\binom{k-1}{2s-1}(1 - 2^{1-k})\zeta(k). \tag{1.1}$$

On the other hand, for $\zeta(\mathbf{k})$, the following identity is known as the Le-Murakami relation ([6]): for even k,

$$\sum_{\mathbf{k} \in I_0(k,s)} (-1)^{\mathrm{dep}(\mathbf{k})}\zeta(\mathbf{k}) = \frac{(-1)^{k/2}}{(k+1)!} \sum_{r=0}^{k/2-s} \binom{k+1}{2r}(2 - 2^{2r})B_{2r}\pi^k,$$

where B_n denotes the Bernoulli number. As Euler discovered, the right-hand side is a rational multiple of the Riemann zeta value $\zeta(k)$.

In this short article, we establish the analogous identities for *finite multiple zeta values*.

For an index set of positive integers $\mathbf{k} = (k_1, \ldots, k_r)$, the finite multiple zeta value $\zeta_{\mathcal{A}}(\mathbf{k})$ and the finite multiple zeta-star value $\zeta^{\star}_{\mathcal{A}}(\mathbf{k})$ are elements in the quotient ring $\mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) / \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$ (p runs over all primes) represented respectively by

$$\left(\sum_{p>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \bmod p\right)_p \quad \text{and} \quad \left(\sum_{p>m_1\geq\cdots\geq m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \bmod p\right)_p.$$

Studies of finite multiple zeta(-star) values go back at least to Hoffman [2] (the preprint was available around 2004) and Zhao [10]. But it was only recently that Zagier proposed (in 2012 to the first-named author) considering them in the (characteristic 0) ring $\mathcal{A}$ ([5], see also [3, 4]). In $\mathcal{A}$, the naive analogue $\zeta_{\mathcal{A}}(k)$ of the Riemann zeta value $\zeta(k)$ is zero because $\sum_{n=1}^{p-1} 1/n^k$ is congruent to 0 modulo p for all sufficiently large primes p. However, the "true" analogue of $\zeta(k)$ in $\mathcal{A}$ is considered to be

$$Z(k) := \left(\frac{B_{p-k}}{k}\right)_p.$$

We note that this value is zero when k is even because the odd-indexed Bernoulli numbers are 0 except B_1. It is still an open problem whether $Z(k) \neq 0$ for any odd $k \geq 3$.

We now state our main theorem, where the role of $Z(k)$ as a finite analogue of $\zeta(k)$ is evident.

Theorem 1.1. *The following identities hold in $\mathcal{A}$:*

$$\sum_{\mathbf{k}\in I_0(k,s)} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})Z(k), \tag{1.2}$$

$$\sum_{\mathbf{k}\in I_0(k,s)} (-1)^{\mathrm{dep}(\mathbf{k})}\zeta_{\mathcal{A}}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})Z(k). \tag{1.3}$$

We should note that the right-hand sides are exactly the same. In the next section, we give a proof of the theorem.

2 Proof of Theorem 1.1

Let $\mathrm{Li}^{\star}_{\mathbf{k}}(t)$ be the 'nonstrict' version of the multiple-polylogarithm:

$$\mathrm{Li}^{\star}_{\mathbf{k}}(t) = \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{t^{m_1}}{m_1^{k_1}\cdots {m_r}^{k_r}}.$$

Aoki and Ohno [1] computed the generating function

$$\Phi_0 := \sum_{k,s\geq 1}\left(\sum_{\mathbf{k}\in I_0(k,s)} \mathrm{Li}^{\star}_{\mathbf{k}}(t)\right) x^{k-2s} z^{2s-2},$$

and, in view of $\mathrm{Li}^\star_{\mathbf{k}}(1) = \zeta^\star(\mathbf{k})$ (if $k_1 > 1$), evaluated it at $t = 1$ to obtain the identity (1.1). For our purpose, the function $\mathrm{Li}^\star_{\mathbf{k}}(t)$ is useful because the truncated sum

$$\sum_{p>m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

used to define $\zeta^\star_{\mathcal{A}}(\mathbf{k})$ is the sum of the coefficients of t^i in $\mathrm{Li}^\star_{\mathbf{k}}(t)$ for $i = 1, \ldots, p-1$. In [1, Section 3], Aoki and Ohno showed that

$$\Phi_0 = \sum_{n=1}^{\infty} a_n t^n,$$

where

$$a_n \;=\; \sum_{l=1}^{n} \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right)$$

and

$$A_{n,l}(z) \;=\; (-1)^l \binom{n-1}{l-1} \frac{(z-l+1)\cdots(z-1)z(z+1)\cdots(z+n-l-1)}{(2z-l+1)\cdots(2z-1)2z(2z+1)\cdots(2z+n-l)}.$$

The problem is then to compute the coefficient of $x^{k-2s}z^{2s-2}$ in $\sum_{n=1}^{p-1} a_n$ modulo p.

We proceed as follows:

$$\begin{aligned}
\sum_{n=1}^{p-1} a_n &= \sum_{n=1}^{p-1}\sum_{l=1}^{n} \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right) \\
&= \sum_{l=1}^{p-1}\sum_{n=l}^{p-1} \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right) \\
&= \sum_{l=1}^{p-1}\sum_{n=0}^{p-l-1} \left(\frac{A_{n+l,l}(z)}{x+z-l} + \frac{A_{n+l,l}(-z)}{x-z-l} \right).
\end{aligned}$$

Writing $A_{n+l,l}(z)$ as

$$A_{n+l,l}(z) \;=\; \frac{(-1)^l}{2z} \frac{(z-l+1)_{l-1}}{(2z-l+1)_{l-1}} \frac{(l)_n (z)_n}{(2z+1)_n n!},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, we have

$$\sum_{n=0}^{p-l-1} A_{n+l,l}(z) \;=\; \frac{(-1)^l}{2z} \frac{(z-l+1)_{l-1}}{(2z-l+1)_{l-1}} \sum_{n=0}^{p-l-1} \frac{(l)_n (z)_n}{(2z+1)_n n!}.$$

We view the sum on the right as

$$\sum_{n=0}^{p-l-1} \frac{(l)_n (z)_n}{(2z+1)_n n!} \;\equiv\; F(-p+l, z; 2z+1; 1) - \frac{(l)_{p-l}(z)_{p-l}}{(2z+1)_{p-l}(p-l)!} \mod p.$$

Here, $F(a,b;c;z)$ is the Gauss hypergeometric series

$$F(a,b;c;z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_n$ for $n \geq 1$ is as before and $(a)_0 = 1$. Note that if a (or b) is a nonpositive integer $-m$, then $F(a,b;c;z)$ is a polynomial in z of degree at most m, and the renowned formula of Gauss

$$F(a,b;c;1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

becomes

$$F(-m,b;c;1) \;=\; \frac{(c-b)_m}{(c)_m}.$$

Hence

$$F(-p+l, z; 2z+1; 1) \;=\; \frac{(z+1)_{p-l}}{(2z+1)_{p-l}} \equiv \frac{z^{p-1}-1}{(2z)^{p-1}-1} \frac{(2z-l+1)_{l-1}}{(z-l+1)_{l-1}} \mod p.$$

We also compute

$$\frac{(l)_{p-l}(z)_{p-l}}{(2z+1)_{p-l}(p-l)!} \equiv (-1)^{l-1} \frac{z(z^{p-1}-1)}{(2z)^{p-1}-1} \frac{(2z-l+1)_{l-1}}{(z-l)_l} \mod p.$$

Since we only need the coefficient of z^{2s-2}, we may work modulo higher powers of z and, in particular, we may replace $(z^{p-1}-1)/((2z)^{p-1}-1)$ by 1, assuming p is large enough. (We may assume this because an identity in $\mathcal{A}$ holds true if the p-components on both sides agree in $\mathbb{Z}/p\mathbb{Z}$ for all large enough p.) Hence,

$$\begin{aligned}\sum_{n=1}^{p-1} a_n \equiv \sum_{l=1}^{p-1} \Bigg\{ & \frac{(-1)^l}{2z} \left(\frac{1}{x+z-l} - \frac{1}{x-z-l} \right) \\ & + \frac{1}{2} \left(\frac{1}{(x+z-l)(z-l)} - \frac{1}{(x-z-l)(z+l)} \right) \Bigg\} \mod p.\end{aligned}$$

By the binomial expansion,

$$\begin{aligned}\sum_{l=1}^{p-1} \frac{(-1)^l}{x+z-l} \;&=\; \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \left(\frac{x+z}{l} \right)^m \\ &=\; \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \frac{1}{l^m} \sum_{i=0}^{m} \binom{m}{i} x^{m-i} z^i \\ &=\; \sum_{m \geq i \geq 0} \binom{m}{i} \left(\sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-i} z^i.\end{aligned}$$

From this we obtain

$$\sum_{l=1}^{p-1} \frac{(-1)^l}{2z} \left(\frac{1}{x+z-l} - \frac{1}{x-z-l} \right) = \sum_{m \geq 2i+1 \geq 0} \binom{m}{2i+1} \left(\sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-2i-1} z^{2i}$$

and, by letting $i \to s-1$ and $m \to k-1$, the coefficient of $x^{k-2s}z^{2s-2}$ in this is

$$\binom{k-1}{2s-1}\sum_{l=1}^{p-1}\frac{(-1)^{l-1}}{l^k}.$$

This is known to be congruent modulo p to

$$2\binom{k-1}{2s-1}(1-2^{1-k})\frac{B_{p-k}}{k}$$

(see for example, [11, Theorem 8.2.7]). Concerning the other term,

$$\begin{aligned}&\sum_{l=1}^{p-1}\frac{1}{2}\left(\frac{1}{(x+z-l)(z-l)}-\frac{1}{(x-z-l)(z+l)}\right)\\ &=\frac{1}{2}\sum_{l=1}^{p-1}\left\{\frac{1}{x}\left(\frac{1}{z-l}-\frac{1}{x+z-l}\right)-\frac{1}{x}\left(\frac{1}{z+l}+\frac{1}{x-z-l}\right)\right\},\end{aligned}$$

every quantity that appears as a coefficient in the expansion into power series in x and z is a multiple of the sum of the form $\sum_{l=1}^{p-1}1/l^m$, and all are congruent to 0 modulo p. This concludes the proof of (1.2).

We may prove (1.3) in a similar manner by using the generating series of Ohno-Zagier [7], but we deduce (1.3) from (1.2) by showing that the left-hand sides of both formulas are equal up to sign.

Set $S_{k,s} := \sum_{\mathbf{k}\in I_0(k,s)}(-1)^{\mathrm{dep}(\mathbf{k})}\zeta_{\mathcal{A}}(\mathbf{k})$ and $S^\star_{k,s} := \sum_{\mathbf{k}\in I_0(k,s)}\zeta^\star_{\mathcal{A}}(\mathbf{k})$.

Lemma 2.1. $S^\star_{k,s} = (-1)^{k-1}S_{k,s}$.

Proof. We use the well-known identity (see, for instance, [8, Corollary 3.16])

$$\sum_{i=0}^{r}(-1)^i\zeta_{\mathcal{A}}(k_i,\dots,k_1)\zeta^\star_{\mathcal{A}}(k_{i+1},\dots,k_r) \;=\; 0. \tag{2.1}$$

Taking the sum of this over all $\mathbf{k}\in I_0(k,s)$ and separating the terms corresponding to $i=0$ and $i=r$, we obtain

$$S^\star_{k,s}+\sum_{\substack{k'+k''=k\\ s'+s''=s}}\left(\sum_{\mathbf{k}'\in I_0(k',s')}(-1)^{\mathrm{dep}(\mathbf{k}')}\zeta_{\mathcal{A}}(\overleftarrow{\mathbf{k}'})\right)\left(\sum_{\mathbf{k}''\in I(k'',s'')}\zeta^\star_{\mathcal{A}}(\mathbf{k}'')\right)+(-1)^kS_{k,s} \;=\; 0.$$

Here, $\overleftarrow{\mathbf{k}'}$ denotes the reversal of $\mathbf{k}'$, and the set $I(k'',s'')$ consists of all indices (no restriction on the first component) of weight k'' and height s''. We have used $\zeta_{\mathcal{A}}(\overleftarrow{\mathbf{k}}) = (-1)^k\zeta_{\mathcal{A}}(\mathbf{k})$ in computing the last term ($i=r$). Since the second sum in the middle is symmetric and hence 0 (by Hoffman [2, Theorem 4.4] and $\zeta_{\mathcal{A}}(k)=0$ for all $k\geq 1$), the lemma follows. □

Since $Z(k)=0$ if k is even, we see from Lemma 2.1 that the formula for $S_{k,s}$ is the same as that for $S^\star_{k,s}$. This concludes the proof of our theorem.

Remark 2.2. K. Yaeo [9] proved the lemma in the case $s=1$ and T. Murakami (unpublished) in general for all odd k.

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