Genus character L-functions of quadratic orders and class numbers

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Dedicated to the memories of Professor Tsuneo Arakawa.

Abstract

An explicit form of genus character L-functions of quadratic orders is presented in full generality. As an application, we generalize a formula due to Hirzebruch and Zagier on the class number of imaginary quadratic fields expressed in term of the continued fraction expansion.

1 Statements of main results

In 1970's, Hirzebruch and Zagier discovered and proved a beautiful formula on the class number of imaginary quadratic fields in terms of the continued fraction expansion [10, 31, 32, 33], a typical special case being the following.

Theorem (Hirzebruch-Zagier)**.** *Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$, and*

$$\sqrt{p} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, \overline{a_1, a_2, a_3, \cdots, a_{2t}}]$$

be the continued fraction expansion of $\sqrt{p}$ with $(a_1, a_2, a_3, \cdots, a_{2t})$ the minimal period. Suppose that the class number $h(4p)$ in wide sense of $\mathbf{Q}(\sqrt{p})$ is equal to 1. Then the class number of $\mathbf{Q}(\sqrt{-p})$ is given by

$$h(-p) = \frac{1}{3}\sum_{i=1}^{2t}(-1)^i a_i.$$

In this paper, we obtain a counterpart of this theorem when p is congruent to 1 mod 4, as a consequence of our computation of an explicit formula for the genus character L-functions of general quadratic orders, together with a generalization of a formula of Zagier on the value at 0 of such an L-function.

Corollary 1 to Theorem 3. *Let p be a prime number such that $p \equiv 1 \pmod 4$, and let $2\sqrt{p} = [a_0, \overline{a_1, a_2, a_3, \cdots, a_{2t}}]$ be the continued fraction expansion of $2\sqrt{p}$. Suppose that the (wide) class number $h(4p)$ of the quadratic order of discriminant $4p$ is 1. Then the class number of $\mathbf{Q}(\sqrt{-p})$ is given by*

$$h(-4p) = \frac{1}{3}\sum_{i=1}^{2t}(-1)^i a_i.$$

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As examples, take $p = 53$ and $p = 73$. Both satisfy $h(4p) = 1$. We compute

$$2\sqrt{53} = [14, \overline{1,1,3,1,1,1,6,1,1,1,3,1,1,28}] \quad \text{and} \quad 2\sqrt{73} = [17, \overline{11,2,1,3,8,3,1,2,11,34}],$$

thereby obtain

$$h(-4\cdot 53) = \frac{1}{3}(-1+1-3+1-1+1-6+1-1+1-3+1-1+28) = \frac{18}{3} = 6$$

and

$$h(-4\cdot 73) = \frac{1}{3}(-11+2-1+3-8+3-1+2-11+34) = \frac{12}{3} = 4.$$

We may obtain a number of results of similar kind. We present here another two.

Corollary 2. *Let p be a prime such that $p \equiv 1 \pmod{12}$ and suppose $h(p) = 1$. Let $(1+3\sqrt{p})/2 = [a_0, \overline{a_1, a_2, a_3, \cdots, a_{2t}}]$ be the continued fraction expansion. Then we have*

$$h(-3p) = \frac{1}{2}\sum_{i=1}^{2t}(-1)^i a_i.$$

For instance, by the continued fraction

$$\frac{1+3\sqrt{73}}{2} = [13, \overline{3,6,12,1,1,1,12,6,3,25}],$$

we deduce

$$h(-3\cdot 73) = \frac{1}{2}(-3+6-12+1-1+1-12+6-3+25) = \frac{8}{2} = 4.$$

Corollary 3. *Let p be a prime such that $p \equiv 1 \pmod{24}$ and suppose $h(p) = 1$. Let $3\sqrt{p} = [a_0, \overline{a_1, a_2, a_3, \cdots, a_{2t}}]$ be the continued fraction expansion. Then we have*

$$h(-3p) = \frac{1}{10}\sum_{i=1}^{2t}(-1)^i a_i.$$

As an example, take the same $p = 73$ as above but this time compute

$$3\sqrt{73} = [25, \overline{1,1,1,2,1,1,5,1,4,1,5,1,1,2,1,1,1,50}],$$

which implies

$$h(-3\cdot 73) = \frac{1}{10}(-1+1-1+2-1+1-5+1-4+1-5+1-1+2-1+1-1+50) = \frac{40}{10} = 4.$$

We are now going to state our main theorems on L-functions.

Let Δ be a quadratic discriminant, that means, Δ is a non-square integer such that $\Delta \equiv 0, 1 \pmod 4$. Suppose that d_1 is a fundamental discriminant which divides Δ and that Δ/d_1 is a discriminant. Then we may write $\Delta = d_1 d_2 f_0^2$ with another fundamental discriminant d_2 and a natural number f_0. With these is associated a genus character $\chi^{(\Delta)}_{d_1,d_2}$ on the narrow class group $\mathcal{C}^+_\Delta$ of the order $\mathcal{O}_\Delta$ of discriminant Δ. We recall precise definitions and basic facts on quadratic orders, quadratic irrationals, and general genus characters in §7 Appendix 1. We then define the genus character L-function for $\Re(s) > 1$ by

$$L(s, \chi^{(\Delta)}_{d_1,d_2}) := \sum_{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta} \frac{\chi^{(\Delta)}_{d_1,d_2}(\mathfrak{a})}{\mathfrak{N}_\Delta(\mathfrak{a})^s},$$

where the sum is taken over all $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{a}$ of $\mathcal{O}_\Delta$ and $\mathfrak{N}_\Delta(\mathfrak{a}) = (\mathcal{O}_\Delta : \mathfrak{a})$ denotes the norm of $\mathfrak{a}$. This function is analytically continued as a meromorphic function to the whole s-plane. We first establish the following.

Theorem 1. *For $\Delta = d_1 d_2 f_0^2$ as above, one has the identity*

$$L(s,\chi_{d_1,d_2}^{(\Delta)}) = L(s,\chi_{d_1})L(s,\chi_{d_2}) \times \prod_{\substack{p \mid f_0 \\ p:\text{prime}}} \frac{(1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}) - p^{m_p-1-2m_p s}(p^{1-s}-\chi_{d_1}(p))(p^{1-s}-\chi_{d_2}(p))}{1-p^{1-2s}}.$$

Here, $L(s,\chi_d)$ is the Dirichlet L-function of the Kronecker character $\chi_d(*) = (\frac{d}{*})$, the product on the right runs over the prime factors of f_0 and m_p is a positive integer such that p^{m_p} is the highest power of p dividing f_0 if $f_0 > 1$, while the empty product is understood as being 1 if $f_0 = 1$.

When at least one of d_i is odd and $2 \nmid f_0$, this is Proposition 4.2 in Chinta-Offen [4]. The formula (4) in [13] can be regarded as a special case when $d_1 = 1$. Note however that no detailed proofs have been given in neither of these. In view of its importance for our purpose, we give a complete proof of Theorem 1 in the next section. Tomoyoshi Ibukiyama has informed the authors that he also proved this result in a different manner ([12]). His motivation comes from an explicit trace formula of the Hecke operator acting on the space of Siegel modular forms.

With the same notation as in Theorem 1, for any prime $p \mid f_0$, put

$$\epsilon_p(s,\chi_{d_1,d_2}^{(\Delta)}) := \frac{(1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}) - p^{m_p-1-2m_p s}(p^{1-s}-\chi_{d_1}(p))(p^{1-s}-\chi_{d_2}(p))}{1-p^{1-2s}}. \tag{1}$$

Then

$$\epsilon_p(1-s,\chi_{d_1,d_2}^{(\Delta)}) = \epsilon_p(s,\chi_{d_1,d_2}^{(\Delta)}) p^{(2s-1)m_p}.$$

Recall the functional equation of the Dirichlet L-function $L(s,\chi_d)$ with fundamental d,

$$\left(\frac{|d|}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_d}{2}\right) L(s,\chi_d) = \left(\frac{|d|}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_d}{2}\right) L(1-s,\chi_d),$$

where $\delta_d = 0$ if $d > 0$ and $\delta_d = 1$ if $d < 0$. Combined, we obtain a functional equation of $L(s,\chi_{d_1,d_2}^{(\Delta)})$ by multiplying suitable gamma factors. We remark here that the local factor $\epsilon_p(s,\chi_{d_1,d_2}^{(\Delta)})$ enjoys the Riemann hypothesis as analogous to [14].

Now suppose Δ is positive and let $\mathbb{X}_\Delta^0$ be the set of all reduced real quadratic irrationals of discriminant Δ. For $\alpha = (b+\sqrt{\Delta})/(2a) \in \mathbb{X}_\Delta^0$, the lattice $\mathfrak{a} = [a, a\alpha] = \mathbf{Z}a + \mathbf{Z}a\alpha$ is an $\mathcal{O}_\Delta$-regular ideal and we put $\chi_{d_1,d_2}^{(\Delta)}(\alpha) := \chi_{d_1,d_2}^{(\Delta)}(\mathfrak{a})$ (see §7 Appendix 1 for basic definitions and the terminologies).

Theorem 2. *The notation being the same as in Theorem 1, suppose that Δ is positive and that d_1 and d_2 are distinct negative fundamental discriminants. Then we have*

$$L(0,\chi_{d_1,d_2}^{(\Delta)}) = \frac{1}{6} \sum_{\alpha \in \mathbb{X}_\Delta^0} \chi_{d_1,d_2}^{(\Delta)}(\alpha) \lfloor \alpha \rfloor,$$

where $\lfloor \alpha \rfloor$ denotes the integer such that $\lfloor \alpha \rfloor \le \alpha < \lfloor \alpha \rfloor + 1$.

Combining Theorems 1 and 2 together with the classical Dirichlet's formula

$$L(0,\chi_d) = \frac{2h(d)}{w(d)},$$

where d is any negative fundamental discriminant and $w(d)$ is the number of units in the order $\mathcal{O}_d$, we obtain the following identity.

Theorem 3. *Under the same assumption and the notation in Theorem 2, we have the identity*

$$24\frac{h(d_1)h(d_2)}{w(d_1)w(d_2)} \prod_{\substack{p\mid f_0\\ p:\text{prime}}} \frac{(1-\chi_{d_1}(p))(1-\chi_{d_2}(p)) - p^{m_p-1}(p-\chi_{d_1}(p))(p-\chi_{d_2}(p))}{1-p} = \sum_{\alpha\in\mathbb{X}^0_\Delta} \chi^{(\Delta)}_{d_1,d_2}(\alpha)\lfloor\alpha\rfloor.$$

The empty product is understood as being 1 *if* $f_0 = 1$.

Zagier proved this result in the case where $(d_1,d_2)=1$ and $f_0=1$. When d_1 and d_2 are not relatively prime (with $d_1 \neq d_2$) and $f_0=1$, the identity has been conjectured experimentally by Y. Kido in his master's thesis [15].

We give some examples. First, take $\Delta = 160$ (this example is taken from [15, p. 39]). For the choice $d_1=-4$, $d_2=-40$, $f_0=1$, we have $h(-4)=1$, $w(-4)=4$, $h(-40)=2$, $w(-40)=2$ and the left-hand side is $24\cdot 1\cdot 2/(4\cdot 2) = 6$, whereas $\sum_{\alpha\in\mathbb{X}^0_{160}} \chi^{(160)}_{-4,-40}(\alpha)\lfloor\alpha\rfloor = 6$ (see the table below). For $d_1=-8$, $d_2=-20$, $f_0=1$, we have $h(-8)=1$, $w(-8)=2$, $h(-20)=2$, $w(-20)=2$ and $24\cdot 1\cdot 2/(2\cdot 2)=12$, whereas $\sum_{\alpha\in\mathbb{X}^0_{160}} \chi^{(160)}_{-8,-20}(\alpha)\lfloor\alpha\rfloor = 12$.

Next take $\Delta = 1440$ and let $d_1=-4$, $d_2=-40$, $f_0=3$. We then have $\chi_{-4}(3)=-1$, $\chi_{-40}(3)=-1$ and hence the Euler factor $((1-(-1))(1-(-1))-(3-(-1))(3-(-1)))/(1-3)=6$. Therefore the left-hand side is equal to $24\cdot 1\cdot 2/(4\cdot 2)\times 6 = 36$, which agrees with the right-hand side, as calculated in the table.

If we take $d_1=-8$, $d_2=-20$, $f_0=3$ with the same $\Delta=1440$, the value of the Euler factor is 2 because $\chi_{-8}(3)=\chi_{-20}(3)=1$ and the left-hand side equals $12\times 2 = 24$. This coincides with the value of the right-hand side computed in the table.

type of α	$\alpha\in\mathbb{X}^0_{160}$	cont.frac.	$\chi^{(160)}_{-4,-40}(\alpha)\lfloor\alpha\rfloor$	$\chi^{(160)}_{-8,-20}(\alpha)\lfloor\alpha\rfloor$
$(3,10,-5)$	$\frac{1}{3}(5+2\sqrt{10})$	$[\overline{3,1,3,2}]$	-3	$+3$
$(5,10,-3)$	$\frac{1}{5}(5+2\sqrt{10})$	$[\overline{2,3,1,3}]$	$+2$	-2
$(8,8,-3)$	$\frac{1}{4}(2+\sqrt{10})$	$[\overline{1,3,2,3}]$	$+1$	-1
$(3,8,-8)$	$\frac{2}{3}(2+\sqrt{10})$	$[\overline{3,2,3,1}]$	-3	$+3$
$(1,12,-4)$	$2(3+\sqrt{10})$	$[\overline{12,3}]$	$+12$	$+12$
$(4,12,-1)$	$\frac{1}{2}(3+\sqrt{10})$	$[\overline{3,12}]$	-3	-3
			6	12

When the class number $h(\Delta)$ is equal to 1, the right-hand side of Theorem 3 is given in terms of the period of a *single* reduced quadratic number, and hence we may obtain such statements as corollaries presented at the beginning, by taking suitable small values of d_1, as many as you like.

Remark. Lu [20] obtained analogous but considerably different formulas for $h(d_1)h(d_2)$ in the case $d_1 \mid d_2$, $f_0=1$. As a consequence, he obtained the same statement as Corollary 1 when $p\equiv 1 \pmod 8$ assuming $h(p)=1$ as an example in [20, p. 1147]. Notice that, as we see in Remark 1 of Section 4, $h(4p)=h(p)$ holds in this case. Unfortunately, the proof in the paper [20] is very short and heavily depends on the author's previous papers which are hardly accessible.

type of α	$\alpha \in \mathbb{X}^0_{1440}$	cont.frac.	$\chi^{(1440)}_{-4,-40}(\alpha)\lfloor\alpha\rfloor$	$\chi^{(1440)}_{-8,-20}(\alpha)\lfloor\alpha\rfloor$
$(13,20,-20)$	$\frac{2}{13}(5+3\sqrt{10})$	$[\overline{2,4,2,1}]$	$+2$	-2
$(20,20,-13)$	$\frac{1}{10}(5+3\sqrt{10})$	$[\overline{1,2,4,2}]$	-1	$+1$
$(8,24,-27)$	$\frac{3}{4}(2+\sqrt{10})$	$[\overline{3,1,6,1}]$	$+3$	-3
$(27,24,-8)$	$\frac{2}{9}(2+\sqrt{10})$	$[\overline{1,6,1,3}]$	-1	$+1$
$(5,30,-27)$	$\frac{3}{5}(5+2\sqrt{10})$	$[\overline{6,1,3,1}]$	$+6$	-6
$(27,30,-5)$	$\frac{1}{9}(5+2\sqrt{10})$	$[\overline{1,3,1,6}]$	-1	$+1$
$(8,32,-13)$	$\frac{1}{4}(8+3\sqrt{10})$	$[\overline{4,2,1,2}]$	-4	$+4$
$(13,32,-8)$	$\frac{2}{13}(8+3\sqrt{10})$	$[\overline{2,1,2,4}]$	$+2$	-2
$(1,36,-36)$	$6(3+\sqrt{10})$	$[\overline{36,1}]$	$+36$	$+36$
$(36,36,-1)$	$\frac{1}{6}(3+\sqrt{10})$	$[\overline{1,36}]$	-1	-1
$(4,36,-9)$	$\frac{3}{2}(3+\sqrt{10})$	$[\overline{9,4}]$	-9	-9
$(9,36,-4)$	$\frac{2}{3}(3+\sqrt{10})$	$[\overline{4,9}]$	$+4$	$+4$
			36	24

The outline of the present paper is as follows. In Section 2, we give a proof of Theorem 1 by direct computations. In Section 3, we recall works of Meyer, Siegel and Zagier, which will be used to prove Theorem 2. Corollaries are proved in Section 4 assuming Theorem 3, Theorem 3 follows from Theorems 1, 2, and Theorem 2 is proved in Section 5. At the final step to prove Theorem 2, we follow Zagier's treatment given in [32, Lemme, p. 90]. After submitting the first draft of this paper, we find that Theorem 2 is known by Lang [18, (2.17), p. 423]. We include his sophistcated proof in Section 8 as Appendix 2, since in our opinion it should be widely known. At the same time, we find other works of Lang [16, 17], Lang and Schertz [19], which might be useful to study Kido's conjecture. Section 6 is devoted to give a new proof of the explicit formulas of the Dirichlet series associated to the primary representation numbers by genera obtained in [11, 23]. In fact, by orthogonality of genus characters, we see that our Theorem 1 and their formulas are equivalent to each other. As a consequence, we present the second proof of Theorem 1. Section 7 (Appendix 1) gives a summary on quadratic orders and irrationals.

We use the following integral formula initiated by Hecke and Meyer to prove Theorem 2. For $z = x + iy \in \mathbf{C}$ with $y = \Im(z) > 0$, and $s \in \mathbf{C}$ with $\Re(s) > 1$, let

$$E(z,s) := \sum_{\substack{m,n\in\mathbf{Z}\\(m,n)\neq(0,0)}} \frac{y^s}{|m+nz|^{2s}}, \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Then, for $\Delta = d_1 d_2 f_0^2 > 0$, $d_1 < 0$, $d_2 < 0$, $d_1 \neq d_2$ (d_i fundamental), we have

$$\sum_{[\mathfrak{b}]^+\in\mathcal{C}^+_\Delta} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})\chi^{(\Delta)}_{d_1,d_2}((\beta_2)) \int_{z_0}^{z_0^*} \frac{\partial}{\partial z} E(z,s)\, dz = 2\Delta^{s/2}\frac{\Gamma((s+1)/2)^2}{i\Gamma(s)} L(s,\chi^{(\Delta)}_{d_1,d_2}).$$

Here the path of integration on the left-hand side is taken as follows. Let $\epsilon_0 > 1$ be the generator of the unit group of positive norm, that is, $\{\epsilon \in \mathcal{O}^\times_\Delta; \mathcal{N}(\epsilon) > 1\} = \langle -1, \epsilon_0\rangle = \pm\epsilon_0^{\mathbf{Z}}$, $\epsilon_0 > 1$. For any $\mathcal{O}_\Delta$-invertible fractional ideal $\mathfrak{b}$ with $\mathfrak{b} = [\beta_1, \beta_2]$ such that $(\beta_1\beta_2' - \beta_1'\beta_2)/\mathcal{N}(\beta_2) > 0$, we define $M_\mathfrak{b} \in SL_2(\mathbf{Z})$ and a real number α by

$$\epsilon_0 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = M_\mathfrak{b} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad M_\mathfrak{b} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}), \qquad \alpha := \frac{\beta_1}{\beta_2}.$$

By means of the linear fractional transformation, the matrix $M_\mathfrak{b}$ has two fixed points $\alpha = \beta_1/\beta_2$ and α', where $\alpha' < \alpha$ by the assumption. Let $C_{M_\mathfrak{b}}$ be the geodesic semi-circle connecting α' and α.

For any fixed $z_0 \in C_{M_{\mathfrak{b}}}$, the integral is taken along the line $C_{M_{\mathfrak{b}}}$ from z_0 to $z_0^* := M_{\mathfrak{b}}\langle z_0\rangle \in C_{M_{\mathfrak{b}}}$, $M_{\mathfrak{b}}\langle z_0\rangle = (az_0+b)(cz_0+d)^{-1}$.

Similarly, when $\Delta = d_1d_2f_0^2 < 0$ with fundamental d_i, we have

$$\sum_{[\mathfrak{b}]^+\in\mathcal{C}_\Delta^+} \chi_{d_1,d_2}^{(\Delta)}(\mathfrak{b})\frac{E(\xi_{\mathfrak{b}},s)}{w(\Delta)} = \left(\frac{|\Delta|}{4}\right)^{s/2} L(s,\chi_{d_1,d_2}^{(\Delta)})$$

with $\xi_{\mathfrak{b}}$ in the upper-half plane satisfying $[\xi_{\mathfrak{b}}]_\sim = \iota_\Delta^{-1}([\mathfrak{b}]^+)$ (see §7 Appendix 1 for notation), while, when $\Delta = d_1d_2f_0^2 > 0$, $d_1 > 0$, $d_2 > 0$, $d_1 \neq d_2$ (d_i fundamental), we have

$$\sum_{[\mathfrak{b}]^+\in\mathcal{C}_\Delta^+} \chi_{d_1,d_2}^{(\Delta)}(\mathfrak{b})\int_{z_0}^{z_0^*} E(z,s)\ dl = 2\Delta^{s/2}\frac{\Gamma(s/2)^2}{\Gamma(s)}L(s,\chi_{d_1,d_2}^{(\Delta)}),$$

where the path of integration is the same as above, and $dl^2 = (dx^2+dy^2)/y^2$ is the hyperbolic line element.

In the above three formulas, the averages on the left-hand sides are taken over $\mathcal{C}_\Delta^+$, in other words, over all classes of "not negative-definite" primitive forms of discriminant Δ. In [5, 6] and [21], similar averages over all classes of "not negative-definite" forms of discriminant Δ are computed. In view of the above three identities and the Möbius inversion formula, Theorem 1 and their results seem to be equivalent (cf. [21, Corollary 1.2.4, p. 13]). We refer the reader to [5, 6, 7, 21] for a recent progress and a survey on related topics. In particular, it turns out that such "not necessarily primitive" averages of $E(z,s)$ are the Fourier coefficients of real analytic modular forms.

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2 Proof of Theorem 1

Let $\sigma_\Delta = 0$ or 1 according as $\Delta = d_1d_2f_0^2$ is even or odd. Put $\omega_\Delta := (\sigma_\Delta + \sqrt{\Delta})/2$, where $\sqrt{\Delta} := i^{(1-\mathrm{sign}(\Delta))/2}\sqrt{|\Delta|}$. We frequently use a canonical parametrization of lattices in $K := \mathbf{Q}(\sqrt{\Delta})$ and that of ideals of $\mathcal{O}_\Delta$ as given in [8, Theorem 5.3.1, p. 129, Theorem 5.4.2, p. 133]. A lattice $\mathfrak{b} = [a, r+\omega_\Delta]$ $(a, r \in \mathbf{Z}, a \neq 0)$ in K is an ideal of $\mathcal{O}_\Delta$ if and only if $a \mid \mathcal{N}(r+\omega_\Delta)$. In this case, this ideal $\mathfrak{b}$ is $\mathcal{O}_\Delta$-primitive and $\mathfrak{N}_\Delta(\mathfrak{b}) = |a|$. An ideal $\mathfrak{b} = [a, r+\omega_\Delta]$ $(a, r \in \mathbf{Z}, a \neq 0)$ of $\mathcal{O}_\Delta$ is $\mathcal{O}_\Delta$-invertible if and only if $(a, \Delta, \mathcal{N}(r+\omega_\Delta)/a) = 1$. By definition, an ideal of $\mathcal{O}_\Delta$ is called $\mathcal{O}_\Delta$-regular if and only if it is an $\mathcal{O}_\Delta$-invertible and $\mathcal{O}_\Delta$-primitive ideal of $\mathcal{O}_\Delta$ (cf. [8, Definition 5.4.1, p. 132]).

Any $\mathcal{O}_\Delta$-regular ideal $\mathfrak{b} = [\prod_p p^{l_p}, r+\omega_\Delta]$ has a unique factrization into $\mathcal{O}_\Delta$-regular ideals with mutually coprime prime-power norms. Explicitly, $\mathfrak{b} = \prod_p \mathfrak{b}^{(p)}$, $\mathfrak{b}^{(p)} := [p^{l_p}, r+\omega_\Delta]$, $\mathfrak{N}_\Delta(\mathfrak{b}^{(p)}) = p^{l_p}$, which can be drawn from [8].* Hence, similarly as in [13], we have the Euler product expression

$$L(s,\chi_{d_1,d_2}^{(\Delta)}) = \prod_{\text{prime } p} L_p(s,\chi_{d_1,d_2}^{(\Delta)}),$$

*Let $a = \prod_{i=1}^m p_i^{l_i}$ be a prime factorization of $a \in \mathbf{N}\setminus\{1\}$. Put $I_{p_i^{l_i}} := \{\mathfrak{b}^{(i)}; \mathcal{O}_\Delta\text{-regular ideal}, \mathfrak{N}_\Delta(\mathfrak{b}^{(i)}) = p_i^{l_i}\}$, $I_a := \{\mathfrak{a}; \mathcal{O}_\Delta\text{-regular ideal}, \mathfrak{N}_\Delta(\mathfrak{a}) = a\}$. It is not difficult to see that the map $\prod_{i=1}^m I_{p_i^{l_i}} \to I_a$ given by $(\mathfrak{b}^{(1)},\cdots,\mathfrak{b}^{(m)}) \mapsto \mathfrak{a} = \prod_{i=1}^m \mathfrak{b}^{(i)}$ is a well-defined bijection using [8, Theorem 5.4.6, p. 137], [8, Exercise 5.4.8, p. 139] and [8, Theorem 5.4.2, p. 133]. For example, to see that this is one-to-one, we may write $\mathfrak{b}^{(i)} = [p_i^{l_i}, \frac{B_i+\sqrt{\Delta}}{2}]$, $\mathfrak{a} = [a, \frac{b+\sqrt{\Delta}}{2}]$. Then, it follows from $\frac{b+\sqrt{\Delta}}{2} \in \mathfrak{a} = \prod_{i=1}^m \mathfrak{b}^{(i)} \subset \mathfrak{b}^{(i)}$ that $b \equiv B_i \pmod{2p_i^{l_i}}$. Hence, $\mathfrak{b}^{(i)}$ has the where

$$L_p(s,\chi^{(\Delta)}_{d_1,d_2}) := \sum_{\substack{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta \\ \mathfrak{N}_\Delta(\mathfrak{a})=p^t}} \frac{\chi^{(\Delta)}_{d_1,d_2}(\mathfrak{a})}{\mathfrak{N}_\Delta(\mathfrak{a})^s} = \sum_{t\geq 0} \frac{\eta_1(p^t)}{p^{ts}}.$$

Here, we define

$$\eta_1(p^t) := \sum_{\substack{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta \\ \mathfrak{N}_\Delta(\mathfrak{a})=p^t}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{a}) = \sum_{j=0}^{\lfloor t/2\rfloor} \sum_{\substack{\mathcal{O}_\Delta-\text{regular ideal } \mathfrak{b} \\ \mathfrak{N}_\Delta(\mathfrak{b})=p^{t-2j}}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b}) = \sum_{j=0}^{\lfloor t/2\rfloor} \theta_1(p^{t-2j}),$$

$$\theta_1(p^a) := \sum_{\substack{\mathcal{O}_\Delta-\text{regular ideal } \mathfrak{b} \\ \mathfrak{N}_\Delta(\mathfrak{b})=p^a}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b}).$$

Let us define $\eta_0(p^t)$ and $\theta_0(p^a)$ analogously by replacing all of the values $\chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})$ with 1, that is, taking $d_1 = 1$, in the definitions of $\eta_1(p^t)$ and $\theta_1(p^a)$.

Let $K = \mathbf{Q}(\sqrt{\Delta})$ and $\Delta = d_K f^2$ with d_K the discriminant of K and f the conductor of Δ. For any prime not dividing the conductor, the treatment is essentially the same as in the case of the maximal order $\mathcal{O}_{d_K}$. The unique factorization into prime ideals holds for any non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$ prime to f, that is, $(\mathfrak{N}_\Delta(\mathfrak{a}), f) = 1$ ([8, Theorem 5.8.1, p. 174]) and we know the factorization of a rational prime $p \nmid f$ into prime ideals of $\mathcal{O}_\Delta$ ([8, Theorem 5.8.8, p. 178]).

For primes p dividing the conductor f, the values $\eta_0(p^t)$ are given in [3, Lemma 4, p. 264]. In addition, in [3, p. 263], we have the list of all regular $\mathcal{O}_\Delta$-ideals $\mathfrak{b}$ such that $\mathfrak{N}_\Delta(\mathfrak{b})$ is any power of prime $p \mid f$. To determine the value of $\chi^{(\Delta)}_{d_1,d_2}$, we compute the associated class of primitive forms $\Phi_\Delta^{-1}([\mathfrak{b}]^+)$, and then apply the definition of $\chi^{(\Delta)}_{d_1,d_2}$, for which we refer the reader to Appendix 1 in §7. A character sum evaluation given in [29, Proposition 4.1, p. 86][†] and the identity $\eta_j(p^{t+2}) = \eta_j(p^t) + \theta_j(p^{t+2})$ $(t \geq 0)$ are useful.

2.1 Odd prime factors with $p \mid f$

Lemma 1. *Let p be an odd prime factor of f and denote by n the largest integer such that $d := \Delta/p^{2n}$ is a discriminant. The values $\theta_j(p^a)$ are given as follows. Here φ is the Euler function.*
(1) For $a < 2n$,
(a) if a is odd, then $\theta_0(p^a) = 0$, $\theta_1(p^a) = 0$,
(b) if $a = 2h$ is even, then $\theta_0(p^a) = \varphi(p^h)$, $\theta_1(p^a) = \varphi(p^h)$.
(2) For $a \geq 2n$, we make a case distinction according to the value of $(\frac{d}{p})$.
Case 1. Suppose $(\frac{d}{p}) = 1$.
(a) If $a > 2n$, then $\theta_0(p^a) = 2\varphi(p^n)$,
while $\theta_1(p^a) = \chi_{d_1}(p)^a \cdot 2\varphi(p^n)$ if $p \nmid d_1$, $\theta_1(p^a) = 0$ if $p \mid d_1$.
(b) If $a = 2n$, then $\theta_0(p^{2n}) = p^{n-1}(p-2)$,
while $\theta_1(p^{2n}) = p^{n-1}(p-2)$ if $p \nmid d_1$, $\theta_1(p^{2n}) = -p^{n-1}$ if $p \mid d_1$.

Case 2. Suppose $(\frac{d}{p}) = -1$.
(a) If $a > 2n$, then $\theta_0(p^a) = 0$, $\theta_1(p^a) = 0$.

form $\mathfrak{b}^{(i)} = [p_i^{l_i}, \frac{b+\sqrt{\Delta}}{2}]$. This means that $\mathfrak{b}^{(i)} \in I_{p_i^{l_i}}$ is determined uniquely from $\mathfrak{a}$. In [8, Exercise 5.4.8, p. 139], in addition to the condition $b \equiv b_i \pmod{2a_i}$ for $i \in \{1,2\}$ given there, it seems better to impose $b^2 \equiv \Delta \pmod{4a_1a_2}$. Such an integer b is uniquely determined modulo $2a_1a_2$ [8, Exercise 6.4.12, p. 219].

[†] [29, Proposition 4.1, p. 86] Let p be an odd prime, a, b, c integers such that $p \nmid a$. Put $f(x) = ax^2 + bx + c$, $D = b^2 - 4ac$, $\chi_{p^*}(n) = \left(\frac{p^*}{n}\right)$. Then, one has $\sum_{m \pmod p} \chi_{p^*}(f(m)) = \begin{cases} -\chi_{p^*}(a), & \text{if } p \nmid D, \\ (p-1)\chi_{p^*}(a), & \text{if } p \mid D. \end{cases}$

(b) If $a=2n$, then $\theta_0(p^{2n})=p^n$,
while $\theta_1(p^{2n})=p^n$ if $p\nmid d_1$, $\theta_1(p^{2n})=-p^{n-1}$ if $p\mid d_1$.

Case 3. Suppose $p\mid d$.
(a) If $a>2n+1$, then $\theta_0(p^a)=0$, $\theta_1(p^a)=0$.
(b) If $a=2n$, then $\theta_0(p^{2n})=\varphi(p^n)$, $\theta_1(p^a)=\varphi(p^n)$.
(c) If $a=2n+1$, then $\theta_0(p^{2n+1})=p^n$,
while $\theta_1(p^{2n+1})=\chi_{d_1}(p)p^n$ if $p\nmid d_1$, $\theta_1(p^{2n+1})=\chi_{d_2}(p)p^n$ if $p\mid d_1$.

Proof. We give the proof for (2) the Case 2 (b), other cases being treated similarly. In general, an $\mathcal{O}_\Delta$-primitive ideal $\mathfrak{b}$ of $\mathcal{O}_\Delta$ has the form $\mathfrak{b}=[a,r+\omega_\Delta]$, where $a,r\in\mathbf{Z}$, $a>0$, $a\mid\mathcal{N}(r+\omega_\Delta)$, $\mathfrak{N}_\Delta(\mathfrak{b})=a$. This is an oriented basis in the sense that $\beta_1=a,\beta_2=r+\omega_\Delta$ satisfy $(\beta_1\beta_2'-\beta_1'\beta_2)/\sqrt{\Delta}<0$, where β' is the conjugate of β. The representative of the equivalence class of binary forms associated to $[\mathfrak{b}]^+$ can be taken as

$$f_{\mathfrak{b}}:=f_{a,r+\omega_\Delta}(x,y)=a^{-1}(ax+(r+\omega_\Delta)y)(ax+(r+\omega_\Delta')y)=[a,2r+\sigma_\Delta,\mathcal{N}(r+\omega_\Delta)/a],$$

that is, we have $[\![f_{\mathfrak{b}}]\!]=\Phi_\Delta^{-1}([\mathfrak{b}]^+)$ through the bijection $\Phi_\Delta:\mathfrak{F}_\Delta\to\mathcal{C}_\Delta^+$ explained in §7.1.

Let us compute $\theta_j(p^a)$ in the case $a=2n$ and $(\frac{d}{p})=-1$. The lattice $\mathfrak{b}=[p^{2n},r+\omega_\Delta]$ is an $\mathcal{O}_\Delta$-regular ideal if and only if there exists $m\in\mathbf{Z}$ such that $2r+\sigma_\Delta=p^nm$, $p\nmid m^2-d$ by Butts-Pall's list. Since we are assuming $(\frac{d}{p})=-1$, this is equivalent to $2r+\sigma_\Delta=p^nm$. Note that $[p^{2n},r_1+\omega_\Delta]=[p^{2n},r_2+\omega_\Delta]$ if and only if $r_1\equiv r_2 \pmod{p^{2n}}$. Hence, m can run through mod $2p^n$ such that $m\equiv\sigma_\Delta \pmod 2$, and we have

$$\theta_0(p^{2n})=\sum_{\substack{\mathcal{O}_\Delta-\text{regular ideal }\mathfrak{b}\\ \mathfrak{N}_\Delta(\mathfrak{b})=p^{2n}}}1=\sum_{\substack{m \pmod{2p^n}\\ m\equiv\sigma_\Delta \pmod 2}}1=p^n.$$

On the other hand, one has $f_{\mathfrak{b}}(x,y)=p^{2n}x^2+mp^nxy+((m^2-d)/4)y^2$ and by definition

$$\chi^{(q^*)}(\mathfrak{b})=\begin{cases}1, & \text{if } (p,q^*)=1,\\ \chi_{p^*}(m^2-d), & \text{if } (p,q^*)\neq 1,\end{cases}\qquad \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})=\begin{cases}1 & \text{if } p\nmid d_1,\\ \chi_{p^*}(m^2-d), & \text{if } p\mid d_1.\end{cases}$$

This implies

$$\theta_1(p^{2n})=\sum_{\substack{\mathcal{O}_\Delta-\text{regular ideal }\mathfrak{b}\\ \mathfrak{N}_\Delta(\mathfrak{b})=p^{2n}}}\chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})=\begin{cases}\displaystyle\sum_{\substack{m \pmod{2p^n}\\ m\equiv\sigma_\Delta \pmod 2}}1=p^n, & \text{if } p\nmid d_1,\\ \displaystyle\sum_{\substack{m \pmod{2p^n}\\ m\equiv\sigma_\Delta \pmod 2}}\chi_{p^*}(m^2-d)=-p^{n-1}, & \text{if } p\mid d_1,\end{cases}$$

as stated. To get the last value when $p\mid d_1$, we put $m=2l+\sigma_\Delta$ with l mod p^n, and apply [29, Proposition 4.1, p. 86]. □

Remark. The sums over m mod $2p^n$ such that $m\equiv\sigma_\Delta \pmod 2$ can be replaced by the sums over m mod p^n if we take representatives m suitably in the sense that $m\equiv\sigma_\Delta \pmod 2$ holds, which is possible because p^n is odd. See a note before [3, Lemma 3, p. 263].

Lemma 2. *Under the same assumption and notation as in Lemma 1, the values $\eta_j(p^t)$ are given as follows.*

Case 1. Suppose $(\frac{d}{p})=1$.
(o) Suppose that t is odd.

(o-i) If $t < 2n$, then $\eta_0(p^t) = 0$, $\eta_1(p^t) = 0$.
(o-ii) If $t \geq 2n$, then $\eta_0(p^t) = (t-2n+1)\varphi(p^n)$, $\eta_1(p^t) = \chi_{d_1}(p)^t(t-2n+1)\varphi(p^n)$.
(e) Suppose that $t = 2s$ is even.
(e-i) If $t < 2n$, then $\eta_0(p^t) = p^s$, $\eta_1(p^t) = p^s$.
(e-ii) If $t \geq 2n$, then $\eta_0(p^t) = (t-2n+1)\varphi(p^n)$, $\eta_1(p^t) = \chi_{d_1}(p)^t(t-2n+1)\varphi(p^n)$.

Case 2. Suppose $(\frac{d}{p}) = -1$.
(o) If t is odd, then $\eta_0(p^t) = 0$, $\eta_1(p^t) = 0$.
(e) Suppose that $t = 2s$ is even.
(e-i) If $t < 2n$, then $\eta_0(p^t) = p^s$, $\eta_1(p^t) = p^s$.
(e-ii) If $t \geq 2n$, then $\eta_0(p^t) = p^n + p^{n-1}$,
while $\eta_1(p^t) = p^n + p^{n-1}$ if $p \nmid d_1$, and $\eta_1(p^t) = 0$ if $p \mid d_1$.

Case 3. Suppose $p \mid d$.
(o) Suppose that t is odd.
(o-i) If $t < 2n$, then $\eta_0(p^t) = 0$, $\eta_1(p^t) = 0$.
(o-ii) If $t \geq 2n$, then $\eta_0(p^t) = p^n$,
while $\eta_1(p^t) = \chi_{d_1}(p)p^n$ if $p \nmid d_1$, and $\eta_1(p^t) = \chi_{d_2}(p)p^n$ if $p \mid d_1$.
(e) Suppose that $t = 2s$ is even.
(e-i) If $t < 2n$, then $\eta_0(p^t) = p^s$, $\eta_1(p^t) = p^s$.
(e-ii) If $t \geq 2n$, then $\eta_0(p^t) = p^n$, $\eta_1(p^t) = p^n$.

The proof of Lemma 2 is a direct application of Lemma 1.

Put $x := p^{-s}$. By Lemma 2, we can compute p-factors explicitly for any odd prime $p \mid f$. The results are as follows. Here n, which can be characterized by $p^n \mid\mid f$, is as in Lemmas 1, 2, and $m_p \geq 0$ is defined by $p^{m_p} \mid\mid f_0$.

Case 1. Suppose $(\frac{d}{p}) = 1$. One has

$$L_p(s, \chi^{(\Delta)}_{d_1,d_2}) = \begin{cases} \dfrac{(1-\chi_{d_1}(p)x)^2 - p^{n-1}x^{2n}(1-\chi_{d_1}(p)px)^2}{(1-px^2)(1-\chi_{d_1}(p)x)^2}, & p \nmid d_1, \\ \dfrac{1-p^nx^{2n}}{1-px^2}, & p \mid d_1. \end{cases}$$

If $p \nmid d_1$, then $p \nmid d_2$, $m_p = n$, $\chi_{d_1}(p)\chi_{d_2}(p) = 1$.
If $p \mid d_1$, then $p \mid d_2$, $m_p = n-1$, $\chi_{d_1}(p) = \chi_{d_2}(p) = 0$.

Case 2. Suppose $(\frac{d}{p}) = -1$. One has

$$L_p(s, \chi^{(\Delta)}_{d_1,d_2}) = \begin{cases} \dfrac{(1+x)(1-x) + p^{n-1}x^{2n}(1-px)(1+px)}{(1-px^2)(1-x^2)}, & p \nmid d_1, \\ \dfrac{1-p^nx^{2n}}{1-px^2}, & p \mid d_1. \end{cases}$$

If $p \nmid d_1$, then $p \nmid d_2$, $m_p = n$, $\chi_{d_1}(p)\chi_{d_2}(p) = -1$.
If $p \mid d_1$, then $p \mid d_2$, $m_p = n-1$, $\chi_{d_1}(p) = \chi_{d_2}(p) = 0$.

Case 3. Suppose $p \mid d$. One has

$$L_p(s, \chi^{(\Delta)}_{d_1,d_2}) = \frac{1-\beta x - p^{n-1}x^{2n}(1-p\beta x)(-p\beta x)}{(1-px^2)(1-\beta x)},$$

where $\beta = \chi_{d_1}(p)$ or $\chi_{d_2}(p)$ according as $p \nmid d_1$ or $p \mid d_1$.

If $p \nmid d_1$, then $p \mid d_2$, $m_p = n$.
If $p \mid d_1$, then $p \nmid d_2$, $m_p = n$.

We remark that m_p can be 0 only when $\chi_{d_1}(p) = \chi_{d_2}(p) = 0$, since $n \geq 1$. Hence, the above formulas coincide with the p-factor of the formula in Theorem 1 for any odd prime $p \mid f$;

$$\frac{(1-\chi_{d_1}(p)x)(1-\chi_{d_2}(p)x) - p^{m_p-1}x^{2m_p}(px-\chi_{d_1}(p))(px-\chi_{d_2}(p))}{(1-\chi_{d_1}(p)x)(1-\chi_{d_2}(p)x)(1-px^2)}.$$

2.2 2-factor with $2 \mid f$

For any discriminant d, we have the decomposition $d = d_0 f^2$ with a fundamental discriminant d_0 and a natural number f. Let us put $\Delta(d) := d_0$ and denote by $\Delta_2(d)$ the 2-part with sign (including 1) of the fundamental discriminant $\Delta(d)$. Hence $\Delta_2(d) \in \{1, -4, \pm 8\}$ and $\Delta(d)/\Delta_2(d)$ is an odd fundamental discriminant.

Let $\Delta = d_1 d_2 f_0^2$ be a quadratic discriminant as given in Theorem 1 and $m_2 \geq 0$ an integer such that $2^{m_2} \mid\mid f_0$. We also write $\Delta = d_K f^2$ as usual, and suppose $2 \mid f$ throughout this section 2.2. Let n denote the largest integer such that $d := \Delta/2^{2n}$ is a discriminant. In other words, $2^n \mid\mid f$ and so $\Delta_2(d) = \Delta_2(d_K)$. This d is an odd discriminant (i.e. $\Delta_2(d) = 1$) if and only if $\Delta_2(d_1) = \Delta_2(d_2)$.

Note that $\Delta_2(d) = -4$ if and only if $(\Delta_2(d_1), \Delta_2(d_2)) \in \{(1,-4), (-4,1), (8,-8), (-8,8)\}$, and in this case one has $d \equiv 12 \pmod{16}$. We see $\Delta_2(d) = 8$ if and only if $(\Delta_2(d_1), \Delta_2(d_2)) \in \{(1,8), (8,1), (-4,-8), (-8,-4)\}$, and in this case one has $d \equiv 8 \pmod{32}$. We also see $\Delta_2(d) = -8$ if and only if $(\Delta_2(d_1), \Delta_2(d_2)) \in \{(1,-8), (-8,1), (-4,8), (8,-4)\}$, and in this case one has $d \equiv 24 \pmod{32}$. In these cases that $2 \mid \Delta_2(d)$, if either $\Delta_2(d_1)$ or $\Delta_2(d_2)$ is equal to 1, then $n = m_2$. Otherwise, $n = m_2 + 2$ if $\Delta_2(d) = -4$, and $n = m_2 + 1$ if $\Delta_2(d) = \pm 8$.

Note that $(\frac{d}{2}) = -1$ if and only if $d \equiv 5 \pmod 8$, and $(\frac{d}{2}) = 1$ if and only if $d \equiv 1 \pmod 8$, because d is a discriminant. Taking these remarks into account, we apply the list in [3, p. 263] of regular $\mathcal{O}_\Delta$-ideals $\mathfrak{b}$ such that $\mathfrak{N}_\Delta(\mathfrak{b})$ is any power of $2 \mid f$. The proof of the following Lemmas 3, 4 are similar to those of Lemmas 1, 2.

Lemma 3. *Suppose that $2 \mid f$. Let n denote the largest integer such that $d := \Delta/2^{2n}$ is a discriminant. Then the values $\theta_j(2^a)$ are given as follows. Here φ is the Euler function.*

Case 1. Suppose $(\frac{d}{2}) = 1$.
(a) If $a < 2n$ is odd, then $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.
(b) If $a = 2h \leq 2n-6$, then $\theta_0(2^{2h}) = \varphi(2^h)$, $\theta_1(2^{2h}) = \varphi(2^h)$.
(c) If $a = 2n-4$, then $\theta_0(2^{2n-4}) = \varphi(2^{n-2})$,
while $\theta_1(2^{2n-4}) = \varphi(2^{n-2})$ if $\Delta_2(d_1) \in \{1,-4\}$, and $\theta_1(2^{2n-4}) = -\varphi(2^{n-2})$ if $\Delta_2(d_1) = \pm 8$.
(d) If $a = 2n-2$, then $\theta_0(2^{2n-2}) = \varphi(2^{n-1})$, while $\theta_1(2^{2n-2})$ takes the value
$\varphi(2^{n-1})$ if $\Delta_2(d_1) = 1$, and $-\varphi(2^{n-1})$ if $\Delta_2(d_1) = -4$, and 0 if $\Delta_2(d_1) = \pm 8$.
(e) If $a = 2n$, then $\theta_0(2^{2n}) = 0$, $\theta_1(2^{2n}) = 0$.
(f) If $a > 2n$, then $\theta_0(2^a) = 2\varphi(2^n)$,
while $\theta_1(2^a) = \chi_{d_1}(2)^a \cdot 2\varphi(2^n)$ if $\Delta_2(d_1) = 1$, and $\theta_1(2^a) = 0$ if $2 \mid \Delta_2(d_1)$.

Case 2. Suppose $(\frac{d}{2}) = -1$.
(a) If $a < 2n$ is odd, then $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.
(b) If $a = 2h \leq 2n-6$, then $\theta_0(2^{2h}) = \varphi(2^h)$, $\theta_1(2^{2h}) = \varphi(2^h)$.
(c) If $a = 2n-4$, then $\theta_0(2^{2n-4}) = \varphi(2^{n-2})$,
while $\theta_1(2^{2n-4}) = \varphi(2^{n-2})$ if $\Delta_2(d_1) \in \{1,-4\}$, and $\theta_1(2^{2n-4}) = -\varphi(2^{n-2})$ if $\Delta_2(d_1) = \pm 8$.
(d) If $a = 2n-2$, then $\theta_0(p^{2n-2}) = \varphi(2^{n-1})$, while $\theta_1(2^{2n-2})$ takes the value
$\varphi(2^{n-1})$ if $\Delta_2(d_1) = 1$, and $-\varphi(2^{n-1})$ if $\Delta_2(d_1) = -4$, and 0 if $\Delta_2(d_1) = \pm 8$.
(e) If $a = 2n$, then $\theta_0(2^{2n}) = \varphi(2^{n+1})$,

while $\theta_1(2^{2n}) = \varphi(2^{n+1})$ *if* $\Delta_2(d_1) = 1$, *and* $\theta_1(2^{2n}) = 0$ *if* $2 \mid \Delta_2(d_1)$.
(f) If $a > 2n$, *then* $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.

Case 3. Suppose $2 \mid d$ *and both* $\Delta_2(d_1)$ *and* $\Delta_2(d_2)$ *are even.*
(a) If $a < 2n$ *is odd, then* $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.
(b) If $a = 2h \leq 2n - 4$, *then* $\theta_0(2^{2h}) = \varphi(2^h)$, $\theta_1(2^{2h}) = \varphi(2^h)$.
(c) If $a = 2n - 2$, *then* $\theta_0(2^{2n-2}) = \varphi(2^{n-1})$,
while $\theta_1(2^{2n-2}) = \varphi(2^{n-1})$ *if* $\Delta_2(d) = \pm 8$, *and* $\theta_1(2^{2n-2}) = -\varphi(2^{n-1})$ *if* $\Delta_2(d) = -4$.
(d) If $a = 2n$, *then* $\theta_0(2^{2n}) = \varphi(2^n)$,
while $\theta_1(2^{2n}) = -\varphi(2^n)$ *if* $\Delta_2(d) = \pm 8$, *and* $\theta_1(2^{2n}) = 0$ *if* $\Delta_2(d) = -4$.
(e) If $a = 2n + 1$, *then* $\theta_0(2^{2n+1}) = 2^n$, $\theta_1(2^{2n+1}) = 0$.
(f) If $a > 2n + 1$, *then* $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.

Case 4. Suppose $2 \mid d$ *and either* $\Delta_2(d_1)$ *or* $\Delta_2(d_2)$ *is equal to* 1.
(a) If $a < 2n$ *is odd, then* $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.
(b) If $a = 2h \leq 2n$, *then* $\theta_0(2^{2h}) = \varphi(2^h)$, $\theta_1(2^{2h}) = \varphi(2^h)$.
(c) If $a = 2n + 1$, *then* $\theta_0(2^{2n+1}) = 2^n$,
while $\theta_1(2^{2n+1}) = \chi_{d_1}(2)2^n$ *if* $2 \nmid \Delta_2(d_1)$, *and* $\theta_1(2^{2n+1}) = \chi_{d_2}(2)2^n$ *if* $2 \mid \Delta_2(d_1)$.
(d) If $a > 2n + 1$, *then* $\theta_0(2^a) = 0$, $\theta_1(2^a) = 0$.

For an integer a, let us denote by $\mathrm{ord}_2(a)$ an integer such that $2^{\mathrm{ord}_2(a)}$ is the highest power of 2 dividing a. Notice that when the discriminant d defined in Lemma 3 is odd, then $n \geq \mathrm{ord}_2(\Delta_2(d_1)) = \mathrm{ord}_2(\Delta_2(d_2))$ by definition.

Lemma 4. *Under the same assumption and notation as in Lemma 3, the values* $\eta_j(2^t)$ *are given as follows.*

Case 1. Suppose $(\frac{d}{2}) = 1$.
(o) Suppose that t *is odd.*
If $t < 2n$, *then* $\eta_0(2^t) = 0$, $\eta_1(2^t) = 0$.
If $t \geq 2n$, *then* $\eta_0(2^t) = (t - 2n + 1)\varphi(2^n)$, $\eta_1(2^t) = \chi_{d_1}(2)(t - 2n + 1)\varphi(2^n)$.
(e) Suppose that $t = 2s$ *is even.*
(e-i) If $t < 2n$ *then* $\eta_0(2^t) = 2^s$, *and if* $t \geq 2n$ *then* $\eta_0(2^t) = (t - 2n + 1)\varphi(2^n)$.
(e-ii) If $\Delta_2(d_1) = 1$, *then* $\eta_1(2^t) = \eta_0(2^t)$ *for any even* t.
(e-iii) Suppose $2 \mid \Delta_2(d_1)$.
If $t \leq 2n - 2\mathrm{ord}_2(\Delta_2(d_1))$, *then* $\eta_1(2^t) = 2^s$.
If $t > 2n - 2\mathrm{ord}_2(\Delta_2(d_1))$, *then* $\eta_1(2^t) = 0$.

Case 2. Suppose $(\frac{d}{2}) = -1$.
(o) Suppose that t *is odd. Then* $\eta_0(2^t) = 0$, $\eta_1(2^t) = 0$.
(e) Suppose that $t = 2s$ *is even.*
(e-i) If $t < 2n$ *then* $\eta_0(2^t) = 2^s$, *and if* $t \geq 2n$ *then* $\eta_0(2^t) = 2^n + 2^{n-1}$.
(e-ii) If $\Delta_2(d_1) = 1$, *then* $\eta_1(2^t) = \eta_0(2^t)$ *for any even* t.
(e-iii) Suppose $2 \mid \Delta_2(d_1)$.
If $t \leq 2n - 2\mathrm{ord}_2(\Delta_2(d_1))$, *then* $\eta_1(2^t) = 2^s$.
If $t > 2n - 2\mathrm{ord}_2(\Delta_2(d_1))$, *then* $\eta_1(2^t) = 0$.

Case 3. Suppose $2 \mid d$ *and both* $\Delta_2(d_1)$ *and* $\Delta_2(d_2)$ *are even.*
(o) Suppose that t *is odd. Then* $\eta_1(2^t) = 0$.
(e) Suppose that $t = 2s$ *is even.*
If $t \leq 2n - 8 + 2\mathrm{ord}_2(\Delta_2(d))$ *then* $\eta_1(2^t) = 2^s$.
If $t \geq 2n - 6 + 2\mathrm{ord}_2(\Delta_2(d))$ *then* $\eta_1(2^t) = 0$.

The formula for $\eta_0(2^t)$ for $t \geq 0$ is the same as given in Case 4 below.

Case 4. Suppose $2 \mid d$ and either $\Delta_2(d_1)$ or $\Delta_2(d_2)$ is equal to 1.
(o) Suppose that t is odd.
(o-i) If $t < 2n$, then $\eta_0(2^t) = 0$, $\eta_1(2^t) = 0$.
(o-ii) If $t \geq 2n$, then $\eta_0(2^t) = 2^n$,
while $\eta_1(2^t) = \chi_{d_1}(2)2^n$ if $2 \nmid \Delta_2(d_1)$, and $\eta_1(2^t) = \chi_{d_2}(2)2^n$ if $2 \mid \Delta_2(d_1)$.
(e) Suppose that $t = 2s$ is even.
(e-i) If $t < 2n$, then $\eta_0(2^t) = 2^s$, $\eta_1(2^t) = 2^s$.
(e-ii) If $t \geq 2n$, then $\eta_0(2^t) = 2^n$, $\eta_1(2^t) = 2^n$.

Put $x := 2^{-s}$. By Lemma 4, we can compute the 2-factor explicitly when $2 \mid f$. The results are as follows. Here n is as in Lemmas 3, 4, that is $2^n \mid\mid f$, and $m_2 \geq 0$ is defined by $2^{m_2} \mid\mid f_0$.

Case 1. Suppose $(\frac{d}{2}) = 1$. One has

$$L_2(s, \chi^{(\Delta)}_{d_1,d_2}) = \begin{cases} \dfrac{(1-\chi_{d_1}(2)x)^2 - 2^{n-1}x^{2n}(1-\chi_{d_1}(2)2x)^2}{(1-2x^2)(1-\chi_{d_1}(2)x)^2}, & \Delta_2(d_1) = 1, \\ \dfrac{1-(2x^2)^{n-\mathrm{ord}_2(\Delta_2(d_1))+1}}{1-2x^2}, & 2 \mid \Delta_2(d_1). \end{cases}$$

If $\Delta_2(d_1) = 1$, then $\Delta_2(d_2) = 1$, $m_2 = n$, $\chi_{d_1}(2)\chi_{d_2}(2) = 1$.
If $2 \mid \Delta_2(d_1)$, then $\Delta_2(d_2) = \Delta_2(d_1)$, $n = m_2 + \mathrm{ord}_2(\Delta_2(d_1))$, $\chi_{d_1}(2) = \chi_{d_2}(2) = 0$.

Case 2. Suppose $(\frac{d}{2}) = -1$. One has

$$L_2(s, \chi^{(\Delta)}_{d_1,d_2}) = \begin{cases} \dfrac{(1+x)(1-x) + 2^{n-1}x^{2n}(1-2x)(1+2x)}{(1-2x^2)(1-x^2)}, & \Delta_2(d_1) = 1, \\ \dfrac{1-(2x^2)^{n-\mathrm{ord}_2(\Delta_2(d_1))+1}}{1-2x^2}, & 2 \mid \Delta_2(d_1). \end{cases}$$

If $\Delta_2(d_1) = 1$, then $\Delta_2(d_2) = 1$, $m_2 = n$, $\chi_{d_1}(2)\chi_{d_2}(2) = -1$.
If $2 \mid \Delta_2(d_1)$, then $\Delta_2(d_2) = \Delta_2(d_1)$, $n = m_2 + \mathrm{ord}_2(\Delta_2(d_1))$, $\chi_{d_1}(2) = \chi_{d_2}(2) = 0$.

Case 3. Suppose $2 \mid d$ and both $\Delta_2(d_1)$ and $\Delta_2(d_1)$ are even. One has

$$L_2(s, \chi^{(\Delta)}_{d_1,d_2}) = \frac{1-(2x^2)^{n-3+\mathrm{ord}_2(\Delta_2(d))}}{1-2x^2}.$$

If $\Delta_2(d) = -4$, then $m_2 + 2 = n$. If $\Delta_2(d) = \pm 8$, then $m_2 + 1 = n$.
Hence, $n - 3 + \mathrm{ord}_2(\Delta_2(d)) = m_2 + 1$.

Case 4. Suppose $2 \mid d$ and either $\Delta_2(d_1)$ or $\Delta_2(d_1)$ is equal to 1. One has

$$L_2(s, \chi^{(\Delta)}_{d_1,d_2}) = \frac{1 - \beta x - 2^{n-1}x^{2n}(1-2\beta x)(-2\beta x)}{(1-2x^2)(1-\beta x)},$$

where $\beta = \chi_{d_1}(2)$ or $\chi_{d_2}(2)$ according as $2 \nmid \Delta_2(d_1)$ or $2 \mid \Delta_2(d_1)$.
If $2 \nmid \Delta_2(d_1)$, then $2 \mid \Delta_2(d_2)$, $n = m_2$.
If $2 \mid \Delta_2(d_1)$, then $2 \nmid \Delta_2(d_2)$, $n = m_2$.

We remark that m_2 can be 0 only when $\chi_{d_1}(2) = \chi_{d_2}(2) = 0$, since $n \geq 1$. Hence, the above formulas coincide with the 2-factor of the formula in Theorem 1 when $2 \mid f$;

$$\frac{(1-\chi_{d_1}(2)x)(1-\chi_{d_2}(2)x) - 2^{m_2-1}x^{2m_2}(2x-\chi_{d_1}(2))(2x-\chi_{d_2}(2))}{(1-\chi_{d_1}(2)x)(1-\chi_{d_2}(2)x)(1-2x^2)}.$$

2.3 p-factor with $p \nmid f$

For any prime not dividing the conductor f, the similar counting argument as given in Sections 2.1, 2.2 works.

Lemma 5. *Let $\Delta = d_1 d_2 f_0^2$ be as in Theorem 1 and write $\Delta = d_K f^2$, where d_K is the discriminant of $K = \mathbf{Q}(\sqrt{\Delta})$ and f is the conductor of Δ. Let p be any prime such that $p \nmid f$. Then, one has*

$$L_p(s, \chi^{(\Delta)}_{d_1,d_2}) = \frac{1}{(1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s})}.$$

Proof. Let $\mathfrak{a}$ be an $\mathcal{O}_\Delta$-invertible ideal of $\mathcal{O}_\Delta$ such that $\mathfrak{N}_\Delta(\mathfrak{a}) = p^t$ $(t \geq 1)$. Let $\mathfrak{q}$ be any prime ideal of $\mathcal{O}_\Delta$ satisfying $\mathfrak{a} \subset \mathfrak{q}$. It follows from $p^t \in p^t\mathcal{O}_\Delta = \mathfrak{a}\mathfrak{a}' \subset \mathfrak{q}$ that $\mathfrak{q} \cap \mathbf{Z} = p\mathbf{Z}$, and thus $\mathfrak{q}$ is exactly the one described in [8, Theorem 5.8.8, p. 178]. Let $Q_\Delta(*) = \left(\frac{\Delta}{*}\right)$ be the quadratic symbol defined in [8, p. 85]. Here $\left(\frac{m}{n}\right)$ in the right-hand side is the Kronecker symbol defined in [8, p. 82]. If $p \nmid f_0$, then $Q_\Delta(p) = \chi_{d_1}(p)\chi_{d_2}(p)$ by [8, Theorem 3.5.1, p. 82]. Note that $Q_\Delta(p) \neq 0$ implies $p \nmid \Delta$, and $p \nmid f_0$. By the unique factorization of $\mathfrak{a}$ into prime ideals ([8, Theorem 5.8.1, p. 174]) together with the factorization of a rational prime $p \nmid f$ into prime ideals of $\mathcal{O}_\Delta$ ([8, Theorem 5.8.8, p. 178]), one can deduce the followings; (1) if $Q_\Delta(p) = 1$ then $\chi_{d_1}(p)\chi_{d_2}(p) = 1$ and $\eta_1(p^t) = (t+1)\chi_{d_1}(p)^t$, (2) if $Q_\Delta(p) = -1$ then $\chi_{d_1}(p)\chi_{d_2}(p) = -1$ and $\eta_1(p^t) = 1$ or 0 according as $2 \mid t$ or $2 \nmid t$, (3) if $Q_\Delta(p) = 0$ then $\chi_{d_1}(p)\chi_{d_2}(p) = 0$, $(\chi_{d_1}(p), \chi_{d_2}(p)) \neq (0,0)$ and $\eta_1(p^t) = \beta^t$, where $\beta = \chi_{d_1}(p)$ or $\chi_{d_2}(p)$ according as $p \nmid d_1$ or $p \nmid d_2$. These results agree with the case $t = 0$, that is $\eta_1(1) = 1$. Simple calculations yield the desired formula as stated in Lemma 5. □

In this way, we get all of the Euler factors of the formula stated in Theorem 1. Notice that any prime $p \nmid f$ satisfies $p \nmid f_0$. This completes the proof of Theorem 1. □

Remarks. 1. We give another proof of Theorem 1 in Section 6.

2. We may put $d_1 = 1$ and $\Delta = d_2 f_0^2$ to get the formula in [13]. See [1, Proposition 10.18, p. 171] for a different proof of this special case.

3. A similar computation is worked out in a paper by K. Wong [30] which appeared when we were finishing this part of our paper. It seems possible to get Theorem 1 by pushing forward his computation, or using results of [5, 6, 21], together with the Möbius inversion as in the manner of [1].

3 Meyer-Siegel-Zagier

The proof of Theorem 2 uses Meyer-Siegel's analytic class number formula. Although, the results in this section can be found in the literatures, we include a summary for convenience of the readers. Let $\mathcal{C}_\Delta$ be the class group and $\mathcal{C}_\Delta^+$ the narrow class group of a quadratic discriminant Δ. We write $\mathfrak{a} \sim \mathfrak{b}$ if $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent and denote by $[\mathfrak{a}]$ the equivalence class of $\mathfrak{a}$. We write $\mathfrak{a} \sim_+ \mathfrak{b}$ for proper equivalence, in other words, equivalence in the narrow sense, and denote a proper equivalence class of $\mathfrak{a}$ by $[\mathfrak{a}]^+$. The principal ideal generated by $\beta \in \mathbf{Q}(\sqrt{\Delta})$ is denoted by $(\beta) := \beta\mathcal{O}_\Delta$. We then define for $\Re(s) > 1$ the L-function associated to a narrow class character χ of $\mathcal{C}_\Delta^+$ by

$$L(s, \chi) := \sum_{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta} \frac{\chi(\mathfrak{a})}{\mathfrak{N}_\Delta(\mathfrak{a})^s},$$

where the sum is taken over all $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{a}$ of $\mathcal{O}_\Delta$ and $\mathfrak{N}_\Delta(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$.

Proposition 1. *[C. Meyer, C. Siegel] Let $\Delta > 0$ be a quadratic discriminant. Suppose that χ is a narrow class character such that $\chi((\beta)) = \mathrm{sign}\mathcal{N}(\beta)$ for any $\beta \in \mathbf{Q}(\sqrt{\Delta})^\times$. We assume $\mathcal{N}(\epsilon_\Delta) = 1$ for the fundamental unit $\epsilon_\Delta > 1$ of the real quadratic order $\mathcal{O}_\Delta$. Under these assumptions, $L(s,\chi)$ can be continued meromorphically to the whole complex s-plane, and the value at $s = 0$ has the form*

$$L(0,\chi) = \sum_{[\mathfrak{b}]^+ \in \mathcal{C}_\Delta^+} \overline{\chi}(\mathfrak{b})\chi((\beta_2))G(\mathfrak{b}) = 2\sum_{[\mathfrak{b}] \in \mathcal{C}_\Delta} \overline{\chi}(\mathfrak{b})\chi((\beta_2))G(\mathfrak{b}),$$

where

$$G(\mathfrak{b}) := \frac{1}{4\pi i}\left[\mathrm{Log}\left((z-\alpha)(z-\alpha')\eta(z)^4\right)\right]_{z_0}^{z_0^*}.$$

Here $\eta(z) := e^{\frac{\pi i z}{12}}\prod_{m=1}^{\infty}(1-e^{2\pi i m z})$ is the Dedekind eta function, z_0 is any point of the upper-half plane, and for any $\mathcal{O}_\Delta$-invertible fractional ideal $\mathfrak{b}$ with $\mathfrak{b} = [\beta_1, \beta_2]$ such that $(\beta_1\beta_2' - \beta_1'\beta_2)/\mathcal{N}(\beta_2) > 0$, the point $z_0^* := M_\mathfrak{b}\langle z_0\rangle$ and the real number α are defined by

$$\epsilon_\Delta \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = M_\mathfrak{b} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad M_\mathfrak{b} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}), \qquad z_0^* := M_\mathfrak{b}\langle z_0\rangle = \frac{az_0+b}{cz_0+d}, \qquad \alpha := \frac{\beta_1}{\beta_2}.$$

The Log *is any fixed branch of the logarithm.*

Proof. The proof closely follows that of [28], where only the case of fundamental discriminant is treated and the value at $s = 1$ instead of $s = 0$ is considered. So we give a sketch of the proof. Suppose that $\Re(s) > 1$. We rearrange the sum

$$L(s,\chi) = \sum_{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta} \chi(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{a})^{-s} = \sum_{[\mathfrak{c}] \in \mathcal{C}_\Delta} \sum_{\mathcal{O}_\Delta-\text{ideal } \mathfrak{a} \in [\mathfrak{c}]} \chi(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{a})^{-s}.$$

Fix representatives $\mathfrak{c}$ so that $\mathfrak{c} \subset \mathcal{O}_\Delta$, and take their conjugate ideals $\mathfrak{c}' \in [\mathfrak{c}]^{-1}$, which are $\mathcal{O}_\Delta$-invertible ideals of $\mathcal{O}_\Delta$ satisfying $\mathfrak{c}\mathfrak{c}' = (\mathfrak{N}_\Delta(\mathfrak{c}))$ ([8, Corollary 5.4.3, p. 136]). For two lattices $\mathfrak{a}$, $\mathfrak{b}$ in $K := \mathbf{Q}(\sqrt{\Delta})$, let $(\mathfrak{a} :_K \mathfrak{b}) := \{\alpha \in K; \alpha\mathfrak{b} \subset \mathfrak{a}\}$ ([8, p. 116]). Using [8, Lemma 5.3.4, p. 130] and the argument in [33, §11], the map $(\mathfrak{c}' \setminus \{0\})/\mathcal{O}_\Delta^\times \xrightarrow{\phi} \{\mathfrak{a} \in [\mathfrak{c}]; \mathfrak{a} \subset \mathcal{O}_\Delta\}$ given by $\phi(\beta) := \beta(\mathcal{O}_\Delta :_K \mathfrak{c}')$ is well-defined and ϕ is a bijection. Note that if $\mathfrak{a} = \phi(\beta)$, then $\mathfrak{a}\mathfrak{c}' = (\beta)$ and thus $|\mathcal{N}(\beta)| = \mathfrak{N}_\Delta(\mathfrak{c}')\mathfrak{N}_\Delta(\mathfrak{a})$, $\chi(\mathfrak{a})\chi(\mathfrak{c}') = \chi((\beta))$ (cf. [8, Theorem 5.1.3, p. 116], [8, Theorem 5.4.6, p. 137])‡. Hence we deduce (cf. [33, p. 88])§

$$\sum_{\mathcal{O}_\Delta-\text{ideal } \mathfrak{a} \in [\mathfrak{c}]} \chi(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{a})^{-s} \quad = \quad \overline{\chi}(\mathfrak{c}')\mathfrak{N}_\Delta(\mathfrak{c}')^s \cdot \frac{1}{2}\, L_{\mathfrak{c}', U_\Delta}(s,\chi),$$

where, for any $\mathcal{O}_\Delta$-invertible fractional ideal $\mathfrak{b}$, we put

$$L_{\mathfrak{b},U_\Delta}(s,\chi) := \sum_{0 \neq \beta \in \mathfrak{b}/U_\Delta} \chi((\beta))|\mathcal{N}(\beta)|^{-s}, \qquad U_\Delta := \langle\epsilon_\Delta\rangle = \epsilon_\Delta^{\mathbf{Z}}. \tag{2}$$

Writing $\mathfrak{c}' = \mathfrak{b}$, we obtain

$$L(s,\chi) = \frac{1}{2}\sum_{[\mathfrak{b}] \in \mathcal{C}_\Delta} \overline{\chi}(\mathfrak{b})\mathfrak{N}_\Delta(\mathfrak{b})^s L_{\mathfrak{b},U_\Delta}(s,\chi).$$

In the sum, each term is independent of the choice of a representative of class in $\mathcal{C}_\Delta$. Here, we extend $\mathfrak{N}_\Delta(\mathfrak{b}) := n^{-2}\mathfrak{N}_\Delta(n\mathfrak{b})$, where n is any natural number such that $n\mathfrak{b}$ is an ideal of $\mathcal{O}_\Delta$ and $\mathfrak{N}_\Delta(n\mathfrak{b}) = (\mathcal{O}_\Delta : n\mathfrak{b})$ is the absolute norm of $n\mathfrak{b}$.

‡In [8, Theorem 5.1.3, p. 116], $(\mathfrak{c} :_K \xi\mathfrak{a}) = |\mathcal{N}(\xi)|(\mathfrak{c} :_K \mathfrak{a})$ should be read $(\mathfrak{c} : \xi\mathfrak{a}) = |\mathcal{N}(\xi)|(\mathfrak{c} : \mathfrak{a})$.

§$\frac{1}{2}$ arises from the difference between $\mathcal{O}_\Delta^\times = \langle -1, \epsilon_\Delta\rangle$ and $U_\Delta = \langle\epsilon_\Delta\rangle$.

For $z = x + iy \in \mathbf{C}$ with $y > 0$, and $s \in \mathbf{C}$ with $\Re(s) > 1$, let

$$E(z,s) := \sum_{\substack{m,n\in\mathbf{Z}\\(m,n)\neq(0,0)}} \frac{y^s}{|m+nz|^{2s}}, \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

As is well-known, the Eisenstein series $E(z,s)$ can be continued meromorphically to the whole complex s-plane. In addition, $s(s-1)\Gamma(s)E(z,s)$ is smooth in (z,s) ($z = x+iy \in \mathbf{C}$, $y > 0$, $s \in \mathbf{C}$), and holomorphic in $s \in \mathbf{C}$ (cf. [9, Theorem 11.6, p. 128], [27, Theorem 8.10, p. 55], [34, Proposition, p. 266]). By Meyer-Siegel [28, pp. 126–129], one has¶

$$\int_{z_0}^{z_0^*} \frac{\partial}{\partial z} E(z,s)\, dz = \frac{\chi((\beta_2))}{2i} \mathfrak{N}_\Delta(\mathfrak{b})^s \sqrt{\Delta}^s \frac{\Gamma((s+1)/2)^2}{\Gamma(s)} L_{\mathfrak{b},U_\Delta}(s,\chi).$$

Here the notation in Proposition 1 is used and the path of integration on the left-hand side is taken as follows. The matrix $M_\mathfrak{b}$ has two fixed points $\alpha := \beta_1/\beta_2$ and α', where $\alpha' < \alpha$ by the assumption. Let $C_{M_\mathfrak{b}}$ be the geodesic semi-circle connecting α' and α. For any fixed $z_0 \in C_{M_\mathfrak{b}}$, the integral is taken along the line $C_{M_\mathfrak{b}}$ from z_0 to $z_0^* := M_\mathfrak{b}\langle z_0\rangle \in C_{M_\mathfrak{b}}$.

Noticing the relations $\overline{\chi((\beta_2))} = \chi((\beta_2))$ and $\Gamma(s+1) = s\Gamma(s)$, we have

$$\begin{aligned} L(s,\chi) &= \sum_{[\mathfrak{b}]\in\mathcal{C}_\Delta} \overline{\chi}(\mathfrak{b}) \cdot i \cdot \chi((\beta_2)) \sqrt{\Delta}^{-s} \frac{\Gamma(s)}{\Gamma((s+1)/2)^2} \int_{z_0}^{z_0^*} \frac{\partial}{\partial z} E(z,s)\, dz \\ &= \frac{i}{2\sqrt{\Delta}^s} \frac{\Gamma(s+1)}{\Gamma((s+1)/2)^2} \sum_{[\mathfrak{b}]^+\in\mathcal{C}_\Delta^+} \overline{\chi}(\mathfrak{b})\chi((\beta_2)) \int_{z_0}^{z_0^*} \frac{\partial}{\partial z}\frac{1}{s} E(z,s)\, dz. \end{aligned}$$

Using the fact that $E(z,0)$ is a constant ($E(z,0) = -1$ cf. [34, Proposition, p. 266]), we deduce

$$L(0,\chi) = \frac{i}{2}\frac{1}{\Gamma(1/2)^2} \sum_{[\mathfrak{b}]^+\in\mathcal{C}_\Delta^+} \overline{\chi}(\mathfrak{b})\chi((\beta_2)) \int_{z_0}^{z_0^*} \frac{\partial}{\partial z}\left(\frac{\partial}{\partial s}E(z,s)\bigg|_{s=0}\right) dz. \tag{3}$$

It is known that (cf. [25, (5), p. 273]), for $z \in \mathbf{C}$, $y = \Im(z) > 0$,

$$\frac{\partial}{\partial s}E(z,s)\bigg|_{s=0} = -\log(y|\eta(z)|^4) - 2\log(2\pi),$$

and that, when $z \in C_{M_\mathfrak{b}}$, one has (cf. [28, p. 132])‖

$$\frac{1}{z-\overline{z}} = \frac{1}{2}\frac{\partial}{\partial z}\log\left((z-\alpha)(z-\alpha')\right).$$

Therefore, for $z \in C_{M_\mathfrak{b}}$, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial s}E(z,s)\bigg|_{s=0}\right) = -\frac{1}{2}\frac{\partial}{\partial z}\log\left((z-\alpha)(z-\alpha')\eta(z)^4\right) = -\frac{1}{2}\frac{d}{dz}\log\left((z-\alpha)(z-\alpha')\eta(z)^4\right),$$

and

$$\int_{z_0}^{z_0^*} \frac{\partial}{\partial z}\left(\frac{\partial}{\partial s}E(z,s)\bigg|_{s=0}\right) dz = -\frac{1}{2}\cdot\left[\mathrm{Log}\left((z-\alpha)(z-\alpha')\eta(z)^4\right)\right]_{z_0}^{z_0^*}.$$

¶Indeed, the computation given in [28, pp. 126–129] for any positive fundamental discriminant works for any general quadratic discriminants $\Delta > 0$.

‖$C_{M_\mathfrak{b}}$ has the form $|z - \frac{\alpha+\alpha'}{2}| = \frac{\alpha-\alpha'}{2}$. If we put $w = z - \frac{\alpha+\alpha'}{2}$ and $\beta = \frac{\alpha-\alpha'}{2}$, then $w\overline{w} = |w|^2 = \beta^2$. Therefore, $z-\overline{z} = w-\overline{w} = w - \frac{\beta^2}{w} = \frac{w^2-\beta^2}{w} = \frac{(z-\alpha)(z-\alpha')}{w}$, $\frac{1}{z-\overline{z}} = \frac{w}{(z-\alpha)(z-\alpha')} = \frac{1}{2}\{\frac{1}{z-\alpha'} + \frac{1}{z-\alpha}\} = \frac{1}{2}\frac{\partial}{\partial z}\{\log(z-\alpha) + \log(z-\alpha')\}$.

This together with (3) completes the proof of Proposition 1. By the remark after [28, Theorem 12, p. 133], the point z_0 can be chosen as any point in the upper-half plane. Indeed, $G(\mathfrak{b})$ in Proposition 1 is holomorphic on the upper-half plane as a function of z_0. In addition, it is constant on $C_{M_\mathfrak{b}}$ as we saw in this proof (cf. Lemma 6 below). $\square$

To compute the value $G(\mathfrak{b})$, put $\delta(z) := 6\mathrm{Log}(\eta(z)^4)$, where we choose and fix the branch by

$$\delta(z) = 6\mathrm{Log}(\eta(z)^4) := 2\pi i z - 24\sum_{n,m=1}^{\infty}\frac{e^{2\pi i n m z}}{m}, \quad \Im(z) > 0.$$

Following [32, (3), p. 82], for any $M \in SL_2(\mathbf{Z})$, we define

$$n_M := \frac{1}{2\pi i}\{\delta(M\langle z\rangle) - \delta(z) - 12\mathrm{Log}(cz+d)\}. \tag{4}$$

Here Log in the right-hand side is defined by the principal value, that is, $\mathrm{Log}(cz+d) := \mathrm{Log}|cz+d| + i\mathrm{Arg}(cz+d)$ with $-\pi < \mathrm{Arg}(cz+d) \le \pi$.**

By Meyer (cf. [32, Theoreme, p. 86]), we have

Lemma 6. *[C. Meyer] Under the same assumption as in Proposition 1, one has*

$$G(\mathfrak{b}) = \frac{n_{M_\mathfrak{b}}}{12}, \quad L_{\mathfrak{b},U_\Delta}(0,\chi) = \chi((\beta_2))\frac{n_{M_\mathfrak{b}}}{3}.$$

Here $L_{\mathfrak{b},U_\Delta}(s,\chi)$ is defined by (2).

Proof. Put $z^* := M_\mathfrak{b}\langle z\rangle$, $\alpha^* := M_\mathfrak{b}\langle\alpha\rangle$ and $\alpha'^* := M_\mathfrak{b}\langle\alpha'\rangle$. The first statement follows from

$$\mathrm{Log}(\eta(z_0^*)^4) - \mathrm{Log}(\eta(z_0)^4) = \frac{1}{6}\{\delta(z_0^*) - \delta(z_0)\} = \frac{\pi i}{3}n_{M_\mathfrak{b}} + 2\mathrm{Log}(cz_0+d),$$

$$z_0^* - \alpha^* = \frac{z_0-\alpha}{(cz_0+d)(c\alpha+d)}, \quad z_0^* - \alpha'^* = \frac{z_0-\alpha'}{(cz_0+d)(c\alpha'+d)},$$

$$\big[\mathrm{Log}(z-\alpha) + \mathrm{Log}(z-\alpha')\big]_{z_0}^{z_0^*} = -2\mathrm{Log}(cz_0+d),$$

where the principal value is chosen for Log. In fact, noticing

$$\alpha = \frac{\beta_1}{\beta_2}, \quad \alpha^* = \alpha, \quad \epsilon_\Delta = c\alpha + d, \quad \epsilon_\Delta\cdot\epsilon'_\Delta = 1, \quad c > 0,$$

we have

$$\begin{aligned}
&\big[\mathrm{Log}(z-\alpha) + \mathrm{Log}(z-\alpha')\big]_{z_0}^{z_0^*}\\
=\ &\mathrm{Log}(z_0^*-\alpha^*) + \mathrm{Log}(z_0^*-\alpha'^*) - \mathrm{Log}(z_0-\alpha) - \mathrm{Log}(z_0-\alpha')\\
=\ &\mathrm{Log}\frac{z_0-\alpha}{(cz_0+d)\epsilon_\Delta} + \mathrm{Log}\frac{z_0-\alpha'}{(cz_0+d)\epsilon'_\Delta} - \mathrm{Log}(z_0-\alpha) - \mathrm{Log}(z_0-\alpha')\\
=\ &-2\mathrm{Log}(cz_0+d).
\end{aligned}$$

The second statement can be deduced from the proof of Proposition 1 together with the first statement (cf. [32, (17), p. 87]). $\square$

**In terms of Rademacher's Φ defined in [26, (71.21), p. 150] for $M \in SL_2(\mathbf{Z})$, one has $n_M = -3\mathrm{sign}(c) + \Phi(M)$ if $c \ne 0$, and $n_M = -3(1-\mathrm{sign}(d)) + \Phi(M)$ if $c = 0$. In fact, this follws from (4) and [26, (71.22), p. 151]. The value n_M is independent of z, and is an integer since $\Phi(M) \in \mathbf{Z}$ by [26, Corollary, p. 155].

Using the reciprocity law of the Dedekind sum, we can evaluate n_M (cf. [32, Lemme, p. 90]). For integers c, d with $(c,d) = 1$, $c > 0$, the Dedekind sum is defined by (cf. [26, (68.3), p. 146])

$$s(d,c) := \sum_{k=1}^{c} \big((k/c)\big)\big((kd/c)\big). \tag{5}$$

Here $((x)) := x - \lfloor x \rfloor - 1/2$ if $x \notin \mathbf{Z}$, and $((x)) := 0$ if $x \in \mathbf{Z}$. When $(c,d) = 1$, $c > 0$, $d > 0$, this satisfies (cf. [26, (69.6), p. 148])

$$s(d,c) + s(c,d) = \frac{c^2 + d^2 + 1}{12cd} - \frac{1}{4}. \tag{6}$$

Moreover $s(-d,c) = -s(d,c)$, and $s(d,c) = s(a,c)$ if $ad \equiv 1 \pmod{c}$ by [26, (68.4)(68.5), p. 146]. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, one has $n_{-M} = n_M + 6\mathrm{sign}(c)$ if $c \neq 0$, and $n_{-M} = n_M - 6\mathrm{sign}(d)$ if $c = 0$ by (4) (cf. [32, (4), p. 82]).

Lemma 7. *[R. Dedekind] If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ satisfies $c \neq 0$, then one has*

$$n_M = \frac{a+d}{c} - \mathrm{sign}(c)\left(3 + 12s(d,|c|)\right).$$

Proof. If $c > 0$, it is given in [32, Theoreme, p. 83] or [26, (67.6), p. 145]. If $c < 0$, it follows from $n_{-M} = n_M + 6\mathrm{sign}(c)$ that $n_M = n_{-M} + 6 = \frac{-a-d}{-c} - (3 + 12s(-d,-c)) + 6 = \frac{a+d}{c} + (3 + 12s(d,|c|))$. This lemma can also be deduced from [26, (71.22), p. 151] and [26, (71.21), p. 150]. $\square$

Lemma 8. *Suppose that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ satisfies $ac \neq 0$. Put $M' = \begin{pmatrix} u & -1 \\ 1 & 0 \end{pmatrix} M \in SL_2(\mathbf{Z})$. Then one has*

$$n_{M'} = n_M + u - 3\mathrm{sign}(a) - 3\mathrm{sign}(ac) + 3\mathrm{sign}(c).$$

In other words,

$$n_{M'} = \begin{cases} u + 9 + n_M & \text{if } a < 0 \text{ and } c > 0, \\ u - 3 + n_M & \text{if } a > 0 \text{ or } c < 0. \end{cases}$$

Proof. Applying Lemma 7 to $M' = \begin{pmatrix} ua - c & ub - d \\ a & b \end{pmatrix}$, one has

$$n_{M'} = \frac{ua - c + b}{a} - \mathrm{sign}(a)\left(3 + 12s(b,|a|)\right).$$

Since $ad - bc = 1$, we have $-bc \equiv 1 \pmod{|a|}$ and $s(b,|a|) = s(-c,|a|) = -\mathrm{sign}(c)s(|c|,|a|)$. By (6),

$$s(|c|,|a|) = -s(|a|,|c|) + \frac{a^2 + c^2 + 1}{12|a||c|} - \frac{1}{4} = -\mathrm{sign}(a)s(d,|c|) + \frac{a^2 + c^2 + 1}{12|a||c|} - \frac{1}{4}.$$

Here, we used $ad \equiv 1 \pmod{|c|}$ and $s(|a|,|c|) = \mathrm{sign}(a)s(a,|c|) = \mathrm{sign}(a)s(d,|c|)$ at the second equal sign. Hence,

$$\begin{aligned} n_{M'} &= u + \frac{-c+b}{a} - \mathrm{sign}(a)\left(3 + 12\mathrm{sign}(a)\mathrm{sign}(c)s(d,|c|) - \frac{a^2+c^2+1}{|a|c} + 3\mathrm{sign}(c)\right) \\ &= u + \frac{-c+b}{a} - 3\mathrm{sign}(a) - 12\mathrm{sign}(c)s(d,|c|) + \frac{a^2+c^2+1}{ac} - 3\mathrm{sign}(a)\mathrm{sign}(c) \\ &= u + \frac{a+d}{c} - 3\mathrm{sign}(a) - 12\mathrm{sign}(c)s(d,|c|) - 3\mathrm{sign}(a)\mathrm{sign}(c). \end{aligned}$$

This yields the desired formula. $\square$

Lemma 9. *Let n be a natural number, and let $u_j \geq 1$, $j = 0, 1, 2, \cdots, 2n-1$ be integers. We define $M \in SL_2(\mathbf{Z})$ by*

$$M = (-1)^n \begin{pmatrix} \widetilde{u}_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{u}_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \widetilde{u}_{2n-1} & -1 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{u}_j := (-1)^j u_j.$$

Then one has

$$n_M = \sum_{j=0}^{2n-1} \widetilde{u}_j.$$

Proof. We prove this lemma by induction on n, that is, the half of the number of matrices defining M. Put $M_j := \begin{pmatrix} \widetilde{u}_j & -1 \\ 1 & 0 \end{pmatrix}$. Notice that the $(1,1)$ component of $-M_1$ is positive. So using Lemma 8 for $M_0(-M_1)$, and then using Lemma 7 for $-M_1$, we see that

$$n_{-M_0 M_1} = u_0 - 3 + n_{-M_1} = u_0 - 3 + (-u_1 + 3) = u_0 - u_1.$$

This confirms the case $n = 1$. Suppose that $n \geq 2$. Put

$$N_j := M_{2j} M_{2j+1} = \begin{pmatrix} \widetilde{u}_{2j} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{u}_{2j+1} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{u}_{2j}\widetilde{u}_{2j+1} - 1 & -\widetilde{u}_{2j} \\ \widetilde{u}_{2j+1} & -1 \end{pmatrix}.$$

All components of N_j are negative, those of $N_1 N_2 \cdots N_{n-1}$ have the same signature $(-1)^{n-1}$, and thus the $(1,1)$ component of $(-1)^n M_1 N_1 \cdots N_{n-1}$ is positive. Using $M = M_0 (-1)^n M_1 N_1 \cdots N_{n-1}$ together with Lemma 8, we get

$$n_{(-1)^n M_0 M_1 \cdots M_{2n-1}} = u_0 - 3 + n_{(-1)^n M_1 N_1 \cdots N_{n-1}} = u_0 - 3 + (-u_1 - 3) + n_{(-1)^n N_1 \cdots N_{n-1}}$$

Noticing that $n_{(-1)^n N_1 \cdots N_{n-1}} = n_{(-1)^{n-1} N_1 \cdots N_{n-1}} + 6$ as mentioned before Lemma 7, we conclude

$$n_{(-1)^n M_0 M_1 \cdots M_{2n-1}} = u_0 - u_1 + n_{(-1)^{n-1} N_1 \cdots N_{n-1}} = u_0 - u_1 + n_{(-1)^{n-1} M_2 M_3 \cdots M_{2n-1}}.$$

This completes the proof of the lemma by induction. □

Remark. Meyer's results [22] on the value $L_{\mathfrak{b}, U_\Delta}(1, \chi)$ are summarized in [18, (2.3), p. 421] and in [32, §III].

4 Proofs of Corollaries

We prepare the following facts.

Proposition 2. *Let d_1, d_2 be any fundamental discriminants such that $d_1 \neq d_2$. For any natural number f_0, put $\Delta = d_1 d_2 f_0^2$. Then,*
(1) $\chi_{d_1,d_2}^{(\Delta)}$ defined in §7 is a character of $\mathcal{C}_\Delta^+$ (a narrow class character).
(2) When $d_1 < 0$ and $d_2 < 0$, it satisfies $\chi_{d_1,d_2}^{(\Delta)}((\alpha)) = \mathrm{sign}\mathcal{N}(\alpha)$ for any $\alpha \in \mathbf{Q}(\sqrt{\Delta})^\times$.

Proof. By the definition of $\chi_{d_1,d_2}^{(\Delta)}$ and the group isomorphism Φ_Δ in §7 together with [8, Theorem 6.5.11, p. 231] and [8, Theorem 6.5.5, p. 227], we see that $\chi_{d_1,d_2}^{(\Delta)}$ is a character of $\mathcal{C}_\Delta^+$. This shows the statement (1). To prove (2), for $\alpha \in \mathbf{Q}(\sqrt{\Delta})$ with $\mathcal{N}(\alpha) > 0$, it is easy to see that $\Phi_\Delta^{-1}([\alpha\mathcal{O}_\Delta]^+) = [\![1, \sigma_\Delta, (\sigma_\Delta - \Delta)/4]\!]$. Hence, $\chi_{d_1,d_2}^{(\Delta)}((\alpha)) = \chi_{d_1}(1) = 1$. While, for $\alpha \in \mathbf{Q}(\sqrt{\Delta})$ with $\mathcal{N}(\alpha) < 0$, we have $\Phi_\Delta^{-1}([\alpha\mathcal{O}_\Delta]^+) = [\![(\Delta - \sigma_\Delta)/4, -\sigma_\Delta, -1]\!]$ and $\chi_{d_1,d_2}^{(\Delta)}((\alpha)) = \chi_{d_1}(-1) = -1$, since $d_1 < 0$. □

Lemma 10. *Let $\epsilon_d > 1$ be the fundamental unit of a real quadratic order $\mathcal{O}_d$.*
(1) For any quadratic discriminant $d > 0$, if $\mathcal{N}(\epsilon_d) = -1$, then d is not divisible by any prime p such that $p \equiv 3 \pmod 4$.
(2) Let d_1, d_2 be any negative fundamental discriminants such that $d_1 \neq d_2$. For any natural number f_0, put $\Delta = d_1 d_2 f_0^2$. Then we have $\mathcal{N}(\epsilon_\Delta) = 1$. Accordingly, the period length of the continued fraction expansion of any $\alpha \in \mathbb{X}_\Delta^0$ must be even [8, Theorem 2.2.9, p. 44].

Proof. The statement (1) is given in [8, Theorem 5.2.2, p. 124]. We prove the statement (2).
Case 1. Suppose that Δ is divisible by a prime $p \equiv 3 \pmod 4$. By (1), we deduce $\mathcal{N}(\epsilon_\Delta) = 1$.
Case 2. Suppose that Δ is not divisible by any prime $p \equiv 3 \pmod 4$. Since $d_i < 0$ and $p \equiv 1 \pmod 4$ is a positive prime discriminant, each d_i must be divisible by the negative prime discriminant -4 or -8. Hence $4^2 \mid \Delta$. In this case, $\mathcal{O}_\Delta = [1, \sqrt{\Delta}/2]$ ([8, p. 118]), and ϵ_Δ has the form $\epsilon_\Delta = x + y\sqrt{\Delta}/2$ for some integers x, y. Then $\mathcal{N}(\epsilon_\Delta) = x^2 - y^2\Delta/4 \equiv x^2 \equiv 0, 1 \pmod 4$. So $\mathcal{N}(\epsilon_\Delta) = -1$ is impossible. $\square$

The idea to prove Lemma 10 (2) in case $f_0 = 1$ is taken from [15, Proposition 5.1], but we simplify Kido's argument.

Lemma 11. *For any quadratic discriminant d, let $K := \mathbf{Q}(\sqrt{d})$ and $d = d_K f^2$. Here d_K is the discriminant of K and f is the conductor of d. Suppose that $f > 1$. Let $\mathcal{O}_{d_K}$ be the ring of integers of K, χ_{d_K} the Kronecker symbol of K, φ the Euler function, and*

$$\nu := (\mathcal{O}_{d_K}^\times : \mathcal{O}_d^\times), \qquad \psi(f) := f \prod_{\text{prime } q \mid f} (1 - \chi_{d_K}(q) q^{-1}).$$

Then, one has
(1) $h(d) = h(d_K)\psi(f)/\nu$.
(2) For $d > 0$, we have $\nu \mid \varphi(f)\psi(f)$.

Proof. The formula (1) can be found *e.g.* in [1, Theorem 6.12 (2), p. 90], [8, Theorem 5.9.7, p. 184]. The statement (2) should also be standard but for convenience we give a sketch of the proof. Let $\epsilon_d > 1$ (resp. $\epsilon_{d_K} > 1$) be the fundamental unit of $\mathcal{O}_d$ (resp. $\mathcal{O}_{d_K}$). [8, Theorem 5.2.3, p. 125] tells us that $\nu = \min\{n \in \mathbf{N} : \epsilon_{d_K}^n \in \mathcal{O}_d^\times\}$ and $\epsilon_d = \epsilon_{d_K}^\nu$. Since $\epsilon_{d_K} \in (\mathcal{O}_{d_K}/f\mathcal{O}_{d_K})^\times$, we have $\epsilon_{d_K}^{\Phi(f)} \equiv 1 \pmod{f\mathcal{O}_{d_K}}$, where $\Phi(f) := |(\mathcal{O}_{d_K}/f\mathcal{O}_{d_K})^\times|$. In view of $f\mathcal{O}_{d_K} \subset \mathcal{O}_d$ ([8, Theorem 5.1.7, p. 118]), we see $\epsilon_{d_K}^{\Phi(f)} \in \mathcal{O}_d \cap \mathcal{O}_{d_K}^\times = \mathcal{O}_d^\times$ (cf. [8, Theorem 5.2.3, p. 125]) and thus ν divides $\Phi(f)$. On the other hand, $\Phi(f) = \varphi(f)\psi(f)$ by [8, Proposition 5.9.4, p. 181]. $\square$

Lemma 12. *If $p \equiv 1 \pmod 4$ is a prime, then $x^2 - py^2 = -1$ has an integral solution (x, y). Accordingly, the fundamental unit $\epsilon_p > 1$ of $\mathcal{O}_p$ satisfies $\mathcal{N}(\epsilon_p) = -1$.*

Proof. See [8, Theorem 5.7.1, p. 160] or [24, Theorem 107, p. 203]. $\square$

We shall prove Corollaries 1, 2 and 3 using Theorem 3. Theorem 3 follows from Theorems 1 and 2, and Theorem 2 will be proved in the next section.

We recall here some facts needed below. Let d_1, d_2 be any negative fundamental discriminants such that $d_1 \neq d_2$. Let $\sigma_\Delta = 0$ or 1 according as $\Delta = d_1 d_2 f^2 > 0$ is even or odd. The basis number is defined by $\omega_\Delta := (\sigma_\Delta + \sqrt{\Delta})/2$. It has the continued fraction expansion of the form $\omega_\Delta = [u_0, \overline{u_1, u_2, u_3, \cdots, u_{2t}}] = [u_0, u_1, \cdots, u_{j-1}, \xi_j]$ when $\Delta > 5$ ([8, Theorem 2.3.5, p. 54]). Indeed, we see $1/(\omega_\Delta - \lfloor \omega_\Delta \rfloor) \in \mathbb{X}_\Delta^0$, which has a periodic continued fraction expansion by [8, Theorem 2.2.2, p. 39], and the period length must be even by Lemma 10 (2). Note that ξ_j is given by $\xi_j = [\overline{u_j, u_{j+1}, u_{j+2}, \cdots, u_{2t}, u_1, u_2, \cdots, u_{j-1}}]$ for $j = 1, 2, \cdots, 2t$, in particular $u_j = \lfloor \xi_j \rfloor$. [8, Theorem 1.3.5, p. 17] (or [8, Theorem 2.2.2, p. 39]) tells us that $\xi_j \in \mathbb{X}_\Delta^0$ and the type (a_j, b_j, c_j)

of ξ_j satisfies $a_j > 0$. The lattice $I(\xi_j) = [a_j, a_j\xi_j]$ is an $\mathcal{O}_\Delta$-regular ideal by [8, Theorem 5.4.5, p. 136].

By [8, Theorem 2.2.2, p. 39], we see that $\xi_j \sim \omega_\Delta$, $\xi_j \sim_+ (-1)^j\omega_\Delta$ for every j. Moreover, $\omega_\Delta \nsim_+ -\omega_\Delta$ by [8, Theorem 1.3.8, p. 19], Lemma 10 (2) and [8, Theorem 2.2.9, p. 44]. We refer Appendix 1 for the symbols $\sim$, $\sim_+$. Therefore, by [8, Theorem 5.5.7, p. 144], we deduce that $[I(\xi_j)] = [I(\omega_\Delta)]$ for all j, $[I(\xi_j)]^+ = [I(\omega_\Delta)]^+$ for any even j, $[I(\omega_\Delta)]^+ \neq [I(\xi_k)]^+ = [\sqrt{\Delta}I(\omega_\Delta)]^+$ for any odd k. Let $g_\Delta = [1, \sigma_\Delta, (\sigma_\Delta - \Delta)/4]$ be the principal form (cf. [8, p. 193]), which satisfies $\Phi_\Delta^{-1}([I(\omega_\Delta)]^+) = [\![g_\Delta]\!]$. For any even j, one has $\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j)) = \chi_{d_1,d_2}^{(\Delta)}(I(\omega_\Delta)) = \chi_{d_1,d_2}^{(\Delta)}(g_\Delta) = 1$. On the other hand, for any odd j, one has $\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j)) = \chi_{d_1,d_2}^{(\Delta)}(\sqrt{\Delta}I(\omega_\Delta)) = \chi_{d_1,d_2}^{(\Delta)}((\sqrt{\Delta}))\chi_{d_1,d_2}^{(\Delta)}(I(\omega_\Delta)) = \mathrm{sign}(\mathcal{N}(\sqrt{\Delta}))\chi_{d_1,d_2}^{(\Delta)}(I(\omega_\Delta)) = -1$ by Proposition 2. Hence, $\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j)) = (-1)^j$ for any j.

Proof of Corollaries 2 and 3. To prove Corollary 2, let $d_1 = -3, d_2 = -3p, f_0 = 1$, thus $\Delta = 9p$. Since $p \equiv 1 \pmod{12}$, the fundamental discriminant d_K of $K = \mathbf{Q}(\sqrt{9p})$ is p, and the conductor of Δ is $f = 3$, $\varphi(3) = \psi(3) = 2$. Put $\nu_p = (\mathcal{O}_p^\times : \mathcal{O}_{9p}^\times)$. By Lemma 11 and the assumption $h(p) = 1$, one has $\nu_p \mid 4$ and $h(9p) = 2/\nu_p$, which is a natural number. While, by Lemmas 10 (2) and 12, we conclude $\nu_p > 1$, $\nu_p = 2$, and $h(9p) = 1$. Hence, all elements in $\mathbb{X}_{9p}^0$ are equivalent to $\omega_{9p} = (1 + 3\sqrt{p})/2$. With the above notation, we deduce that $\mathbb{X}_{9p}^0 = \{\xi_j; j = 1, 2, \cdots, 2t\}$ (cf. [8, Theorem 2.2.2, p. 39]) and that

$$\sum_{\alpha \in \mathbb{X}_{9p}^0} \chi_{-3,-3p}^{(9p)}(\alpha)\lfloor\alpha\rfloor = \sum_{j=1}^{2t} \chi_{-3,-3p}^{(9p)}(\xi_j)\lfloor\xi_j\rfloor = \sum_{j=1}^{2t}(-1)^j u_j,$$

where we put $\chi_{-3,-3p}^{(9p)}(\alpha) := \chi_{-3,-3p}^{(9p)}(I(\alpha))$ for $\alpha \in \mathbb{X}_{9p}^0$.

Corollary 2 follows from this combined with Theorem 3 and $h(-3) = 1$, $w(-3) = 6$, $w(-3p) = 2$. Corollary 3 is the case $d_1 = -3, d_2 = -3p, f_0 = 2$, and can be proved in the same manner. □

Proof of Corollary 1. Let $d_1 = -4, d_2 = -4p, f_0 = 1$, thus $\Delta = 16p$. Since $p \equiv 1 \pmod 4$, the fundamental discriminant d_K of $K = \mathbf{Q}(\sqrt{16p}) = \mathbf{Q}(\sqrt{4p})$ is p. Put $\nu_p' = (\mathcal{O}_p^\times : \mathcal{O}_{16p}^\times)$, $\nu_p = (\mathcal{O}_p^\times : \mathcal{O}_{4p}^\times)$. Lemma 11 implies $h(4p) = h(p)(2 - \chi_p(2))/\nu_p$ and $\nu_p \mid (2 - \chi_p(2))$, namely $h(p) \mid h(4p)$. Since we are assuming $h(4p) = 1$, we get $h(p) = 1$. Accordingly, we have $\nu_p = 1$ if $p \equiv 1 \pmod 8$, and $\nu_p = 3$ if $p \equiv 5 \pmod 8$. On the other hand, Lemma 11 implies $h(16p) = 2(2 - \chi_p(2))/\nu_p'$ and $\nu_p' \mid 4(2 - \chi_p(2))$. By Lemmas 10 (2) and 12, $\nu_p' > 1$ is even. It follows from $\mathcal{O}_{16p}^\times \subset \mathcal{O}_{4p}^\times \subset \mathcal{O}_p^\times$ that $\nu_p \mid \nu_p'$. These observations imply $h(16p) = 1$ as we will see below.
Case 1. When $p \equiv 1 \pmod 8$, since $h(16p) = 2/\nu_p'$ is a natural number, an even $\nu_p' > 1$ must equal to 2 and $h(16p) = 1$ as desired.
Case 2. When $p \equiv 5 \pmod 8$, $\nu_p' \mid 12$ and ν_p' is divisible by $\nu_p = 3$, so $\nu_p' \in \{3, 6, 12\}$. Since ν_p' must be even and $h(16p) = 6/\nu_p'$ is a natural number, we conclude $\nu_p' = 6$ and $h(16p) = 1$.

By the same reasoning as the proof of Corollary 2, Corollary 1 follows from $h(16p) = 1$, Theorem 3 together with $\omega_{16p} = 2\sqrt{p}$ and the known values $h(-4) = 1$, $w(-4) = 4$, $w(-4p) = 2$. □

Remarks. 1. When $p \equiv 1 \pmod 8$, $h(4p) = 1$ is equivalent to $h(p) = 1$. Indeed, Lemma 11 tells us that $h(4p) = h(p)(2 - \chi_p(2))/(\mathcal{O}_p^\times : \mathcal{O}_{4p}^\times)$ and that $(\mathcal{O}_p^\times : \mathcal{O}_{4p}^\times) \mid (2 - \chi_p(2))$. Since $\chi_p(2) = 1$, we deduce that $(\mathcal{O}_p^\times : \mathcal{O}_{4p}^\times) = 1$, $h(4p) = h(p)$. In particular, we have $h(4 \cdot 73) = h(73)$, which equals to 1. See the table in [8, pp. 22–23].

2. By $(1 + \sqrt{53})/2 = [4, \overline{7}]$, we find $[\overline{7}] = (7 + \sqrt{53})/2 \in \mathbb{X}_{53}^0$ ([8, Theorem 2.2.2, p. 39]) and $\epsilon_{53} = (7 + \sqrt{53})/2$ ([8, Theorem 2.2.9, p. 44]). Using $\nu_{53} := (\mathcal{O}_{53}^\times : \mathcal{O}_{4\cdot 53}^\times) \mid 3$ and $\epsilon_{53} \notin \mathcal{O}_{4\cdot 53}^\times$, we see $\nu_{53} = 3$, and $h(4 \cdot 53) = h(53)$ (cf. Lemma 11), which equals to 1. See the table in [8, pp. 22–23]. Note that the value $\nu_{53} = 3$ can be obtained from [8, Theorem 5.2.3, p. 125].

5 Proof of Theorem 2

Suppose that Δ is of the form $\Delta = d_1 d_2 f_0^2$ as in Theorem 2. In view of the bijection ι_Δ (cf. §7 Appendix 1), it holds that $\mathcal{C}_\Delta = \{[I(\xi)]; [\xi]_\sim \in \mathfrak{X}_\Delta\}$, where we take ξ with $\xi \in \mathbb{X}_\Delta^0$ [8, Theorem 5.5.7, p. 144]. Let (a, b, c) be the type of ξ. The $\mathcal{O}_\Delta$-regular ideal $\mathfrak{b} := I(\xi) = [a\xi, a]$ has the basis $\beta_1 = a\xi$, $\beta_2 = a$ satisfying $(\beta_1\beta_2' - \beta_1'\beta_2)/\mathcal{N}(\beta_2) = \xi - \xi' > 0$.

Let $\xi_0 := \xi$ and $\xi = [\overline{u_0, u_1, u_2, \cdots, u_{2t-1}}] = [u_0, u_1, \cdots, u_{j-1}, \xi_j]$ be the continued fraction expansion, where $2t$ is the period length (cf. Lemma 10 (2), [8, Theorem 2.2.2, p. 39]). [8, Theorem 1.3.5, p. 17] tells us that $\xi_j \in \mathbb{X}_\Delta^0$ and that the type (a_j, b_j, c_j) of ξ_j satisfies $a_j > 0$. One has $\xi_j \sim \xi$, $\xi_j \sim_+ (-1)^j\xi$ for every j by [8, Theorem 2.2.2, p. 39], and $\xi \nsim_+ -\xi$ by [8, Theorem 1.3.8, p. 19], Lemma 10 (2) and [8, Theorem 2.2.9, p. 44]. Hence by [8, Theorem 5.5.7, p. 144], we see that $[I(\xi_j)] = [I(\xi)]$ for all j, $[I(\xi_j)]^+ = [I(\xi)]^+$ for any even j, and $[I(\xi)]^+ \neq [I(\xi_k)]^+ = [I(\xi_1)]^+ = [\sqrt{\Delta} I(\xi)]^+$ for any odd k. For $\mathfrak{b} = I(\xi)$, it follows from [8, Theorem 5.5.4, p. 142] that $[\mathfrak{b}] = [I(\xi)]^+ \cup [I(\xi_1)]^+$. Notice that $\chi_{d_1,d_2}^{(\Delta)}(I(\xi_1)) = -\chi_{d_1,d_2}^{(\Delta)}(I(\xi))$, as (cf. Proposition 2)

$$\chi_{d_1,d_2}^{(\Delta)}(I(\xi_1)) = \chi_{d_1,d_2}^{(\Delta)}((\sqrt{\Delta}))\chi_{d_1,d_2}^{(\Delta)}(I(\xi)) = \mathrm{sign}(\mathcal{N}(\sqrt{\Delta}))\chi_{d_1,d_2}^{(\Delta)}(I(\xi)) = -\chi_{d_1,d_2}^{(\Delta)}(I(\xi)).$$

Hence, one has $\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j)) = (-1)^j\chi_{d_1,d_2}^{(\Delta)}(I(\xi))$ and $\chi_{d_1,d_2}^{(\Delta)}(I(\xi))\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j)) = (-1)^j$ for any j.

By the fourth displayed identity in [8, p. 45] (cf. [8, (1), p. 26]), the matrix $M_\mathfrak{b}$ defined in Proposition 1 for $\mathfrak{b} = I(\xi)$ with the basis $\beta_1 = a\xi$, $\beta_2 = a$ is given by

$$M_\mathfrak{b} = \begin{pmatrix} u_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} u_{2t-1} & 1 \\ 1 & 0 \end{pmatrix}.$$

Put $J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Using the identities

$$\begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} J = -\begin{pmatrix} u & -1 \\ 1 & 0 \end{pmatrix}, \quad J\begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -u & -1 \\ 1 & 0 \end{pmatrix}, \quad JJ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$M_\mathfrak{b} = (-1)^t\begin{pmatrix} \widetilde{u}_0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{u}_1 & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \widetilde{u}_{2t-1} & -1 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{u}_j := (-1)^j u_j.$$

This[††] together with Lemma 9 implies that

$$n_{M_\mathfrak{b}} = \sum_{j=0}^{2t-1}(-1)^j u_j = \chi_{d_1,d_2}^{(\Delta)}(I(\xi))\sum_{j=0}^{2t-1}\chi_{d_1,d_2}^{(\Delta)}(I(\xi_j))\lfloor \xi_j \rfloor.$$

By summing over $\mathcal{C}_\Delta = \{[I(\xi)]; [\xi]_\sim \in \mathfrak{X}_\Delta\}$, it follows from Proposition 1 and Lemma 6 together with Proposition 2 and Lemma 10 (2) that

$$L(0, \chi_{d_1,d_2}^{(\Delta)}) = 2\sum_{[\xi]_\sim \in \mathfrak{X}_\Delta}\chi_{d_1,d_2}^{(\Delta)}(I(\xi))\chi_{d_1,d_2}^{(\Delta)}((a))\frac{n_{M_{I(\xi)}}}{12} = \frac{1}{6}\sum_{\alpha \in \mathbb{X}_\Delta^0}\chi_{d_1,d_2}^{(\Delta)}(\alpha)\lfloor \alpha \rfloor,$$

where we put $\chi_{d_1,d_2}^{(\Delta)}(\alpha) := \chi_{d_1,d_2}^{(\Delta)}(I(\alpha))$ for $\alpha \in \mathbb{X}_\Delta^0$. Here we used the fact that the set $\mathbb{X}_\Delta^0$ consists of all ξ_j $(j = 0, 1, 2, \cdots, 2t-1)$ with $[\xi]_\sim$ running through $\mathfrak{X}_\Delta$, where ξ_j are defined as above from given $\xi \in \mathbb{X}_\Delta^0$ (cf. [8, Theorem 2.2.2, p. 39]). This completes the proof of Theorem 2.

[††] We owe this identity to Kido's exposition [15] of Zagier's works [31, 32].

6 A remark on Dirichlet series of the primary representation numbers by genera

In this section, we shall show that our Theorem 1 is equivalent to the formulas in [11, Theorem 10.1, p. 295] for $\Delta < 0$ and [23, Theorem 3, p. 52] for $\Delta > 0$. Let Δ be a quadratic discriminant. Let $\mathcal{G}_\Delta := \mathcal{C}_\Delta^+/\mathcal{C}_\Delta^{+2} \cong \mathfrak{F}_\Delta/\mathfrak{F}_\Delta^2$ be the genus group of discriminant Δ, and $\widehat{\mathcal{G}_\Delta} \cong \widehat{\mathfrak{F}_\Delta/\mathfrak{F}_\Delta^2}$ the group of genus characters. By [8, Theorem 6.5.2, p. 223], we have $|\mathcal{G}_\Delta| = |\widehat{\mathcal{G}_\Delta}| = 2^{\mu(\Delta)-1}$ with an explicit natural number $\mu(\Delta)$, which can be found in [8, p. 222]. Following [11, p. 277], we define

$$F(\Delta) := \{d_1; d_1 \text{ is a fundamental discriminant, } d_1 \mid \Delta, \text{ and } \Delta/d_1 \text{ is a discriminant}\}.$$

It is known that $F(\Delta)$ is a group under a suitable binary operation, $|F(\Delta)| = 2^{\mu(\Delta)}$, and that the set $\{\chi_{d_1,d_2}^{(\Delta)}; d_1 \in F(\Delta)\}$ covers $\widehat{\mathcal{G}_\Delta} \cong \widehat{\mathfrak{F}_\Delta/\mathfrak{F}_\Delta^2}$ exactly twice. We refer to [11, p. 279] for the discussion about the map $F(\Delta) \to \widehat{\mathfrak{F}_\Delta/\mathfrak{F}_\Delta^2} \cong \widehat{\mathcal{G}_\Delta}$ sending d_1 to $\chi_{d_1,d_2}^{(\Delta)}$, which turns out to be a group epimorphism with the kernel $\{1, d_K\}$, where d_K is the discriminant of $K = \mathbf{Q}(\sqrt{\Delta})$. In particular, $\chi_{d_1,d_2}^{(\Delta)} = \chi_{e_1,e_2}^{(\Delta)}$ if and only if $(d_1,d_2) \in \{(e_1,e_2),(e_2,e_1)\}$.

For any $\chi \in \widehat{\mathcal{G}_\Delta}$ and $s \in \mathbf{C}$ with $\Re(s) > 1$, we obtain

$$L(s,\chi) = \sum_{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta} \chi(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{a})^{-s} = \sum_{[\mathfrak{c}]^+ \in \mathcal{C}_\Delta^+} \chi(\mathfrak{c}) \sum_{\mathcal{O}_\Delta-\text{ideal } \mathfrak{a} \in [\mathfrak{c}]^+} \mathfrak{N}_\Delta(\mathfrak{a})^{-s}.$$

Fix representatives $\mathfrak{c}$ so that $\mathfrak{c} \subset \mathcal{O}_\Delta$, and take their conjugate ideals $\mathfrak{c}' \in ([\mathfrak{c}]^+)^{-1}$, which are $\mathcal{O}_\Delta$-invertible ideals of $\mathcal{O}_\Delta$ satisfying $\mathfrak{c}\mathfrak{c}' = (\mathfrak{N}_\Delta(\mathfrak{c}))$ ([8, Corollary 5.4.3, p. 136]). For two lattices $\mathfrak{a}$, $\mathfrak{b}$ in $K = \mathbf{Q}(\sqrt{\Delta})$, let $(\mathfrak{a} :_K \mathfrak{b}) := \{\alpha \in K; \alpha\mathfrak{b} \subset \mathfrak{a}\}$ ([8, p. 116]), and let $\mathcal{O}_\Delta^{\times+} := \{\epsilon \in \mathcal{O}_\Delta^\times; \mathcal{N}(\epsilon) > 0\}$. Using [8, Lemma 5.3.4, p. 131] and the argument in §11 [33], the map $\{\beta \in \mathfrak{c}'; \mathcal{N}(\beta) > 0\}/\mathcal{O}_\Delta^{\times+} \stackrel{\phi}{\to} \{\mathfrak{a} \in [\mathfrak{c}]^+; \mathfrak{a} \subset \mathcal{O}_\Delta\}$ given by $\phi(\beta) := \beta(\mathcal{O}_\Delta :_K \mathfrak{c}')$ is well-defined and ϕ is a bijection. Note that if $\mathfrak{a} = \phi(\beta)$, then $\mathfrak{a}\mathfrak{c}' = (\beta)$ and thus $\mathcal{N}(\beta) = \mathfrak{N}_\Delta(\mathfrak{c}')\mathfrak{N}_\Delta(\mathfrak{a})$ (cf. [8, Theorem 5.1.3, p. 116], [8, Theorem 5.4.6, p. 137]). Hence we deduce

$$\sum_{\mathcal{O}_\Delta-\text{ideal } \mathfrak{a} \in [\mathfrak{c}]^+} \mathfrak{N}_\Delta(\mathfrak{a})^{-s} = \mathfrak{N}_\Delta(\mathfrak{c}')^s L^+_{\mathfrak{c}',\mathcal{O}_\Delta^{\times+}}(s),$$

where, for any $\mathcal{O}_\Delta$-invertible fractional ideal $\mathfrak{b}$, we put

$$L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) := \sum_{\beta \in \mathfrak{b}/\mathcal{O}_\Delta^{\times+},\ \mathcal{N}(\beta)>0} \mathcal{N}(\beta)^{-s}.$$

Putting $\mathfrak{c}' = \mathfrak{b}$, we obtain

$$L(s,\chi) = \sum_{[\mathfrak{b}]^+ \in \mathcal{C}_\Delta^+} \overline{\chi}(\mathfrak{b})\mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) = \sum_{G \in \mathcal{G}_\Delta} \overline{\chi}(G) \sum_{[\mathfrak{b}]^+ \in G} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s). \tag{7}$$

In the innermost sum, each term is independent of the choice of a representative of class in G. Here, we extend $\mathfrak{N}_\Delta(\mathfrak{b}) := n^{-2}\mathfrak{N}_\Delta(n\mathfrak{b})$, where n is any natural number such that $n\mathfrak{b}$ is an ideal of $\mathcal{O}_\Delta$. For any fixed genus $G_0 \in \mathcal{G}_\Delta$, by the orthogonality relation, one has

$$\begin{aligned}
\sum_{\chi_{d_1,d_2}^{(\Delta)} \in \widehat{\mathcal{G}_\Delta}} \chi_{d_1,d_2}^{(\Delta)}(G_0) L(s,\chi_{d_1,d_2}^{(\Delta)}) &= \sum_{\chi_{d_1,d_2}^{(\Delta)} \in \widehat{\mathcal{G}_\Delta}} \sum_{G \in \mathcal{G}_\Delta} \chi_{d_1,d_2}^{(\Delta)}(G_0)\chi_{d_1,d_2}^{(\Delta)}(G) \sum_{[\mathfrak{b}]^+ \in G} \mathfrak{N}_\Delta(\mathfrak{b})^s L_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) \\
&= \sum_{G \in \mathcal{G}_\Delta} \left(\sum_{\chi_{d_1,d_2}^{(\Delta)} \in \widehat{\mathcal{G}_\Delta}} \chi_{d_1,d_2}^{(\Delta)}(G_0G) \right) \sum_{[\mathfrak{b}]^+ \in G} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) \\
&= 2^{\mu(\Delta)-1} \sum_{[\mathfrak{b}]^+ \in G_0} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s).
\end{aligned} \tag{8}$$

The sum in the right-hand side of (8) is the Dirichlet series of the number of primary representation by the genus G_0 studied in [11, Theorem 10.1, p. 295] for $\Delta < 0$ and [23, Theorem 3, p. 52] for $\Delta > 0$. In terms of our notations, their results can be stated as

$$2^{\mu(\Delta)} \sum_{[\mathfrak{b}]^+ \in G_0} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times +}}(s) = \sum_{m|f} m^{1-2s} \prod_{\text{prime } p|m,\ p\nmid(f/m)} \left(1-\chi_{d_K}(p)p^{-1}\right)$$
$$\times \sum_{d_1 \in F(\Delta/m^2)} \chi^{(\Delta)}_{d_1,d_2}(G_0) L(s,\chi_{d_1}) L(s,\chi_{d_2}) \prod_{\text{prime } p|(f/m)} (1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}). \quad (9)$$

Here we write $\Delta = d_K f^2$, where d_K is the discriminant of $K = \mathbf{Q}(\sqrt{\Delta})$, f is the conductor of Δ, and for $d_1 \in F(\Delta/m^2) \subset F(\Delta)$ (cf. [23, Lemma 1 (b), p. 29]), we let d_2 be the discriminant of $\mathbf{Q}(\sqrt{\Delta/d_1}) = \mathbf{Q}(\sqrt{\Delta/(m^2 d_1)})$. The empty products here and in the followings are understood as being 1.

To get (9) for $\Delta < 0$, we used [1, Theorem 6.12 (2), p. 90] or [8, Theorem 5.9.7, p. 184], and we multiplied $\frac{1}{2}$ instead of $d_1 > 0$ posed in [11, Theorem 10.1, p. 295].

Notice that $\prod_{\text{prime } p|f,\ p\nmid(f/m)} \left(1-\chi_{d_K}(p)p^{-1}\right) = \prod_{\text{prime } p|m,\ p\nmid(f/m)} \left(1-\chi_{d_K}(p)p^{-1}\right)$. While, to get (9) for $\Delta > 0$, we used [23, (96), p. 56].

In (9), let f_0 be a natural number defined by $\Delta/d_1 = d_2 f_0^2$. Here, d_2 and f_0 depend only on d_1, since Δ is fixed. By [23, Lemma 1 (e), p. 29], for any natural number m and any fundamental discriminant d_1, the condition "$m \mid f$ and $d_1 \in F(\Delta/m^2)$" is equivalent to "$d_1 \in F(\Delta)$ and $m \mid f_0$". Hence (9) can be written as

$$2^{\mu(\Delta)} \sum_{[\mathfrak{b}]^+ \in G_0} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times +}}(s) = \sum_{d_1 \in F(\Delta)} \chi^{(\Delta)}_{d_1,d_2}(G_0) L(s,\chi_{d_1}) L(s,\chi_{d_2}) \sum_{m|f_0} m^{1-2s}$$
$$\times \prod_{\text{prime } p|m,\ p\nmid(f/m)} \left(1-\chi_{d_K}(p)p^{-1}\right) \prod_{\text{prime } p|(f/m)} (1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}). \quad (10)$$

If $f_0 > 1$, let m_p be a positive integer such that p^{m_p} is the highest power of p dividing f_0 as in Theorem 1. For any $m \mid f_0$ with $m = \prod_{p|f_0} p^{r_p}$, we put

$$h_p(r_p,s) := \begin{cases} 1-\chi_{d_1}(p)\chi_{d_2}(p)p^{-1}, & r_p = m_p, \\ (1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}), & 0 \le r_p < m_p. \end{cases}$$

It is elementary to see that the right-hand side of (10) is given by

$$\sum_{d_1 \in F(\Delta)} \chi^{(\Delta)}_{d_1,d_2}(G_0) L(s,\chi_{d_1}) L(s,\chi_{d_2}) \prod_{p|f_0} \sum_{r_p=0}^{m_p} p^{-(2s-1)r_p} h_p(r_p,s)$$

with the convention of the empty product mentioned after (9). Indeed, using the same notations as above, we can write $d_1 d_2 = d_K f_1^2$ with $f_1 \in \mathbf{N}$, so that $f = f_0 f_1$. Noticing that $p \mid f_1$ is equivalent to "$p \mid d_1$ and $p \mid d_2$" for any prime p, we deduce

$$\prod_{\text{prime } p|(f/m)} (1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}) = \prod_{\text{prime } p|f_0} g_p(r_p,s),$$

$$g_p(r_p,s) := \begin{cases} 1, & r_p = m_p, \\ (1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}), & 0 \le r_p < m_p. \end{cases}$$

Similarly, noticing that $\chi_{d_1}(p)\chi_{d_2}(p)$ is 0 or $\chi_{d_K}(p)$ according as $p \mid f_1$ or $p \nmid f_1$, we have

$$\prod_{\text{prime } p|m,\ p\nmid(f/m)} \left(1-\chi_{d_K}(p)p^{-1}\right) = \prod_{\text{prime } p|f_0} k_p(r_p),$$

$$k_p(r_p) := \begin{cases} 1-\chi_{d_1}(p)\chi_{d_2}(p)p^{-1}, & r_p = m_p, \\ 1, & 0 \le r_p < m_p. \end{cases}$$

Defining $h_p(r_p,s) = g_p(r_p,s)k_p(r_p)$, we get the desired expression as stated.

Using $\epsilon_p(s,\chi^{(\Delta)}_{d_1,d_2})$ defined in (1), we obtain $\sum_{r_p=0}^{m_p} p^{-(2s-1)r_p} h_p(r_p,s) = \epsilon_p(s,\chi^{(\Delta)}_{d_1,d_2})$ and conclude that the formula (9) has the form

$$\begin{aligned} & 2^{\mu(\Delta)} \sum_{[\mathfrak{b}]^+ \in G_0} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) \\ = \; & 2 \sum_{\chi^{(\Delta)}_{d_1,d_2} \in \widehat{\mathcal{G}_\Delta}} \chi^{(\Delta)}_{d_1,d_2}(G_0) L(s,\chi_{d_1}) L(s,\chi_{d_2}) \prod_{\text{prime } p|f_0} \epsilon_p(s,\chi^{(\Delta)}_{d_1,d_2}). \end{aligned} \tag{11}$$

In view of (8), the formula (11) is a consequence of our Theorem 1, and we get a new proof of (9). Conversely, (11) gives another proof of Theorem 1. In fact, using (7) for given $\chi^{(\Delta)}_{e_1,e_2} \in \widehat{\mathcal{G}_\Delta}$, then using (11) and the orthogonality relation, we deduce that

$$\begin{aligned} L(s,\chi^{(\Delta)}_{e_1,e_2}) &= \sum_{G_0 \in \mathcal{G}_\Delta} \chi^{(\Delta)}_{e_1,e_2}(G_0) \sum_{[\mathfrak{b}]^+ \in G_0} \mathfrak{N}_\Delta(\mathfrak{b})^s L^+_{\mathfrak{b},\mathcal{O}_\Delta^{\times+}}(s) \\ &= 2^{1-\mu(\Delta)} \sum_{\chi^{(\Delta)}_{d_1,d_2} \in \widehat{\mathcal{G}_\Delta}} \left(\sum_{G_0 \in \mathcal{G}_\Delta} \chi^{(\Delta)}_{e_1,e_2}(G_0) \chi^{(\Delta)}_{d_1,d_2}(G_0) \right) L(s,\chi_{d_1}) L(s,\chi_{d_2}) \prod_{\text{prime } p|f_0} \epsilon_p(s,\chi^{(\Delta)}_{d_1,d_2}) \\ &= L(s,\chi_{e_1}) L(s,\chi_{e_2}) \prod_{\text{prime } p|f_0} \epsilon_p(s,\chi^{(\Delta)}_{e_1,e_2}). \end{aligned} \tag{12}$$

This is the statement of Theorem 1.

We conclude that our Theorem 1 is equivalent to (9) obtained in [11, 23].

7 Appendix 1: quadratic irationals

7.1 Correspondences

A discriminant d (that is, d is an integer with $d \neq 0$, $d \equiv 0,1 \pmod 4$) is called fundamental if it is a discriminant of a quadratic field or 1, and discriminants $(-1)^{(p-1)/2}p$ (p odd primes) together with -4, 8, -8 are called prime discriminants. It is known that d is a fundamental discriminant if and only if d is a product of mutually coprime prime discriminants, except for $d = 1$. An integer Δ is called a quadratic discriminant if $\Delta \equiv 0,1 \pmod 4$, $\Delta \neq \square$. For any quadratic discriminant Δ, we have the unique decomposition $\Delta = d_K f^2$ with d_K the discriminant of the field $K = \mathbf{Q}(\sqrt{\Delta})$ and $f \in \mathbf{N}$, which is called by the conductor of Δ (cf. [8, Theorem 1.1.6, p. 4], [8, Theorem 1.1.9, p. 6]). Let $\sigma_\Delta = 0$ or 1 according as Δ is even or odd. Put $\sqrt{\Delta} := i^{(1-\mathrm{sign}(\Delta))/2}\sqrt{|\Delta|}$ and $\omega_\Delta := (\sigma_\Delta + \sqrt{\Delta})/2$. Let $\mathcal{O}_\Delta := [1,\omega_\Delta]$ be the associated order ([8, Definition 5.1.6, p. 118]). For any non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$, we put $\mathfrak{N}_\Delta(\mathfrak{a}) = (\mathcal{O}_\Delta : \mathfrak{a})$ the absolute norm of $\mathfrak{a}$. By definition, $\mathfrak{a} \subset K = \mathbf{Q}(\sqrt{\Delta})$ is a fractional $\mathcal{O}_\Delta$-ideal if and only if $\mathfrak{a}$ is a lattice in K such that $\mathcal{O}_\Delta\mathfrak{a} \subset \mathfrak{a}$. A fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{a}$ is $\mathcal{O}_\Delta$-invertible if and only if there exists a fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{a}_1$ such that $\mathfrak{a}\mathfrak{a}_1 = \mathcal{O}_\Delta$ ([8, Definition 5.3.3, p. 130]). For any fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{a}$, we extend $\mathfrak{N}_\Delta(\mathfrak{a}) := n^{-2}\mathfrak{N}_\Delta(n\mathfrak{a})$, where n is any natural number such that $n\mathfrak{a}$ is an ideal of $\mathcal{O}_\Delta$. We can take such an n by [8, Lemma 5.3.4, p. 130], and this definition is independent of the choice of n. For lattices $\mathfrak{a}$, $\mathfrak{b}$ in $K = \mathbf{Q}(\sqrt{\Delta})$, let $(\mathfrak{a} :_K \mathfrak{b}) := \{\alpha \in K; \alpha\mathfrak{b} \subset \mathfrak{a}\}$ ([8, p. 116]). For any fractional $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\lambda \in K^\times$, it is not difficult to check the following relations; $\mathfrak{N}_\Delta(\mathfrak{a}\mathfrak{b}) = \mathfrak{N}_\Delta(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{b})$, $\mathfrak{a}\mathfrak{a}' = (\mathfrak{N}_\Delta(\mathfrak{a}))$, $\mathfrak{N}_\Delta((\lambda)) = |\mathcal{N}(\lambda)|$, and $(\mathfrak{a} :_K \mathfrak{b}) = \mathfrak{a}(\mathcal{O}_\Delta :_K \mathfrak{b})$, $\mathfrak{N}_\Delta((\mathfrak{a} :_K \mathfrak{b})) = \mathfrak{N}_\Delta(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{b})^{-1}$.

Two fractional $\mathcal{O}_\Delta$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent, denoted by $\mathfrak{a} \sim \mathfrak{b}$, if and only if there exists $\lambda \in \mathbf{Q}(\sqrt{\Delta})^\times$ such that $\mathfrak{a} = \lambda\mathfrak{b}$. We denote by $[\mathfrak{a}]$ the equivalence class of $\mathfrak{a}$, and by $\mathcal{C}_\Delta$ the set of all equivalence classes of $\mathcal{O}_\Delta$-invertible fractional $\mathcal{O}_\Delta$-ideals. Two fractional $\mathcal{O}_\Delta$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are properly equivalent (in other words, equivalent in the narrow sense), denoted by $\mathfrak{a} \sim_+ \mathfrak{b}$, if and only if there exists $\lambda \in \mathbf{Q}(\sqrt{\Delta})^\times$ with $\mathcal{N}(\lambda) = \lambda\lambda' > 0$ such that $\mathfrak{a} = \lambda\mathfrak{b}$. We denote by $[\mathfrak{a}]^+$ the proper equivalence class of $\mathfrak{a}$, and by $\mathcal{C}_\Delta^+$ the set of all proper equivalence classes of $\mathcal{O}_\Delta$-invertible fractional $\mathcal{O}_\Delta$-ideals ([8, pp. 140–141]).

Let $\mathbb{X}_\Delta := \{\xi = (b + \sqrt{\Delta})/(2a); a \neq 0,\ b, c \in \mathbf{Z}, (a, b, c) = 1, b^2 - 4ac = \Delta\}$ be the set of all quadratic irrationals of discriminant Δ. A quadratic irrational $\xi = (b + \sqrt{\Delta})/(2a) \in \mathbb{X}_\Delta$ is called of type (a, b, c) and discriminant Δ. Put (whenever it is meaninful)

$$M\langle z\rangle := \frac{\alpha z + \beta}{\gamma z + \delta} \quad \text{for} \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Two numbers $z_1,\ z_2 \in \mathbb{X}_\Delta$ are equivalent $(z_1 \sim z_2)$ if and only if there exists $M \in GL_2(\mathbf{Z})$ such that $z_1 = M\langle z_2\rangle$. Denote by $[\xi]_\sim$ the equivalence class of ξ, and by $\mathfrak{X}_\Delta$ the set of all equivalence classes of $\mathbb{X}_\Delta$. Two numbers $z_1,\ z_2 \in \mathbb{X}_\Delta$ are properly equivalent $(z_1 \sim_+ z_2)$ if and only if there exists $M \in SL_2(\mathbf{Z})$ such that $z_1 = M\langle z_2\rangle$. Denote by $[\xi]_{\sim_+}$ the proper equivalence class of ξ, and by $\mathfrak{X}_\Delta^+$ the set of all proper equivalence classes of $\mathbb{X}_\Delta$ ([8, p. 12]).

For $\Delta > 0$, let $\mathbb{X}_\Delta^0 := \{\xi \in \mathbb{X}_\Delta; -1 < \xi' < 0, 1 < \xi\}$ be the set of all reduced irrationals ([8, Definition 1.3.1, p. 16]). For any $\xi \in \mathbb{X}_\Delta$, there exists $\eta \in \mathbb{X}_\Delta^0$ such that $\xi \sim_+ \eta$ ([8, Theorem 1.3.5, p. 17]). For $\xi = (b + \sqrt{\Delta})/(2a) \in \mathbb{X}_\Delta$, the lattice $I(\xi) := [a, a\xi]$ is an $\mathcal{O}_\Delta$-regular ideal and $\mathfrak{N}_\Delta(I(\xi)) = |a|$ ([8, Definition 5.4.1, p. 132], [8, Theorem 5.4.5, p. 136]). By definition, an ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$ is called reduced if $\mathfrak{a} = I(\xi)$ for some $\xi \in \mathbb{X}_\Delta^0$ ([8, p. 143]). Similar statements hold in the case of $\Delta < 0$.

We denote by $f = [a, b, c]$ the quadratic form $f(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbf{Z}$. For such an f, the action $f \mapsto Af$ of $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbf{Z})$ is defined by $(Af)(x, y) := f(\alpha x + \gamma y, \beta x + \delta y)$ ([8, Definition 6.1.2, p. 194]). For any quadratic discriminant Δ, let $\mathfrak{F}_\Delta$ be the set of all equivalence classes with respect to the action $f \mapsto Af$ of $A \in SL_2(\mathbf{Z})$ on "not negative-definite" primitive forms of discriminant Δ ([8, p. 197]). We denote by $[\![f]\!]$ (or $[\![a, b, c]\!]$) the equivalent class of $f = [a, b, c]$. Let $\xi_f := (b + \sqrt{\Delta})/(2a)$ for $f = [a, b, c]$ of discriminant Δ $(\neq \square)$. This ξ_f is a solution of $f(X, -1) = 0$. Hence, when $\Delta > 0$, one has $\xi_f' < \xi_f$ if $a > 0$, while $\xi_f < \xi_f'$ if $a < 0$. When $\Delta < 0$, one has $\Im(\xi_f) > 0$ if f is positive-definite.

Fact 1. For $\Delta > 0$, the map $\vartheta_\Delta : \mathfrak{F}_\Delta \to \mathfrak{X}_\Delta^+$ defined by $\vartheta_\Delta([\![f]\!]) := [\xi_f]_{\sim_+}$ is bijective. For $\Delta < 0$, the map $\vartheta_\Delta : \mathfrak{F}_\Delta \to \mathfrak{X}_\Delta$ defined by $\vartheta_\Delta([\![f]\!]) := [\xi_f]_\sim$ is bijective. See [8, Theorem 6.1.7, p. 197].

Fact 2. For any Δ, the map $\iota_\Delta : \mathfrak{X}_\Delta \to \mathcal{C}_\Delta$ defined by $\iota_\Delta([\xi]_\sim) := [I(\xi)]$ is bijective. For $\Delta > 0$, the map $\iota_\Delta : \mathfrak{X}_\Delta^+ \to \mathcal{C}_\Delta^+$ defined by $\iota_\Delta([\xi]_{\sim_+}) := [I(\xi)]^+$, where ξ is chosen to be of type (a, b, c) with $a > 0$, is bijective. See [8, Theorem 5.5.8, p. 146].

Fact 3. The map $\Phi_\Delta : \mathfrak{F}_\Delta \to \mathcal{C}_\Delta^+$ defined by $\Phi_\Delta([\![f]\!]) := [I(\xi_f)\sqrt{\Delta}^{(1-\mathrm{sign}(a))/2}]^+$ is bijective. Here $f = [a, b, c]$. See [8, Theorem 6.4.2, p. 214].

The set $\mathfrak{F}_\Delta$ has a group structure compatible with that of $\mathcal{C}_\Delta^+$. The product of $F_1,\ F_2 \in \mathfrak{F}_\Delta$ is given by $F_1 * F_2 := \Phi_\Delta^{-1}(\Phi_\Delta(F_1)\Phi_\Delta(F_2))$. Hence Φ_Δ becomes a group isomorphism.

Following [8, Theorem 1.3.10, p. 20], we define $h(\Delta) := |\mathfrak{X}_\Delta|$, $h^+(\Delta) := \begin{cases} |\mathfrak{X}_\Delta| & \text{if } \Delta < 0, \\ |\mathfrak{X}_\Delta^+| & \text{if } \Delta > 0. \end{cases}$

Hence,

$$h(\Delta) = |\mathcal{C}_\Delta|, \quad h^+(\Delta) = |\mathfrak{F}_\Delta| = |\mathcal{C}_\Delta^+|.$$

Remark 1. For $\Delta < 0$ and $\xi = (b + \sqrt{\Delta})/(2a) \in \mathbb{X}_\Delta$, we have $\vartheta_\Delta^{-1}([\xi]_\sim) = [\![|a|, b, (b^2 - \Delta)/(4|a|)]\!]$.

Remark 2. In general, for $\Delta > 0$, the map ι_Δ has the form $\iota_\Delta([\xi]_{\sim_+}) := [I(\xi)\sqrt{\Delta}^{(1-\mathrm{sign}(a))/2}]^+$, where ξ is of type (a, b, c) and discriminant Δ.

Remark 3. For any fractional $\mathcal{O}_\Delta$-invertible ideal $\mathfrak{a}$, take an integral basis $\mathfrak{a} = [\beta_1, \beta_2]$ satisfying $(\beta_1\beta_2' - \beta_1'\beta_2)/\sqrt{\Delta} < 0$ (an oriented basis), where β' is the conjugate of β. We can associate a "not negative-definite" primitive form of discriminant Δ corresponding to this basis by $f_{\beta_1,\beta_2}(x, y) := \mathfrak{N}_\Delta(\mathfrak{a})^{-1}(\beta_1 x + \beta_2 y)(\beta_1' x + \beta_2' y)$. The equivalence class $[\![f_{\beta_1,\beta_2}]\!]$ is determined uniquely from $\mathfrak{a}$ (more precisely, from $[\mathfrak{a}]^+$), and is independent of the typical choice of the oriented basis. Moreover, $\Phi_\Delta^{-1}([\mathfrak{a}]^+) = [\![f_{\beta_1,\beta_2}]\!]$ (cf. [33, §10]). We see that $\iota_\Delta = \Phi_\Delta \circ \vartheta_\Delta^{-1}$ and that $\iota_\Delta^{-1}([[\beta_1, \beta_2]]^+) = [\beta_2/\beta_1]_{\sim_+}$ if $\Delta > 0$, while $\iota_\Delta^{-1}([[\beta_1, \beta_2]]) = [\beta_2/\beta_1]_\sim$ if $\Delta < 0$ using any oriented basis β_1, β_2. Notice that $\xi_{f_{\beta_1,\beta_2}} = \beta_2/\beta_1$.

7.2 Genus characters

In this section, we recall the definition of genus characters for general discriminant. Let d_1 and d_2 be two fundamental discriminants, which are not necessarily relatively prime. Then $\Delta := d_1 d_2 f_0^2$ is a discriminant for any natural number f_0. Conversely, for any discriminant Δ, take any fundamental discriminant $d_1 \mid \Delta$ such that Δ/d_1 is a discriminant. Then a fundamental discriminant d_2 and a natural number f_0 are determined uniquely by the expression $\Delta/d_1 = d_2 f_0^2$. Note that Δ is a quadratic discriminant if and only if $d_1 \neq d_2$. When $\Delta = d_1 d_2 f_0^2$ is a quadratic discriminant, we associate the genus character $\chi_{d_1,d_2}^{(\Delta)}$ on the set of all "not negative-definite" primitive binary quadratic forms of discriminant Δ as follows (see [2, §3], [11, §2], and [8, §6.5] (in particular, p. 226, Theorem 6.5.11 in p. 231, and Theorem 6.5.5 in p. 227 of [8]));

$$\chi_{d_1,d_2}^{(\Delta)}([a, b, c]) := \prod_{\text{prime discriminant } q^* \mid d_1} \chi^{(q^*)}([a, b, c]),$$

$$\chi^{(q^*)}([a, b, c]) := \begin{cases} \chi_{q^*}(a) & \text{if } (a, q^*) = 1, \\ \chi_{q^*}(c) & \text{if } (c, q^*) = 1, \end{cases}$$

where $\chi_{q^*}(m) := \left(\frac{q^*}{m}\right)$ is the Kronecker character. When $d_1 = 1$, we understand that $\chi_{d_1,d_2}^{(\Delta)}$ takes the value 1 identically. This $\chi_{d_1,d_2}^{(\Delta)}$ becomes a character on $\mathfrak{F}_\Delta$ by putting $\chi_{d_1,d_2}^{(\Delta)}(F) = \chi_{d_1,d_2}^{(\Delta)}(f)$ with $F = [\![f]\!]$. For any quadratic discriminant Δ, let us define

$$F(\Delta) := \{d_1; d_1 \text{ is a fundamental discriminant, } d_1 \mid \Delta, \text{and } \Delta/d_1 \text{ is a discriminant}\}.$$

It is known that $F(\Delta)$ is a group under a suitable binary operation, and that the map $F(\Delta) \to \widehat{\mathfrak{F}_\Delta/\mathfrak{F}_\Delta^2}$ sending d_1 to $\chi_{d_1,d_2}^{(\Delta)}$ turns out to be a group epimorphism with the kernel $\{1, d_K\}$. Here d_K the discriminant of $K = \mathbf{Q}(\sqrt{\Delta})$. In particular, $\chi_{d_1,d_2}^{(\Delta)} = \chi_{e_1,e_2}^{(\Delta)}$ if and only if $(d_1, d_2) \in \{(e_1, e_2), (e_2, e_1)\}$. Note that if q^* is odd then $q^* = (-1)^{(q-1)/2} q$ with an odd prime $q := |q^*|$, and we have $\chi_{q^*}(m) = \left(\frac{m}{q}\right)$ (Legendre's symbol) for any integer m.[‡‡]

[‡‡]Here we give additional remarks. Notice that if an odd prime discriminant $q^* \mid d_1$ satisfies $q^* \mid a$, then $q^* \mid \Delta + 4ac = b^2$ and $q^* \mid b$. We have $q^* \nmid c$ since $(a, b, c) = 1$. If $q^* \mid c$, we deduce $q^* \nmid a$ in the same manner. When $q^* \mid d_1$ and q^* is an even prime discriminant, then $\Delta + 4ac = b^2$ is even. Hence $2 \nmid a$ or $2 \nmid c$ because of $(a, b, c) = 1$. Therefore either $(a, q^*) = 1$ or $(c, q^*) = 1$ is fulfilled for any prime discriminant $q^* \mid d_1$. If $(a, q^*) = 1$ and $(c, q^*) = 1$, we have $\chi_{q^*}(a) = \chi_{q^*}(c)$ by [8, Theorem 6.5.5, p. 227]. An equivalent definition of $\chi_{d_1,d_2}^{(\Delta)}$ is $\chi_{d_1,d_2}^{(\Delta)}(f) := \chi_{d_1}(m)$, where m is any integer such that $(m, \Delta) = 1$ and represented by f. Such an m always exists by [8, Theorem 6.2.1, p. 199], and the value of $\chi_{d_1,d_2}^{(\Delta)}$ is independent of the specific choice of m by [8, Theorem 6.5.3, p. 223]. For such an m, it follows that $m \in \mathrm{Ker}(Q_\Delta)$ by [8, Theorem 6.5.3, p. 223], and that $1 = Q_\Delta(m) = \left(\frac{\Delta}{m}\right) = \left(\frac{d_1}{m}\right)\left(\frac{d_2}{m}\right) = \chi_{d_1}(m)\chi_{d_2}(m) = \chi_{d_1,d_2}^{(\Delta)}(f)\chi_{d_2,d_1}^{(\Delta)}(f)$. Hence, we have $\chi_{d_1,d_2}^{(\Delta)} = \chi_{d_2,d_1}^{(\Delta)}$. Similarly, we conclude $\chi_{d_1,d_2}^{(\Delta)}(F_1 * F_2) = \chi_{d_1,d_2}^{(\Delta)}(F_1)\chi_{d_1,d_2}^{(\Delta)}(F_2)$ for any $F_i \in \mathfrak{F}_\Delta$. Indeed, when $F_i \in \mathfrak{F}_\Delta$ represents m_i with $(\Delta, m_i) = 1$, then $F_1 * F_2$ represents $m_1 m_2$ by [8, Corollary 6.4.8, p. 217], and thus one has $\chi_{d_1,d_2}^{(\Delta)}(F_1 * F_2) = \left(\frac{d_1}{m_1 m_2}\right) = \left(\frac{d_1}{m_1}\right)\left(\frac{d_1}{m_2}\right) = \chi_{d_1,d_2}^{(\Delta)}(F_1)\chi_{d_1,d_2}^{(\Delta)}(F_2)$.

By means of the group isomorphism $\Phi_\Delta : \mathfrak{F}_\Delta \to \mathcal{C}_\Delta^+$, the character $\chi_{d_1,d_2}^{(\Delta)}$ gives a character of $\mathcal{C}_\Delta^+$. We then define the genus character L-function for $\Re(s) > 1$ by

$$L(s, \chi_{d_1,d_2}^{(\Delta)}) := \sum_{\mathcal{O}_\Delta-\text{invertible ideal } \mathfrak{a} \subset \mathcal{O}_\Delta} \frac{\chi_{d_1,d_2}^{(\Delta)}(\mathfrak{a})}{\mathfrak{N}_\Delta(\mathfrak{a})^s}.$$

Here the sum is taken over all $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{a}$ of $\mathcal{O}_\Delta$.

8 Appendix 2: H. Lang's evaluation of n_M

According to H. Lang [18, (2.14), p. 423], for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we define $\Psi(M)$ by

$$-12\Psi(M) = \Phi(M) - 3\text{sign}(c),$$

where we set $\text{sign}(c) = 0$ if $c = 0$, and $\Phi(M)$ is Rademacher's Φ (cf. [26, (71.21), p. 150]), that is, $\Phi(M) = b/d$ if $c = 0$, while $-12\Psi(M) = n_M$ if $c \neq 0$ (cf. Lemma 7). In terms of the notations in Lemma 6, Meyer's formula can be stated as

$$L_{\mathfrak{b},U_\Delta}(1,\chi) = \frac{-4\pi^2}{\mathfrak{N}_\Delta(\mathfrak{b})\sqrt{\Delta}}\chi((\beta_2))\Psi(M_\mathfrak{b}), \qquad L_{\mathfrak{b},U_\Delta}(0,\chi) = -4\chi((\beta_2))\Psi(M_\mathfrak{b}).$$

Suppose that $N'' = \left(\begin{smallmatrix} * & * \\ c'' & * \end{smallmatrix}\right)$, $N' = \left(\begin{smallmatrix} * & * \\ c' & * \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ satisfy $N'' = N'N$. By [26, Theorem, p. 152], one has $\Phi(N) + \Phi(N') - \Phi(N'') = 3\text{sign}(cc'c'')$, in other words,

$$\Psi(N'') - \Psi(N) - \Psi(N') = -\frac{1}{4}\{\text{sign}(c) + \text{sign}(c') - \text{sign}(c'') - \text{sign}(cc'c'')\}. \tag{13}$$

Let n be a natural number, and let $u_j \geq 1$, $j = 0, 1, 2, \cdots, 2n-1$ be integers. We define $M \in SL_2(\mathbf{Z})$ by

$$M = \begin{pmatrix} u_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} u_{2n-1} & 1 \\ 1 & 0 \end{pmatrix}.$$

According to [18, p. 422], we put

$$A_j = \begin{pmatrix} u_j & 1 \\ 1 & 0 \end{pmatrix}, \quad A_j^* = \begin{pmatrix} u_j & (-1)^{j+1} \\ (-1)^j & 0 \end{pmatrix},$$

$$M_\nu^* := A_0^* A_1^* \cdots A_{\nu-1}^*, \quad M_\nu := A_0 A_1 \cdots A_{\nu-1} \quad (1 \leq \nu \leq 2n).$$

Noticing the identity $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$, it hold that $M_\nu^* = M_\nu$ for any even $\nu \geq 2$ (in particular $M_{2n}^* = M$), and $M_\nu^* = M_{\nu-1}A_{\nu-1}^*$ for any odd $\nu \geq 2$ (cf. [18, p. 423]). By definition, all components of M_ν are positive when $\nu \geq 2$.

In view of $M_\nu^* = M_{\nu-1}^* A_{\nu-1}^*$, the relation (13) for $(N'', N', N) = (M_\nu^*, M_{\nu-1}^*, A_{\nu-1}^*)$ tells us that

$$\Psi(M_\nu^*) = \Psi(M_{\nu-1}^*) + \Psi(A_{\nu-1}^*) \qquad (\nu \geq 2). \tag{14}$$

Indeed, when ν is even, $c'' > 0$ follows from $M_\nu^* = M_\nu$, and $\text{sign}(c) = (-1)^{\nu-1} = -1$ by the definition of $A_{\nu-1}^*$. If $\nu \geq 4$ is even, since $M_{\nu-1}^* = M_{\nu-2}^* A_{\nu-2}^* = M_{\nu-2} A_{\nu-2}^*$, one has $c' > 0$ by noticing $u_{\nu-2} > 0$. If $\nu = 2$, by $M_{\nu-1}^* = A_0^*$, one has $c' = 1 > 0$. Hence (13) implies (14). When ν is odd, $c' > 0$ follows from $M_{\nu-1}^* = M_{\nu-1}$, and $\text{sign}(c) = (-1)^{\nu-1} = 1$ by the definition of $A_{\nu-1}^*$. Since $M_\nu^* = M_{\nu-1}^* A_{\nu-1}^* = M_{\nu-1} A_{\nu-1}^*$, we see that $c'' > 0$ by $u_{\nu-1} > 0$. Hence (13) implies (14).

It follows that

$$\Psi(M) = \Psi(M_{2n}^*) = \sum_{j=0}^{2n-1} \Psi(A_j^*) = -\frac{1}{12} \sum_{j=0}^{2n-1} \left((-1)^j u_j - (-1)^j 3\right) = -\frac{1}{12} \sum_{j=0}^{2n-1} (-1)^j u_j.$$

Here at the third equal sign, we used $-12\Psi(A_j^*) = n_{A_j^*}$, Lemma 7 and $s(0,1) = 0$ (cf. (5)). This gives another proof of Lemma 9 by noticing that two definitions of M coincide with each other (cf. Section 5).

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