ON MULTIPLE ZETA VALUES OF LEVEL TWO

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Abstract. We study a variant of multiple zeta values of level 2, which forms a subspace of the space of alternating multiple zeta values. This variant, which is regarded as the 'shuffle counterpart' of Hoffman's 'odd variant', exhibits nice properties such as duality, shuffle product, parity results, etc., like ordinary multiple zeta values. We also give some conjectures on relations between our values, Hoffman's values, and multiple zeta values.

1. Introduction

In this paper, we study the following variant of the multiple zeta value,

$$\sum_{\substack{0<m_1<\cdots<m_r \\ m_i\equiv i \bmod 2}} \frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}},$$

which was introduced in [10, Section 5] in connection with a 'level 2' generalization of the zeta function studied by Arakawa and the first named author in [2]. We regard this value as a level 2 multiple zeta value because of the congruence condition in the summation and of the (easily proved) fact that this value can be written as a linear combination of alternating multiple zeta values (also referred to as Euler sums or colored multiple zeta values)

$$\sum_{0<m_1<\cdots<m_r} \frac{(\pm 1)^{m_1}(\pm 1)^{m_2}\cdots(\pm 1)^{m_r}}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}. \tag{1.1}$$

It turned out that the value with the normalizing factor 2^r,

$$T(k_1,k_2,\ldots,k_r) := 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i\equiv i \bmod 2}} \frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}, \tag{1.2}$$

was more natural and convenient, and we often refer this value as the 'multiple T-value' (MTV).

This is in contrast to Hoffman's multiple t-value defined by

$$t(k_1,\ldots,k_r) = \sum_{\substack{0<m_1<\cdots<m_r \\ \forall m_i\,:\,\mathrm{odd}}} \frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}, \tag{1.3}$$

which was introduced and studied in his recent paper [7] as another variant of multiple zeta values of level 2. In the next section and in §5, we discuss in more detail the comparison between T- and t-values.

In the case of depth $r=2$, Tasaka and the first named author studied in [9] both versions in connection to modular forms of level 2, and gave some results generalizing the previous work by Gangle-Kaneko-Zagier [5]. We do not pursue any modular aspects in this paper.

In the following sections, we show several properties of MTVs such as an integral expression, the duality relation, certain sum formulas, the parity result, and the generating

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series of 'height one' MTVs, all similar to those properties for classical multiple zeta values (MZVs)

$$\zeta(k_1,\dots,k_r)=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}, \tag{1.4}$$

and give some conjectures concerning the space spanned by the multiple T-values and a speculation on a basis of multiple zeta values in terms of Hoffman's t-values.

2. The space of multiple T-values

As in the study of multiple zeta values, let us introduce the $\mathbb{Q}$-vector space

$$\mathcal{T}^{ш}=\sum_{k=0}^{\infty}\mathcal{T}_k^{ш}$$

spanned by all MTVs, where

$$\mathcal{T}_0^{ш}=\mathbb{Q},\quad \mathcal{T}_1^{ш}=\{0\},\quad \mathcal{T}_k^{ш}=\sum_{\substack{1\le r\le k-1\\ k_1,\dots,k_{r-1}\ge 1,\ k_r\ge 2\\ k_1+\cdots+k_r=k}}\mathbb{Q}\cdot T(k_1,\dots,k_r)\quad (k\ge 2).$$

The space $\mathcal{T}^{ш}$ becomes a $\mathbb{Q}$-algebra, the product of two MTVs being described by the *shuffle product*. This is clear from the following integral expression of MTVs, which is exactly parallel to that of multiple zeta values.

For a given tuple of numbers $\varepsilon_i\in\{0,1\}$ $(1\le i\le k)$ with $\varepsilon_1=1$ and $\varepsilon_k=0$, set

$$I(\varepsilon_1,\dots,\varepsilon_k)=\int\cdots\int_{0<t_1<\cdots<t_k<1}\Omega_{\varepsilon_1}(t_1)\cdots\Omega_{\varepsilon_k}(t_k),$$

where

$$\Omega_0(t)=\frac{dt}{t},\quad \Omega_1(t)=\frac{2\,dt}{1-t^2}.$$

Recall that an index set $\mathbf{k}=(k_1,k_2,\dots,k_r)\in\mathbb{N}^r$ is *admissible* if $k_r\ge 2$. This ensures the convergence of the series (1.2) (as well as (1.3) and (1.4)).

Theorem 2.1 (cf. Sasaki [15]). *For any admissible index set* $(k_1,k_2,\dots,k_r)$*, we have*

$$T(k_1,k_2,\dots,k_r)=I(1,\underbrace{0,\dots,0}_{k_1-1},1,\underbrace{0,\dots,0}_{k_2-1},\cdots,1,\underbrace{0,\dots,0}_{k_r-1}). \tag{2.1}$$

This can be seen by expanding $1/(1-t^2)$ into geometric series and integrate from left to right, just as in the standard iterated integral expression of the multiple zeta value (1.4), which is given in exactly the same form with $\Omega_1(t)$ replaced by $dt/(1-t)$. We should remark that this integral expression (2.1) is essentially given in [15], although one needs some change of variables to obtain the current form.

From this integral expression, we immediately see that the *same* shuffle product rule holds for MTVs as for MZVs, an example being

$$T(2)^2=4T(1,3)+2T(2,2)\quad\text{and}\quad \zeta(2)^2=4\zeta(1,3)+2\zeta(2,2).$$

Another immediate consequence of the integral expression is the duality. We state and prove this in the next section, but remark here that the formula is again exactly the same as the duality relation for ordinary MZVs.

Returning to the space $\mathcal{T}^{ш}$, the first question would be the dimension d_k^T over $\mathbb{Q}$ of each subspace $\mathcal{T}_k^{ш}$ of weight k elements. We have conducted numerical experiments with Pari-GP, and obtained the following conjectural table.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k^T	1	0	1	1	2	2	4	5	9	10	19	23	42	49	91	110

Interestingly enough, the Fibonacci-like relation $d_k^T = d_{k-1}^T + d_{k-2}^T$ can be read off from the table for *even* k, but no immediate pattern is recognizable for general k.

Remark 2.2. After we gave a talk on this subject at a workshop (Feb. 2019 at Kindai University), Nobuo Sato and Takahiro Kubota pointed out that the series d_k^T might be given by the generating series

$$\sum_{k=0}^{\infty} d_k^T x^k \overset{?}{=} \frac{1 - x\prod_{n=1}^{\infty}(1-x^{2n+2})^{\frac{(-1)^n-2^n}{3}}}{1-x-x^2}.$$

This series conjecturally gives d_{16}^T=201, d_{17}^T=241, d_{18}^T=442, d_{19}^T=541, ...

Recall that the conjectural dimension of the space of alternating MZVs of weight k (spanned by the numbers (1.1) with all possible signs and k_i's with $k_1+\cdots+k_r=k$, with r varying) is given by the Fibonacci number F_k with $F_0=F_1=1$ and $F_k=F_{k-1}+F_{k-2}$.

We also note that Hoffman in [7] conjectures that the dimension d_k^t of the space spanned by his t-values (1.3) of weight k is given by the Fibonacci number F_{k-1} (for $k \geq 2$).

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k^t	1	0	1	2	3	5	8	13	21	34	55	89	144	233	377	610
F_k	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

In §5, we present some speculations on relations between multiple T-values, Hoffman's t-values, and multiple zeta values.

3. Several identities among T-values

In this section, we describe some formulas we have obtained so far.

3.1. **Duality.** We first recall the definition of the dual index set. We can write any admissible index set $\mathbf{k}$ as

$$\mathbf{k} = (\underbrace{1,\ldots,1}_{a_1-1}, b_1+1, \underbrace{1,\ldots,1}_{a_2-1}, b_2+1, \ldots, \underbrace{1,\ldots,1}_{a_m-1}, b_m+1)$$

with $a_1, b_1, a_2, b_2, \ldots, a_m, b_m \in \mathbb{Z}_{\geq 1}$. We call

$$\mathbf{k}^\dagger = (\underbrace{1,\ldots,1}_{b_m-1}, a_m+1, \underbrace{1,\ldots,1}_{b_{m-1}-1}, a_{m-1}+1, \ldots, \underbrace{1,\ldots,1}_{b_1-1}, a_1+1)$$

the "dual" index set of $\mathbf{k}$. It is well-known that $\zeta(\mathbf{k}^\dagger) = \zeta(\mathbf{k})$ holds which is called the duality relation for MZVs (see for instance the textbook of Zhao [21]).

Theorem 3.1. *For any admissible index set* $\mathbf{k}$*, we have*

$$T(\mathbf{k}^\dagger) = T(\mathbf{k}). \tag{3.1}$$

Proof. The involution $t \to (1-t)/(1+t)$, which interchanges the differential forms $\Omega_0(t)$ and $\Omega_1(t)$ and sends the interval $(0,1)$ to itself (with opposite orientation), plays the role for the involution $t \to 1-t$ in the case of MZVs. That is to say, the change of variables

$$s_i = \frac{1-t_{k-i+1}}{1+t_{k-i+1}} \quad (1 \leq i \leq k)$$

in (2.1) immediately gives

$$I(\varepsilon_1, \ldots, \varepsilon_k) = I(1-\varepsilon_k, \ldots, 1-\varepsilon_1),$$

which is the required duality. □

3.2. **Sum formulas.** For multiple zeta values, the classical sum formula is widely known and its variants are enormous (see [21, Chapter 5] for some of them). For our T-values, we only have certain formulas in depths 2 and 3. In depth 2, we obtain an analogue of the weighted sum formula of Ohno and Zudilin [14], but in depth 3, we only obtain a formula which looks incomplete to be called as a sum formula.

Theorem 3.2. *For $k \in \mathbb{Z}_{\geq 3}$, we have*

$$\sum_{j=2}^{k-1} 2^{j-1}\, T(k-j,j) = (k-1)T(k). \tag{3.2}$$

Theorem 3.3. *For $k \in \mathbb{Z}_{\geq 4}$,*

$$\sum_{\substack{a+b+c=k \\ a,b\geq 1, c\geq 2}} T(a,b,c) + \sum_{j=2}^{k-2} T(1,k-1-j,j) = \frac{2}{3} T(2)T(k-2). \tag{3.3}$$

We give proofs of these two theorems in the next section, and also present a conjectural (weighted) sum formula in depth 3.

3.3. **Parity result.** The so-called *parity result*, proved in the case of MZVs in [8, 16], also holds for MTVs.

Theorem 3.4 ([18]). *Let $\mathbf{k} = (k_1, \ldots, k_r)$ be an admissible index set and assume that its depth r and weight $k_1 + \cdots + k_r$ have opposite parity. Then $T(\mathbf{k})$ can be expressed as a $\mathbb{Q}$-linear combination of multiple T-values of lower depths and products of multiple T-values with sum of depths not exceeding r.*

This was essentially proved in a previous paper [17] of the second named author. Actually, what we have shown there was a reduction of T-values having depth and weight of opposite parity into a mixture of T-values and a certain multiple L-values with the character of conductor 4 of lower depth. But by checking carefully the proof of [17, Theorem 1], we see that Theorem 3.4 is in principle already proved there. We plan to write a detailed proof separately in [18].

Example 3.5. For the case of depth 2, we obtain the following formulas. For $p \geq 1$ and $q \geq 2$ with $p+q$ odd, we have

$$\begin{aligned} (-1)^q\, T(p,q) &= \binom{p+q-1}{q} T(p+q) \\ &\quad - \sum_{\substack{\mu=1 \\ \mu\equiv q \bmod 2}}^{q-2} \binom{p+\mu-1}{\mu} \frac{1}{2^{q-\mu}-1} T(p+\mu)T(q-\mu) \\ &\quad - \sum_{\substack{\mu=0 \\ \mu\equiv p \bmod 2}}^{p-2} \binom{q+\mu-1}{\mu} T(p-\mu)T(q+\mu). \end{aligned} \tag{3.4}$$

We discuss a bit more of a special case in depth 3 in §5.

3.4. **Height one T-values.** It is well-known ([1, 4], see also [13]) that the generating function of the 'height one' multiple zeta values is given in terms of the gamma function:

$$1 - \sum_{m,n=1}^{\infty} \zeta(\underbrace{1,\ldots,1}_{n-1}, m+1) X^m Y^n = \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}, \tag{3.5}$$

which immediately gives the height one duality relation

$$\zeta(\underbrace{1,\ldots,1}_{n-1}, m+1) = \zeta(\underbrace{1,\ldots,1}_{m-1}, n+1) \qquad (m,n \in \mathbb{Z}_{\geq 1}).$$

We can give the following T-version of (3.5).

Theorem 3.6. *We have the generating series identity*

$$1-\sum_{m,n=1}^{\infty} T(\underbrace{1,\dots,1}_{n-1},m+1)X^mY^n = \frac{2\,\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}F(1-X,1-Y;1-X-Y;-1),$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function and we assume $|X|<1, -1<Y<0$.

Proof. From the integral expression (2.1), we have

$$\begin{aligned} T(\underbrace{1,\dots,1}_{n-1},m+1) &= \int\cdots\int_{0<t_1<\cdots<t_n<u_1<\cdots<u_m<1} \frac{2dt_1}{1-t_1^2}\cdots\frac{2t_n}{1-t_n^2}\frac{du_1}{u_1}\cdots\frac{du_m}{u_m} \\ &= \int_{0<t_n<1}\frac{1}{(n-1)!}\left(\int_0^{t_n}\frac{2}{1-t^2}\,dt\right)^{n-1}\frac{1}{m!}\left(\int_{t_n}^1\frac{1}{u}\,du\right)^m\frac{2dt_n}{1-t_n^2} \\ &= \frac{1}{(n-1)!\,m!}\int_0^1\left\{\log\left(\frac{1+t_n}{1-t_n}\right)\right\}^{n-1}\left\{\log\left(\frac{1}{t_n}\right)\right\}^m\frac{2dt_n}{1-t_n^2}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\sum_{m,n=1}^{\infty} T(\underbrace{1,\dots,1}_{n-1},m+1)X^mY^{n-1} \\ &= \int_0^1\left(\frac{1+t}{1-t}\right)^Y(t^{-X}-1)\frac{2dt}{1-t^2} \\ &= 2\int_0^1 t^{-X}(1-t)^{-Y-1}(1+t)^{Y-1}dt - \int_0^1\left(\frac{1+t}{1-t}\right)^Y\frac{2dt}{1-t^2}. \end{aligned}$$

Denote the two integrals on the last line by I_1 and I_2 respectively. It follows from the Euler integral

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt \quad (0<\Re a<\Re c)$$

that

$$I_1 = \frac{\Gamma(1-X)\Gamma(-Y)}{\Gamma(1-X-Y)}F(1-X,1-Y;1-X-Y;-1).$$

As for I_2, setting $w=\log\bigl((1+t)/(1-t)\bigr)$, we have

$$I_2 = \int_0^\infty e^{Yw}dw = -\frac{1}{Y} \quad (\text{if } Y<0).$$

Thus, multiplying $-Y$ and using $(-Y)\Gamma(-Y)=\Gamma(1-Y)$, we obtain the desired formula. $\square$

Remark 3.7. Similar to the case of MZVs, Theorem 3.6 gives the height one duality relation

$$T(\underbrace{1,\dots,1}_{n-1},m+1) = T(\underbrace{1,\dots,1}_{m-1},n+1) \qquad (m,n\in\mathbb{Z}_{\geq 1}),$$

which is a special case of (3.1).

Remark 3.8. Theorem 3.6 may be regarded as giving an expression of the expansion of $F(1-X,1-Y;1-X-Y;-1)$ in terms of multiple zeta values (via (3.5)) and multiple T-values. The authors do not know any explicit formula for $F(1-X,1-Y;1-X-Y;-1)$.

4. Proofs of the sum formulas

In this section, we prove Theorems 3.2 and 3.3, and give a conjectural sum formula for depth 3.

4.1. **Proof of Theorem 3.2.** We use two formulas of the function

$$\psi(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{A(k_1,\ldots,k_r;\tanh(t/2))}{\sinh(t)}\,dt \tag{4.1}$$

which was studied in our previous paper [10]. Here, $A(k_1,\ldots,k_r;z)$ is given by

$$A(k_1,\ldots,k_r;z)=2^r\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \bmod 2}}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

(In [10], $2^{-r}A(k_1,\ldots,k_r;z)$ is denoted by $\mathrm{Ath}(k_1,\ldots,k_r;z)$.) The formulas we need are special cases of [10, Theorems 5.3 and 5.5], which read (by letting $k=2$, $r\to k-2$ and $m=0$)

$$\begin{aligned}\psi(\underbrace{1,\ldots,1}_{k-3},2;s)&\\ &=-\sum_{j=2}^{k-1}\binom{s+j-2}{j-1}T(k-j,j-1+s)-T(k-1,s)+T(k-1)T(s)\end{aligned}\tag{4.2}$$

and

$$\psi(\underbrace{1,\ldots,1}_{k-3},2;1)=T(1,k-1). \tag{4.3}$$

We also use the fact that the function $\psi(\underbrace{1,\ldots,1}_{k-3},2;s)$ is holomorphic everywhere. Since each function $T(k-j,j-1+s)$ in the sum on the right of (4.2) is holomorphic at $s=1$, the remaining sum $-T(k-1,s)+T(k-1)T(s)$ should be holomorphic at $s=1$ (each of $T(k-1,s)$ and $T(k-1)T(s)$ has a pole of order 1 at $s=1$). To evaluate the value of $-T(k-1,s)+T(k-1)T(s)$ at $s=1$, we compute the 'stuffle product'

$$\begin{aligned}&\frac{1}{2}T(k-1)\cdot 2^{-s}\zeta(s)\\ &=\sum_{\substack{m=1\\ m:\mathrm{odd}}}\frac{1}{m^{k-1}}\sum_{\substack{n=2\\ n:\mathrm{even}}}\frac{1}{n^s}=\sum_{\substack{0<m<n\\ m:\mathrm{odd},\,n:\mathrm{even}}}\frac{1}{m^{k-1}n^s}+\sum_{\substack{0<n<m\\ n:\mathrm{even},\,m:\mathrm{odd}}}\frac{1}{n^sm^{k-1}}\\ &=\frac{1}{4}T(k-1,s)+\zeta^{eo}(s,k-1),\end{aligned}\tag{4.4}$$

($\zeta^{eo}(s,k-1)$ is the last sum in (4.4)) from which we have

$$\begin{aligned}&-T(k-1,s)+T(k-1)T(s)\\ &\quad=4\zeta^{eo}(s,k-1)-2T(k-1)\cdot 2^{-s}\zeta(s)+T(k-1)T(s)\\ &\quad=4\zeta^{eo}(s,k-1)-2T(k-1)\cdot 2^{-s}\zeta(s)+T(k-1)\cdot 2(1-2^{-s})\zeta(s)\\ &\quad=4\zeta^{eo}(s,k-1)+T(k-1)\cdot 2(1-2^{1-s})\zeta(s).\end{aligned}$$

We then see that $\zeta^{eo}(s,k-1)$ is finite at $s=1$ and so is

$$(1-2^{1-s})\zeta(s)=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots$$

whose value at $s=1$ is $\log 2$. Hence we have

$$\lim_{s\to1}\left(-T(k-1,s)+T(k-1)T(s)\right)=4\zeta^{eo}(1,k-1)+(2\log 2)T(k-1).$$

To compute the value $\zeta^{eo}(1,k-1)$, we consider the specific alternating MZV

$$\sigma_a(1,k-1)=\sum_{1\le m<n}\frac{(-1)^{m-1}}{mn^{k-1}}=\left(1-2^{-k+1}\right)\zeta(1,k-1)-2\zeta^{eo}(1,k-1). \tag{4.5}$$

We use the formula by Borwein et al [3, §4] (we are using the notation there)

$$\sigma_a(1,k-1) = (2\log 2)(1-2^{-k+1})\zeta(k-1) - \frac{k-1}{2}\zeta(k) + \frac{1}{2}\sum_{j=2}^{k-2}(1-2^{1-j})(1-2^{j-k+1})\zeta(j)\zeta(k-j)$$

and by Euler

$$\zeta(1,k-1) = \frac{k-1}{2}\zeta(k) - \frac{1}{2}\sum_{j=2}^{k-2}\zeta(j)\zeta(k-j) \tag{4.6}$$

to conclude

$$\begin{aligned}
4\zeta^{eo}(1,k-1) &= 2(1-2^{-k+1})\zeta(1,k-1) - 2\sigma_a(1,k-1) \\
&= (2-2^{-k+1})(k-1)\zeta(k) - (4\log 2)(1-2^{-k+1})\zeta(k-1) \\
&\quad - \sum_{j=2}^{k-2}\Big\{(1-2^{-k+1}) + (1-2^{1-j})(1-2^{j-k+1})\Big\}\zeta(j)\zeta(k-j) \\
&= (k-1)T(k) - (2\log 2)T(k-1) - \frac{1}{2}\sum_{j=2}^{k-2}T(j)T(k-j).
\end{aligned}$$

We have used $2(1-2^{-m})\zeta(m) = T(m)$ and

$$(1-2^{-k+1}) + (1-2^{1-j})(1-2^{j-k+1}) = 2(1-2^{-j})(1-2^{-k+j}).$$

We therefore have

$$\lim_{s\to 1}\left(-T(k-1,s) + T(k-1)T(s)\right) = (k-1)T(k) - \frac{1}{2}\sum_{j=2}^{k-2}T(j)T(k-j),$$

and by letting $s \to 1$ in (4.2) together with (4.3) we obtain

$$\sum_{j=2}^{k-1}T(k-j,j) + T(1,k-1) = (k-1)T(k) - \frac{1}{2}\sum_{j=2}^{k-2}T(j)T(k-j). \tag{4.7}$$

Now, recall the shuffle product expansion of $T(j)T(k-j)$ has the same form as that of $\zeta(j)\zeta(k-j)$ given in *e.g.* [5, p. 72, (3)], which is

$$T(j)T(k-j) = \sum_{\nu=2}^{k-1}\left\{\binom{\nu-1}{j-1} + \binom{\nu-1}{k-j-1}\right\}T(k-\nu,\nu).$$

Summing up, we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{j=2}^{k-2}T(j)T(k-j) &= \frac{1}{2}\sum_{\nu=2}^{k-1}\left(\sum_{j=2}^{k-2}\left\{\binom{\nu-1}{j-1} + \binom{\nu-1}{k-j-1}\right\}\right)T(k-\nu,\nu) \\
&= \sum_{\nu=2}^{k-1}\left(\sum_{j=2}^{k-2}\binom{\nu-1}{j-1}\right)T(k-\nu,\nu) \\
&= \sum_{\nu=2}^{k-2}(2^{\nu-1}-1)T(k-\nu,\nu) + (2^{k-2}-2)T(1,k-1).
\end{aligned} \tag{4.8}$$

Here, we have used

$$\sum_{j=2}^{k-2}\binom{\nu-1}{j-1} = \begin{cases} 2^{\nu-1}-1 & (\nu \le k-2), \\ 2^{k-2}-2 & (\nu = k-1). \end{cases}$$

Combining (4.7) and (4.8), we obtain Theorem 3.2.

Remark 4.1. The weighted sum formula for the double zeta values ([14, Theorem 3]) is

$$\sum_{j=2}^{k-1} 2^{j-1}\,\zeta(k-j,j)=\frac{k+1}{2}\zeta(k).$$

Our proof above uses essentially the same idea as in the proof given in [14].

4.2. **Triple T-values.** The method of proof here is different from that in the previous subsection and uses partial fraction decompositions. We start with a lemma.

Lemma 4.2. *For $q\in\mathbb{N}$, it holds*

$$\sum_{l=-\infty}^{\infty}\sum_{m,n=0}^{\infty}\frac{1}{(2l+1)(2m+1)(2n+1)^q(2l+2m+2n+3)}=0. \tag{4.9}$$

Proof. It is well-known that

$$\sum_{l=0}^{\infty}\frac{\sin((2l+1)x)}{2l+1}=\frac{\pi}{4},$$

which is uniformly convergent for $0<x<\pi$ (see [19, §2.2]). Setting $x=\pi/2+\theta$, we have

$$\lim_{L\to\infty}\sum_{l=-L}^{L}\frac{(-1)^l e^{(2l+1)i\theta}}{2l+1}=\frac{\pi}{2}\qquad\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right),$$

where $i=\sqrt{-1}$. For simplicity we write the limit of the left-hand side as $\sum_{l=-\infty}^{\infty}$. Hence, for $q\in\mathbb{N}$ and $z\in(-1,1)$, we have

$$\begin{aligned}0&=\left(\sum_{l=-\infty}^{\infty}\frac{(-1)^l e^{(2l+1)i\theta}}{2l+1}-\frac{\pi}{2}\right)\sum_{m,n=0}^{\infty}\frac{(-z)^{m+n}e^{(2m+2n+2)i\theta}}{(2m+1)(2n+1)^q}\\ &=\sum_{l=-\infty}^{\infty}\frac{(-1)^l}{2l+1}\sum_{m,n=0}^{\infty}\frac{(-z)^{m+n}e^{(2l+2m+2n+3)i\theta}}{(2m+1)(2n+1)^q}-\frac{\pi}{2}\sum_{m,n=0}^{\infty}\frac{(-z)^{m+n}e^{(2m+2n+2)i\theta}}{(2m+1)(2n+1)^q}\end{aligned}$$

for $-\pi/2<\theta<\pi/2$. Integrating the both sides from $-t$ to t $(-\pi/2<t<\pi/2)$, we obtain

$$\begin{aligned}0=&2\sum_{l=-\infty}^{\infty}\frac{(-1)^l}{2l+1}\sum_{m,n=0}^{\infty}\frac{(-z)^{m+n}\sin((2l+2m+2n+3)t)}{(2m+1)(2n+1)^q(2l+2m+2n+3)}\\ &-\pi\sum_{m,n=0}^{\infty}\frac{(-z)^{m+n}\sin((2m+2n+2)t)}{(2m+1)(2n+1)^q(2m+2n+2)}.\end{aligned}$$

We can easily see that the right-hand side is absolutely and uniformly convergent for $t\in[-\pi/2,\pi/2]$ and $z\in[-1,1]$. Hence, letting $t\to\pi/2$ and $z\to1$, we obtain (4.9). □

We can write (4.9) as

$$\begin{aligned}&\sum_{l,m,n\geq0}\frac{1}{(2l+1)(2m+1)(2n+1)^q(2l+2m+2n+3)}\\ &\quad-\sum_{l,m,n\geq0}\frac{1}{(2l+1)(2m+1)(2n+1)^q(-2l+2m+2n+1)}=0.\end{aligned}\tag{4.10}$$

Denote the sums on the left-hand side by I_1 and I_2 respectively, so that $I_1=I_2$ holds. We shall write down each I_1 and I_2 in terms of T-values, by using the following partial fraction decomposition formulas.

Lemma 4.3. *For $q \in \mathbb{N}$,*

$$\frac{1}{xy^q} = \sum_{j=0}^{q-1} \frac{1}{y^{q-j}(x+y)^{j+1}} + \frac{1}{x(x+y)^q}, \tag{4.11}$$

$$\begin{aligned}\frac{1}{xyz^q} = \sum_{j=0}^{q-1} \Bigg\{ &\sum_{\nu=0}^{q-j-1} \frac{1}{z^{q-j-\nu}(x+z)^{\nu+1}(x+y+z)^{j+1}} + \frac{1}{x(x+z)^{q-j}(x+y+z)^{j+1}} \\ &+ \sum_{\nu=0}^{q-j-1} \frac{1}{z^{q-j-\nu}(y+z)^{\nu+1}(x+y+z)^{j+1}} + \frac{1}{y(y+z)^{q-j}(x+y+z)^{j+1}} \Bigg\} \\ &+ \frac{1}{x(x+y)(x+y+z)^q} + \frac{1}{y(x+y)(x+y+z)^q}.\end{aligned} \tag{4.12}$$

Proof. Equation (4.11) immediately follows from the factorization

$$\frac{1}{y^q} - \frac{1}{(x+y)^q} = \left(\frac{1}{y} - \frac{1}{x+y}\right) \sum_{j=0}^{q-1} \frac{1}{y^{q-1-j}(x+y)^j} = \frac{x}{y(x+y)} \sum_{j=0}^{q-1} \frac{1}{y^{q-1-j}(x+y)^j}.$$

Replacing y by z and x by $x+y$ in (4.11) and then multiplying $(x+y)/xy = 1/x + 1/y$, we have

$$\frac{1}{xyz^q} = \sum_{j=0}^{q-1} \left(\frac{1}{xz^{q-j}} + \frac{1}{yz^{q-j}}\right) \frac{1}{(x+y+z)^{j+1}} + \frac{1}{xy(x+y+z)^q}.$$

Applying (4.11) to $1/xz^{q-j}$ and $1/yz^{q-j}$ and writing $1/xy$ as $1/x(x+y) + 1/y(x+y)$ in the last term, we obtain (4.12). □

Proof of Theorem 3.3. Using (4.12) with $x = 2l+1, y = 2m+1, z = 2n+1$, we readily have (note $2l+2m+2n+3 = x+y+z$)

$$\begin{aligned} I_1 &= \frac{1}{4} \sum_{j=0}^{q-1} \left\{ \sum_{\nu=0}^{q-1-j} T(q-j-\nu, \nu+1, j+2) + T(1, q-j, j+2) \right\} + \frac{1}{4} T(1,1,q+1) \\ &= \frac{1}{4} \sum_{\substack{a+b+c=q+3 \\ a,b \geq 1, c \geq 2}} T(a,b,c) + \frac{1}{4} \sum_{j=2}^{q+1} T(1, q+2-j, j) + \frac{1}{4} T(1,1,q+1). \end{aligned} \tag{4.13}$$

As for I_2, set $d = n-l$ or $e = l-n$ according as $l < n$ or $l \geq n$. Then

$$\begin{aligned} I_2 = &\sum_{\substack{d \geq 1 \\ l,m \geq 0}} \frac{1}{(2l+1)(2m+1)(2d+2l+1)^q(2d+2m+1)} \\ &+ \sum_{e,m,n \geq 0} \frac{1}{(2e+2n+1)(2m+1)(2n+1)^q(-2e+2m+1)}. \end{aligned} \tag{4.14}$$

The first sum on the right is equal to

$$\begin{aligned} &\sum_{\substack{d \geq 1 \\ l \geq 0}} \frac{1}{(2l+1)(2d+2l+1)^q} \frac{1}{(2d)} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} - \frac{1}{2m+2d+1}\right) \\ &= \sum_{\substack{d \geq 1, l \geq 0 \\ 0 \leq m \leq d-1}} \frac{1}{(2l+1)(2d+2l+1)^q(2d)(2m+1)} \\ &= \sum_{l,m,k \geq 0} \frac{1}{(2l+1)(2m+1)(2m+2k+2)(2l+2m+2k+3)^q} \quad (d = m+k+1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l,m,k\geq 0} \frac{1}{(2l+1)(2m+1)(2l+2m+2k+3)^{q+1}} \\
&\quad + \sum_{l,m,k\geq 0} \frac{1}{(2m+1)(2m+2k+2)(2l+2m+2k+3)^{q+1}} \\
&= \sum_{l,m,k\geq 0} \frac{1}{(2l+1)(2l+2m+2)(2l+2m+2k+3)^{q+1}} \\
&\quad + \sum_{l,m,k\geq 0} \frac{1}{(2m+1)(2l+2m+2)(2l+2m+2k+3)^{q+1}} \\
&\qquad + \sum_{l,m,k\geq 0} \frac{1}{(2m+1)(2m+2k+2)(2l+2m+2k+3)^{q+1}} \\
&= \frac{3}{8}T(1,1,q+1).
\end{aligned}
$$

The second sum in (4.14) is, by setting $f=m-e$ or $g=e-m$ according as $e\leq m$ or $e>m$, transformed into

$$
\begin{aligned}
&\sum_{e,f,n\geq 0} \frac{1}{(2e+2n+1)(2e+2f+1)(2n+1)^q(2f+1)} \\
&\quad + \sum_{\substack{g\geq 1\\ m,n\geq 0}} \frac{1}{(2g+2m+2n+1)(2m+1)(2n+1)^q(-2g+1)}.
\end{aligned}
$$

The second sum of this is equal to $-I_1$ as seen by setting $g=l+1$. We write the first sum, first by separating the terms with $e=0$ and $e>0$, as

$$
\begin{aligned}
&\sum_{f,n\geq 0} \frac{1}{(2f+1)^2(2n+1)^{q+1}} + \sum_{e\geq 1,f,n\geq 0} \frac{1}{(2e+2n+1)(2n+1)^q(2e+2f+1)(2f+1)} \\
&= \frac{1}{4}T(2)T(q+1) + \sum_{\substack{e\geq 1\\ n\geq 0}} \frac{1}{(2e+2n+1)(2n+1)^q}\,\frac{1}{(2e)}\sum_{f=0}^{\infty}\left(\frac{1}{2f+1}-\frac{1}{2f+2e+1}\right) \\
&= \frac{1}{4}T(2)T(q+1) + \sum_{\substack{e\geq 1,n\geq 0\\ 0\leq f\leq e-1}} \frac{1}{(2e+2n+1)(2n+1)^q(2e)(2f+1)} \\
&= \frac{1}{4}T(2)T(q+1) + \sum_{f,l,n\geq 0} \frac{1}{(2f+1)(2f+2l+2)(2n+1)^q(2f+2l+2n+3)}
\end{aligned}
$$

$(e=f+l+1)$. Using (4.11) repeatedly, we have

$$
\begin{aligned}
&\sum_{f,l,n\geq 0} \frac{1}{(2f+1)(2f+2l+2)(2n+1)^q(2f+2l+2n+3)} \\
&= \sum_{f,l,n\geq 0} \left\{ \sum_{j=0}^{q-1} \frac{1}{(2f+1)(2n+1)^{q-j}(2f+2l+2n+3)^{j+2}} \right. \\
&\quad \left. + \frac{1}{(2f+1)(2f+2l+2)(2f+2l+2n+3)^{q+1}} \right\} \\
&= \sum_{f,l,n\geq 0} \sum_{j=0}^{q-1} \left\{ \sum_{\nu=0}^{q-j-1} \frac{1}{(2n+1)^{q-j-\nu}(2f+2n+2)^{\nu+1}(2f+2l+2n+3)^{j+2}} \right. \\
&\qquad \left. + \frac{1}{(2f+1)(2f+2n+2)^{q-j}(2f+2l+2n+3)^{j+2}} \right\}
\end{aligned}
$$

$$+\frac{1}{8}T(1,1,q+1)$$
$$=\frac{1}{8}\sum_{j=0}^{q-1}\left\{\sum_{\nu=0}^{q-1-j}T(q-j-\nu,\nu+1,j+2)+T(1,q-j,j+2)\right\}+\frac{1}{8}T(1,1,q+1)$$
$$=\frac{1}{8}\sum_{\substack{a+b+c=q+3\\ a,b\geq1,c\geq2}}T(a,b,c)+\frac{1}{8}\sum_{j=2}^{q+1}T(1,q+2-j,j)+\frac{1}{8}T(1,1,q+1).$$

We therefore have

$$I_2=\frac{3}{8}T(1,1,q+1)-I_1+\frac{1}{4}T(2)T(q+1)$$
$$+\frac{1}{8}\sum_{\substack{a+b+c=q+3\\ a,b\geq1,c\geq2}}T(a,b,c)+\frac{1}{8}\sum_{j=2}^{q+1}T(1,q+2-j,j)+\frac{1}{8}T(1,1,q+1).$$

Combining this and (4.13) together with $I_1=I_2$ and setting $q+3=k$ gives the theorem. □

Example 4.4. The case $k=5$ of Theorem 3.3 is

$$2T(1,1,3)+2T(1,2,2)+T(2,1,2)=\frac{2}{3}T(2)T(3). \tag{4.15}$$

This is not quite parallel to the case of ordinary MZVs, where the identity

$$2\zeta(1,1,3)+2\zeta(1,2,2)+\zeta(2,1,2)=2\zeta(2)\zeta(3)-\frac{5}{2}\zeta(5)$$

holds. It is unlikely that the right-hand side is a multiple of $\zeta(2)\zeta(3)$.

We end this section by proposing the following conjecture as an analogue of Machide's formula [11, Corollary 4.1].

Conjecture 4.5. *For $k\geq 4$, we have*

$$\sum_{\substack{a+b+c=k\\ a,b\geq1,c\geq2}}2^b(3^{c-1}-1)T(a,b,c)=\frac{2}{3}(k-1)(k-2)T(k).$$

5. Relations among multiple T-, t-, and zeta values

If we denote by $\mathcal{T}^*$ the $\mathbb{Q}$-vector space spanned by all Hoffman's multiple t-values, then, as can be directly seen from the definition (1.3), the space $\mathcal{T}^*$ also becomes a $\mathbb{Q}$-algebra by the *stuffle* (or *harmonic*) product, an example being $t(2)^2=2t(2,2)+t(4)$. Hence, we have two $\mathbb{Q}$-subalgebas $\mathcal{T}^{ш}$ and $\mathcal{T}^*$ of the algebra of alternating multiple zeta values, one being closed under the shuffle product and the other under the stuffle product. There are both shuffle and stuffle product structures on the whole space of alternating multiple zeta values.

It seems that the sum $\mathcal{T}^{ш}+\mathcal{T}^*$ does not exhaust all alternating MZVs, and that the seemingly smaller space $\mathcal{T}^{ш}$ is not contained in $\mathcal{T}^*$, as the following table (numerically computed, only up to weight 8) suggests.

k	0	1	2	3	4	5	6	7	8
$\dim(\mathcal{T}_k^{ш}+\mathcal{T}_k^*)$	1	0	1	2	4	5	9	14	24
$\dim(\mathcal{T}_k^{ш}\cap\mathcal{T}_k^*)$	1	0	1	1	1	2	3	4	6

Let $\mathcal{Z}$ be the space of usual multiple zeta values. The well-known conjectural dimension (Zagier [20]) of the subspace $\mathcal{Z}_k$ of weight k is given by the sequence d_k which satisfies $d_k=d_{k-2}+d_{k-3}$ with $d_0=1,d_1=0,d_2=1$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28

It appears that $d_k^T \geq d_k$ holds for all k ($F_{k-1} \geq d_k$ is certainly true, where F_{k-1} is conjectured to be equal to d_k^t). Moreover, we conjecture (also based on our numerical experiments) that the space $\mathcal{T}^{ш}$ as well as $\mathcal{T}^*$ contains the space $\mathcal{Z}$.

Conjecture 5.1. *Both $\mathcal{T}^{ш}$ and $\mathcal{T}^*$ contain $\mathcal{Z}$ as a $\mathbb{Q}$-subalgebra.*

The intersection $\mathcal{T}^{ш} \cap \mathcal{T}^*$ seems strictly larger than $\mathcal{Z}$, as the tables above suggest. If this conjecture is true, then both $\mathcal{T}^{ш}$ and $\mathcal{T}^*$ are modules over $\mathcal{Z}$. What are the structures of these modules?

Specific elements show an interesting pattern. By definition, the single T-values $T(k)$ and t-values $t(k)$ are multiples of $\zeta(k)$ and hence contained in $\mathcal{Z}$. And by our parity result ([18]), every double T-value of odd weight is also contained in $\mathcal{Z}$. For higher depths, we conjecture the following. Since we have the duality for T-values, we may restrict ourselves to the T-values with depth smaller than or equal to weight/2.

Conjecture 5.2. *1) For even weights, $T(p,q,r)$ with p,r : odd ≥ 3 and q : even (and their duals) are in $\mathcal{Z}$. These values together with the single T-values (and their duals) are the only T-values contained in $\mathcal{Z}$.*

2) For odd weights, $T(p,1,r)$ with p,r : even (and their duals) are in $\mathcal{Z}$. These values together with the single and double T-values (and their duals) are the only T-values contained in $\mathcal{Z}$.

Recall that, from our parity result, the triple T-value $T(p,q,r)$ of even weight can be written in terms of single and the double T-values. From an explicit formula for such an expression (see [18] for the detail), we surmise that the following is true.

Conjecture 5.3. *For $m \geq 1, p \geq 1, q \geq 2$ with $p+q+m$ even, we have*

$$\sum_{\substack{i+j=m\\ i,j\geq 0}} \binom{p+i-1}{i}\binom{q+j-1}{j} T(p+i,q+j) \in \mathcal{Z}.$$

For instance, the case $m=1$ predicts $qT(p,q+1)+pT(p+1,q) \in \mathcal{Z}$.

Remark 5.4. Denoting the sum in the conjecture above by $s(p,q,m)$, the form of the parity reduction for $T(2p+1,2q,2r+1)$ is

$$\begin{aligned} &T(2p+1,2q,2r+1)\\ &= -\sum_{j=0}^{p-1} T(2p-2j)s(2q-1,2r+1,2j+1) - \sum_{j=0}^{r-1} T(2r-2j)s(2q,2j+2,2p)\\ &\quad + \text{sum of products of single } T(n)\text{'s}. \end{aligned}$$

As for t-values, we experimentally observe that any $t(k_1,\ldots,k_r)$ with $\forall k_i \geq 2$ is in $\mathcal{Z}$. Among those, we may choose the following elements as linear and algebraic bases of $\mathcal{Z}$.

Conjecture 5.5. *1) A linear basis of the space $\mathcal{Z}_k$ of multiple zeta values of weight k is given by*

$$\{t(2)^n t(k_1,\ldots,k_r) \mid n,r \geq 0, \forall k_i : \text{odd} \geq 3,\ 2n+k_1+\cdots+k_r = k\}$$

2) An algebra basis of $\mathcal{Z}$ is given by $t(2)$ and $t(k_1,\ldots,k_r)$ with $\forall k_i$: odd ≥ 3 and the sequence $(k_1,\ldots,k_r)$ being Lyndon.

With the usual order by magnitude, a sequence $(k_1,\ldots,k_r)$ is Lyndon if any right subsequence $(k_i,\ldots,k_r)$ $(i \geq 2)$ is greater than $(k_1,\ldots,k_r)$ in lexicographical order.

Remark 5.6. Quite recently, T. Murakami [12] proved our observation $t(k_1,\dots,k_r)\in\mathcal{Z}$ if $\forall k_i\geq 2$, by using the motivic method employed in [6]. Moreover, he showed that each multiple zeta value can be written as a linear combination of $t(k_1,\dots,k_r)$'s with all k_i being equal to 2 or 3, thereby proving Conjecture 5.1 for $\mathcal{T}^*$. Also he proved Conjecture 5.3 and Conjecture 5.2, except that in Conjecture 5.2 he did not prove that those values are the only T-values contained in $\mathcal{Z}$.

6. Description of the space $\mathcal{T}_k^{ш}$ for low weights

Obviously $\mathcal{T}_2^{ш}=\mathbb{Q}\cdot T(2)$ is one dimensional, and by the duality (3.1) the space

$$\mathcal{T}_3^{ш}=\mathbb{Q}\cdot T(3)+\mathbb{Q}\cdot T(1,2)=\mathbb{Q}\cdot T(3)$$

is also one dimensional.

At weight 4, we have $T(1,1,2)=T(4)$ by the duality and $T(2,2)=\frac{1}{2}T(4)-2T(1,3)$ by the sum formula (3.2), and thus we see that

$$\mathcal{T}_4^{ш}=\mathbb{Q}\cdot T(4)+\mathbb{Q}\cdot T(1,3).$$

According to our conjecture (see the table in §2), this would give a basis of $\mathcal{T}_4^{ш}$.

We conjecture that the space $\mathcal{T}_5^{ш}$ of weight 5 is also two dimensional. By the duality, we see that $\mathcal{T}_5^{ш}$ is spanned by $T(5)$ and elements of depth 2. We have two independent relations

$$\begin{aligned}&4T(1,4)+2T(2,3)+T(3,2)=2T(5),\\&2T(1,4)+2T(2,3)+T(3,2)=4T(1,4)+2T(2,3)+\frac{2}{3}T(3,2)\end{aligned}$$

coming from (3.2) and (4.15) (as for the latter, we used the duality on the left-hand side and the shuffle product on the right), and from these we obtain

$$T(3,2)=6T(1,4)\quad\text{and}\quad T(2,3)=T(5)-5T(1,4).$$

Hence we conclude

$$\mathcal{T}_5^{ш}=\mathbb{Q}\cdot T(5)+\mathbb{Q}\cdot T(1,4).$$

Already at weight 6, known identities appear not to be enough to reduce the dimension to the conjectural 4. Using Theorems 3.1 through 3.4 and relations obtained by applying the shuffle product to lower weight relations, we may deduce

$$\begin{aligned}T(1,2,3)&=-\frac{25}{12}T(6)+12T(1,5)+6T(2,4)+2T(3,3)-2T(1,1,4),\\T(1,3,2)&=\frac{55}{12}T(6)-24T(1,5)-12T(2,4)-4T(3,3)-T(1,1,4),\\T(2,1,3)&=\frac{55}{12}T(6)-24T(1,5)-12T(2,4)-4T(3,3)-T(1,1,4),\\T(2,2,2)&=-\frac{35}{4}T(6)+48T(1,5)+24T(2,4)+8T(3,3)+6T(1,1,4),\\T(3,1,2)&=\frac{5}{6}T(6)-T(1,1,4),\\T(4,2)&=\frac{5}{2}T(6)-8T(1,5)-4T(2,4)-2T(3,3).\end{aligned}$$

One missing relation would be supplied by Conjecture 5.3, which predicts for instance

$$\begin{aligned}3T(2,4)+2T(3,3)&=-\frac{15}{7}T(6)+\frac{10}{7}T(3)^2\\&=-\frac{15}{7}T(6)+\frac{120}{7}T(1,5)+\frac{60}{7}T(2,4)+\frac{20}{7}T(3,3).\end{aligned}$$

(Note that the space of multiple zeta values of weight 6 is spanned by $\zeta(6) = \frac{32}{63}T(6)$ and $\zeta(3)^2 = \frac{16}{49}T(3)^2$.) From this we could conclude

$$\mathcal{T}_6^{ш} = \mathbb{Q}\cdot T(6) + \mathbb{Q}\cdot T(1,5) + \mathbb{Q}\cdot T(2,4) + \mathbb{Q}\cdot T(1,1,4).$$

In a similar vein, we may deduce by using proven relations that the space $\mathcal{T}_7^{ш}$ is at most 6 dimensional, and by assuming Conjecture 4.5, we may reduce the dimension to the conjectural 5.

Since it becomes more and more tedious to write down the parity reduction explicitly as the depth gets larger, we have not checked if all relations obtained and conjectured in this paper are enough to give the conjectural upper bound of the dimension of $\mathcal{T}_k^{ш}$ for k greater than 7. To find any other families of relations among MTV's, and ideally, to find even conjecturally a complete set of relations would be an important future problem.

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