

ELLIPTIC NORMAL CURVES OF EVEN DEGREE AND THETA FUNCTIONS

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ABSTRACT. An elliptic curve E can be immersed in \mathbf{P}^{N-1} as a curve of degree N by means of the linear system of $|NO|$, where O is the origin of E . Well-known classical results going back to Bianchi and Klein say that if N is odd, this immersion is uniquely determined by specifying a full-level N structure. In this paper we show that if N is even, uniqueness of immersion is ensured by specifying a level structure associated with a certain congruence subgroup between $\Gamma(N)$ and $\Gamma(2N)$. Moreover, we construct, over the complex number field, an immersion by means of suitably chosen theta functions, and write down the quadratic equations satisfied by them.

1. INTRODUCTION

An elliptic normal curve of degree N means an elliptic curve E together with an immersion into a projective space $E \hookrightarrow \mathbf{P}^{N-1}$ as a degree N curve that is contained in no hyperplane. We assume $N \geq 4$, and that the base field K is a field of characteristic not dividing N . By the Riemann-Roch theorem, any elliptic curve E over K can be realized as an elliptic normal curve of degree N in \mathbf{P}^{N-1} by means of a complete linear system $|D|$ with any effective divisor D of degree N . In particular, we may take $D = NO$, where O is the origin of the group structure of E . Moreover, it is known that the image of E in \mathbf{P}^{N-1} is defined by a system of $N(N-3)/2$ quadratic equations.

Classically, for the case $N = 5$ over the complex number field \mathbf{C} , Bianchi [1] first wrote down defining equations of an elliptic curve in the following “normal form”:

$$E_\phi : \begin{cases} x_0^2 + \phi x_2 x_3 - \phi^{-1} x_1 x_4 = 0, \\ x_1^2 + \phi x_3 x_4 - \phi^{-1} x_2 x_0 = 0, \\ x_2^2 + \phi x_4 x_0 - \phi^{-1} x_3 x_1 = 0, \\ x_3^2 + \phi x_0 x_1 - \phi^{-1} x_4 x_2 = 0, \\ x_4^2 + \phi x_1 x_2 - \phi^{-1} x_0 x_3 = 0. \end{cases}$$

These five quadratic equations define an elliptic curve E_ϕ as a projective curve of degree 5 in \mathbf{P}^4 , and the translations by the 5-torsion points are realized as linear transformations of the projective space. The curve E_ϕ may be viewed as the universal elliptic curve over the modular curve parametrizing triples (E, S, T) where E is an elliptic curve and (S, T) is a basis of $E[5]$ such that the Weil pairing $e_5(S, T)$ is equal to a fixed primitive 5th root of unity ζ_5 . The parameter $\phi = \phi(\tau)$ is a modular function associated with

the congruence subgroup $\Gamma(5)$. Klein [10] generalized Bianchi's results to obtain such a normal form of quadratic equations for general odd integers N , and studied their relation to modular functions of level N . Later, Vélú [13] studied the immersion of elliptic curves into \mathbf{P}^{N-1} as elliptic normal curves of degree N in terms of schemes. In the end, he constructed the modular curve $X(p)$ for odd prime number p as a subscheme in \mathbf{P}^{p-1} over $\text{Spec } \mathbf{Z}[1/p]$, and the universal elliptic curve $E(p)$ over $X(p)$. (See also [3].)

In the case where N is even, Hurwitz [6] generalized Klein's results to a certain extent. In this case, however, it is not possible to obtain the universal family associated with $\Gamma(N)$. Hurwitz's family is over the modular curve associated with some congruence subgroup of level $2N$ or $4N$ depending on $N \bmod 4$. In this paper we use the work of Vélú [13], and construct a slightly different immersion from that of Hurwitz based on an analysis of the level structures associated with congruence subgroups between $\Gamma(N)$ and $\Gamma(2N)$. We obtain a canonical coordinate system compatible with the translations by the N -torsion points and the universal elliptic curve associated with a certain congruence subgroup.

Over \mathbf{C} , we go one step further and realize the above immersion $E \rightarrow \mathbf{P}^{N-1}$ in terms of certain theta functions that become the coordinate functions of the system described above, and write down the quadratic equations satisfied by the image of E using the classical relations among theta functions.

Let us describe our results in more detail. Let E be an elliptic curve over a field of characteristic not dividing N , and consider the immersion $E \rightarrow \mathbf{P}^{N-1}$ using the complete linear system $|NO|$. For an N -torsion point $T \in E[N]$, let us denote by τ_T the translation-by- T map $P \mapsto P + T$. It is easy to see that τ_T can be lifted to an automorphism of the ambient space \mathbf{P}^{N-1} . By choosing a basis (S, T) of $E[N]$, we can find a coordinate system such that the translations τ_S and τ_T are realized as linear transformations of the ambient space \mathbf{P}^{N-1} .

If N is odd, the choice of such a coordinate system is unique once we fix a $\Gamma(N)$ structure, that is, a basis (S, T) of $E[N]$ such that $e_N(S, T)$ is equal to a fixed primitive N th root of unity ζ_N (Proposition 4.4). Thus, we obtain an immersion of the family of elliptic curves with $\Gamma(N)$ structure to a single \mathbf{P}^{N-1} , and by associating the origin of the elliptic curves, we obtain a morphism from the modular curve $X(N)$ to \mathbf{P}^{N-1} .

The situation is not the same if N is even. In this case, the coordinate system mentioned above is not unique even if we specify a $\Gamma(N)$ structure. So, we define an intermediate subgroup $\Gamma^{(N)}(2N)$ between $\Gamma(N)$ and $\Gamma(2N)$ (see Definition 4.10), and show that the coordinate system mentioned above is determined uniquely once we fix a $\Gamma^{(N)}(2N)$ structure (Theorem 4.20). As a consequence we obtain the universal elliptic curve over the modular curve associated with $\Gamma^{(N)}(2N)$.

Furthermore, over \mathbf{C} , we realize the immersion $E \hookrightarrow \mathbf{P}^{N-1}$ using the theta functions, denoted by $\theta_k^{(N)}(z, \tau)$ for $k = 0, \dots, N-1$ (see Definition 5.3), that serve as the coordinate functions of the coordinate system described above, and we obtain quadratic equations of the image (Theorems 6.4 and 6.8) using

the relations between the theta functions coming from Jacobi's identity (6.4) or (6.7), which is essentially the addition formula for the elliptic curve.

In §§7–9 we work out in detail the cases $N = 4, 6$, and 8 . There, we show the explicit equations of the modular curves and the universal curves over them.

Since our theta functions and Hurwitz's σ -functions [6] look quite different, it is not easy to see the exact relationship between them. In Appendix A, we describe the connections and differences between these two sets of functions in detail.

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2. PRELIMINARIES

From this section until §4 we fix a positive integer N greater than 3 , and suppose that the base field K is a field of characteristic not dividing N . Let K_s be a separable closure of K , and let $\Gamma_K = \text{Gal}(K_s/K)$.

Let E be an elliptic curve defined over K . Denote by $E(K)$ the group of K -rational points of E , and $E[N] = \{P \in E(K_s) \mid NP = O\}$ the subgroup of N -torsion points. Since we use the notation and results of Vélú [13] extensively, we give a summary of parts of [13] necessary for later use.

2.1. The Weil pairing. We first fix some notation.

- $K(E)^\times$: the multiplicative group of the function field $K(E)$ of E .
- $\text{Div}_K(E)$: the group of K -rational divisors of E ; i.e., the group of formal \mathbf{Z} -linear combination[†] $D = \sum_{P \in E(K_s)} n_P \{P\}$ with $n_P \in \mathbf{Z}$, such that $n_P = 0$ for all but finitely many $P \in E(K_s)$, and $n_{\sigma P} = n_P$ for all $\sigma \in \Gamma_K = \text{Gal}(K_s/K)$.
- $\text{Div}_K^0(E)$: the kernel of $\text{deg} : \text{Div}_K(E) \rightarrow \mathbf{Z}$.
- $\text{Nul}_K(E)$: the kernel of the homomorphism $\text{Sum} : \text{Div}_K(E) \rightarrow E(K)$ defined by $\sum_P n_P \{P\} \mapsto \text{Sum}_P n_P P$, where Sum means the addition in E .
- $P_K(E) = \text{Div}_K^0(E) \cap \text{Nul}_K(E)$.

We have an obvious homomorphism $\text{div} : K(E)^\times \rightarrow \text{Div}_K(E)$ that sends a function to its divisor, and the following is fundamental.

Theorem 2.1 (Abel-Jacobi). *The following sequence is exact.*

$$(2.1) \quad 1 \longrightarrow K^\times \longrightarrow K(E)^\times \xrightarrow{\text{div}} P_K(E) \longrightarrow 0.$$

In this section and the next we assume $E[N] \subset E(K)$, so in particular, K contains the N th roots of unity. Let $G \subset E[N]$ be a cyclic group of order N .

[†]For a point $P \in E(K_s)$, we denote by $\{P\}$ the base of the formal sums associated with P . Thus, the sum $\{P\} + \{Q\}$ is a formal sum, while the sum in $\{P + Q\}$ means the addition in the elliptic curve E .

We consider (2.1) as an exact sequence of G -modules. Define an action of G on various groups as follows:

- trivially on K^\times , \mathbf{Z} , and $E(K)$.
- by translation on $K(E)$; i.e., if $f \in K(E)$ and $T \in G$, define ${}^T f(X) = f(X - T)$, where X is a generic point of E .
- by translation on $\text{Div}_K(E)$; i.e., if $D = \sum_P n_P \{P\}$ and $T \in G$, define

$${}^T D = \sum_P n_P \{P + T\}.$$

Clearly, the action of G on $\text{Div}_K(E)$ induces actions on $\text{Div}_K^0(E)$ and $P_K(E)$. The Weil pairing is obtained essentially as the connecting homomorphism δ of the long exact sequence of the group cohomology induced from (2.1):

$$(2.2) \quad K^\times \longrightarrow K(E)^{\times G} \xrightarrow{\text{div}} P_K(E)^G \xrightarrow{\delta} H^1(G, K^\times).$$

Theorem 2.2 (Weil pairing). (1) *The homomorphism δ in (2.2) induces a Γ_K -isomorphism $\Psi : G' = E[N]/G \xrightarrow{\cong} \text{Hom}(G, \mu_N)$.*
 (2) *For $S' \in G'$ and $T \in G$, define $e_G(S', T)$ by*

$$e_G(S', T) = \Psi(S')(T).$$

Then e_G is a Γ_K compatible non degenerate bilinear form $G' \times G \rightarrow \mu_N$.

- (3) *For $S' \in G'$, choose $S \in E[N]$ such that $S \bmod G = S'$, and define $D_S = \{S\} - \{O\} \in \text{Div}_{K_s}^0(E)$. Choose $f_S \in K_s(E)^\times$ such that*

$$\text{div } f_S = \sum_{T \in G} {}^T D_S = \sum_{T \in G} (\{S + T\} - \{T\}).$$

Then, we have

$$e_G(S', T) = {}^T f_S / f_S = f_S(X - T) / f_S(X).$$

Proof. Since G is a subgroup of $E[N](K)$ by assumption, it follows that $\mu_N \in K^\times$, and thus we have $H^1(G, K^\times) \simeq \text{Hom}(G, \mu_N)$. The proof is a standard diagram chase. See Vélú [13, Ch. 1] for detail. \square

The usual Weil pairing $e_N : E[N] \times E[N] \rightarrow \mu_N$ is defined in a similar manner. The relation between e_G and e_N is given by

$$e_N(S, T) = e_G(\phi(S), T) \quad \text{for } S \in E[N], T \in G,$$

where ϕ is the isogeny $E \rightarrow E/G$.

2.2. Central extension $E[N](D)$. A central extension of a group G is a short exact sequence of groups $1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1$ such that A is in the center of the group H . Here, we consider central extensions of $E[N]$ by K^\times , that is, short exact sequences of groups

$$1 \longrightarrow K^\times \longrightarrow H \longrightarrow E[N] \longrightarrow 0.$$

(Note that the operation of K^\times is written multiplicatively, while that of $E[N]$ is written additively.)

Given a divisor $D \in \text{Div}_K(E)$, we may construct such a central extension.

Definition 2.3. Let $D \in \text{Div}_K(E)$ be a divisor of degree divisible by N . Define

$$E[N](D) = \{(T, f) \in E[N] \times K(E)^\times \mid \text{div } f = {}^T D - D\},$$

with a group operation on $E[N](D)$ given by

$$(S, g)(T, f) = (S + T, g \cdot {}^S f).$$

Indeed, we have a natural inclusion $K^\times \rightarrow E[N](D)$ given by $c \mapsto (O, c)$, and an exact sequence

$$1 \longrightarrow K^\times \longrightarrow E[N](D) \longrightarrow E[N] \longrightarrow 0.$$

From now on, we identify $(O, c) \in E[N](D)$ with $c \in K^\times$ and consider K^\times as a subgroup in $E[N](D)$ (to ease the notation).

Lemma 2.4. *If two divisors D_1 and D_2 are linearly equivalent, then the extensions $E[N](D_1)$ and $E[N](D_2)$ are isomorphic.*

Proof. Straightforward. □

In general $E[N](D)$ is not commutative and contains elements of infinite order. For $(S, g), (T, f) \in E[N](D)$, we denote the commutator and the N th power by

$$(2.3) \quad \langle S, T \rangle = (S, g)(T, f)(S, g)^{-1}(T, f)^{-1} = (g/{}^T g) \cdot ({}^S f/f)$$

$$(2.4) \quad v(T) = (T, f)^N = f \cdot {}^T f \cdot {}^{2T} f \cdots \cdots ({}^{(N-1)T} f).$$

Recall that we identify (O, c) with c . As this notation suggests, the facts that both $\langle S, T \rangle$ and $v(T)$ mod n th powers are independent of the choice of f and g are shown by the following proposition and lemma of Vélu.

Proposition 2.5 (Vélu [13, Prop. 2.3]). *Let D be a divisor of degree divisible by N , and let $S, T \in E[N]$. Then, $\langle S, T \rangle$ is a bilinear form on $E[N]$ with its value in μ_N , and*

$$\langle S, T \rangle = e_N(T, S)^{\frac{\deg D}{N}}.$$

Lemma 2.6 (Vélu [13, Lemma 2.4]). *Let D be a divisor of degree divisible by N , and let $T \in E[N]$. Then, $v(T)$ is in K^\times , and $v(T)$ mod $K^{\times N}$ is independent of the choice of f .*

The case where $D = N\{O\}$ is of particular interest. In this case, the results depend on the parity of N . The following lemma by Vélu will be needed in §4.2.

Lemma 2.7. *Suppose $(T, f) \in E[N](N\{O\})$, and $\tilde{T} \in E[2N]$ is a point such that $2\tilde{T} = T$. Then,*

- (1) (Vélu [13, Lemma 2.8]) $v(T) = (T, f)^N = f(\tilde{T})^N$.
- (2) (Vélu [13, Lemma 2.7]) *If $U_2 \in E[2] - \{O\}$, then*

$$U_2 f(\tilde{T}) = (-1)^N e_2(U_2, N\tilde{T}) f(\tilde{T}).$$

3. PROJECTIVE IMMERSION ASSOCIATED WITH A CYCLIC SUBGROUP

Let $D \in \text{Div}_K(E)$ be a divisor. If $\deg D \geq 3$, it is well known that the complete linear system $|D|$ gives an immersion $E \hookrightarrow |D|$ in a projective space. Here, we take a closer look at this fact. The following is a summary of results of Vélú [13].

Definition 3.1. For $D \in \text{Div}_K(E)$ and nonnegative integer d , define

$$\begin{aligned}\mathcal{L}^d(D) &= \{h \in K(E)^\times \mid \text{div } h + dD \geq 0\} \cup \{0\} \\ \mathcal{L}^*(D) &= \bigoplus_{d \geq 0} \mathcal{L}^d(D), \\ \mathcal{S}^*(D) &= \text{Sym}^* \mathcal{L}^1(D).\end{aligned}$$

Here, $\text{Sym}^* V$ means the symmetric algebra on a vector space V . By the Riemann-Roch theorem, $\mathcal{L}^d(D)$ is a K -vector space of dimension $d \deg D$, and the space $\mathcal{L}^*(D)$ is equipped naturally with a structure of graded algebra by the multiplication of functions. There is a canonical homomorphism of graded algebras $\mathcal{S}^*(D) \rightarrow \mathcal{L}^*(D)$, and let $\mathcal{I}^*(D)$ be its kernel. We have the exact sequence

$$(3.1) \quad 0 \longrightarrow \mathcal{I}^*(D) \longrightarrow \mathcal{S}^*(D) \longrightarrow \mathcal{L}^*(D).$$

Proposition 3.2. *Let D be an effective divisor of degree $N \geq 4$.*

- (1) (Vélú [13, Th. 3.3]) *The exact sequence (3.1) extends to the following exact sequence:*

$$0 \longrightarrow \mathcal{I}^*(D) \longrightarrow \mathcal{S}^*(D) \longrightarrow \mathcal{L}^*(D) \longrightarrow 0.$$

- (2) $\dim \mathcal{I}^2(D) = N(N-3)/2$.
(3) (Vélú [13, Th. 3.9]) $\mathcal{I}^*(D)$ is generated by $\mathcal{I}^2(D)$, i.e., $\mathcal{I}^*(D) = \mathcal{I}^2(D) \cdot \mathcal{S}^*(D)$.
(4) *The image of the map $E \hookrightarrow \text{Proj } \mathcal{S}^*(D) \simeq \mathbf{P}^{N-1}$ is the scheme-theoretic intersection of the quadrics that contain the image of E .*

Proof. (1) The exactness at $\mathcal{L}^*(D)$ is one of the equivalent definitions of projective normality. For a proof see Vélú [13, Th. 3.3]. See also Hartshorne [5, Ex. IV.4.2] and Mumford [11, p. 55]. The formula (2) follows immediately from (1) as $\dim \mathcal{I}^2(D) = N(N+1)/2$ and $\dim \mathcal{L}^2(D) = 2N$.

(3) See Vélú [13, Th. 3.9]. The assertion (4) is just a paraphrase of (3) as the image of E is defined by an ideal generated by $\mathcal{I}^2(D)$. \square

The central extension $E[N](D)$ has a representation on $\mathcal{L}^*(D)$. More precisely, Vélú showed the following.

Proposition 3.3 (Vélú [13, Prop. 2.13]). (1) *For $h \in \mathcal{L}^d(D)$ and $(T, f) \in E[N](D)$, the function ${}^T h \cdot f^d$ is in $\mathcal{L}^d(D)$.*

- (2) *Let $\tau_d(T, f)$ be the automorphism defined by $h \mapsto f^d \cdot {}^T h$ of $\mathcal{L}^d(D)$. Then, $\tau_d : (T, f) \mapsto \tau_d(T, f)$ is a representation of $E[N](D)$ on $\mathcal{L}^d(D)$, and the representations τ_d extend to a representation τ of $E[N](D)$ on the graded ring $\mathcal{L}^*(D)$.*

- (3) *If D is an effective divisor of degree ≥ 2 and $d \geq 1$, the kernel of the representation τ_d consists of the elements (O, f) , where f is a constant satisfying $f^d = 1$. In particular, τ_1 is faithful.*

Let $C \subset E[N]$ be a cyclic subgroup of order N . Suppose D is an effective divisor of degree N invariant under the translations by C . Then, we may choose a particular coordinate system of \mathbf{P}^{N-1} so that the immersion $E \hookrightarrow |D| \simeq \mathbf{P}^{N-1}$ is expressed in a simple way. To do so, we consider the decomposition of $\mathcal{L}^1(D)$ into eigenspaces.

Definition 3.4. For any character $\chi \in \text{Hom}(C, \mu_N)$ and a positive integer d , define

$$\mathcal{L}^d(D, \chi) = \{h \in \mathcal{L}^d(D) \mid U^d h = \chi(U) \cdot h \text{ for all } U \text{ in } C\}.$$

Since $\mathcal{L}^d(D)$ is a finite dimensional vector space on which the cyclic group C acts, it decomposes into eigenspaces. Thus, we have

$$\mathcal{L}^d(D) = \bigoplus_{\chi} \mathcal{L}^d(D, \chi), \quad \text{and} \quad \mathcal{L}^*(D) = \bigoplus_{d, \chi} \mathcal{L}^d(D, \chi).$$

Definition 3.5. Let $C \subset E[N]$ be a cyclic subgroup of order N . For any $S \in E[N]$, define the character $\chi_S \in \text{Hom}(C, \mu_N)$ by

$$\chi_S(U) = e_N(S, U) \quad \text{for all } U \in C.$$

Proposition 3.6. Let $C \subset E[N]$ be a cyclic subgroup of order N , and D an effective divisor of degree N invariant under the translations by C .

- (1) The space $\mathcal{L}^1(D, \chi)$ is 1-dimensional.
- (2) If $(S, f_S) \in E[N](D)$, then f_S is a basis of $\mathcal{L}^1(D, \chi_S)$.
- (3) The map $\tau_d(S, f_S) : \mathcal{L}^d(D, \chi) \rightarrow \mathcal{L}^d(D, \chi \chi_S^d)$ is an isomorphism for all $d \geq 0$.

Proof. Let (S, f_S) be an element of $E[N](D)$. Since $\text{div } f_S + D = {}^S D \geq 0$, f_S is in $\mathcal{L}^1(D)$. Since ${}^U D - D = 0$ for $U \in C$, we have $(U, 1) \in E[N](D)$. By (2.3), we have $\langle U, S \rangle = {}^U f_S / f_S$, and by Proposition 2.5, we have $\langle U, S \rangle = e_N(S, U) = \chi_S(U)$. Thus, we have ${}^U f_S = \chi_S(U) f_S$, which shows $f_S \in \mathcal{L}^1(D, \chi_S)$.

If $h \in \mathcal{L}^d(D, \chi)$, then

$${}^U ({}^S h \cdot f_S^d) = {}^{U+S} h \cdot ({}^U f_S)^d = \chi(U) \cdot {}^S h \cdot ({}^U f_S)^d = \chi(U) \chi_S(U)^d ({}^S h \cdot f_S^d).$$

This implies the automorphism $\tau_d(S, f_S)$ maps $\mathcal{L}^d(D, \chi)$ onto $\mathcal{L}^d(D, \chi \chi_S^d)$, and (3) is proved. As a consequence, τ_d permutes the spaces $\mathcal{L}^d(D, \chi)$ transitively as long as d is relatively prime to N , and in that case the spaces $\mathcal{L}^d(D, \chi)$ have the same dimension independent of χ . In particular, this is the case for $\mathcal{L}^1(D, \chi)$.

By the Riemann-Roch theorem, $\dim \mathcal{L}^1(D) = \deg D = N$. On the other hand

$$\begin{aligned} \dim \mathcal{L}^1(D) &= \sum_{\chi} \dim \mathcal{L}^1(D, \chi) = \# \text{Hom}(C, \mu_N) \cdot \dim \mathcal{L}^1(D, \chi_0) \\ &= N \dim \mathcal{L}^1(D, \chi_0), \end{aligned}$$

where χ_0 is the trivial character. Thus, we conclude that $\dim \mathcal{L}^1(D, \chi) = 1$ for any χ , which proves (1). Since any nonzero function in a one dimensional space is its basis, f_S is a basis of $\mathcal{L}^1(D, \chi_S)$. \square

4. CANONICAL COORDINATE SYSTEM

In this section we consider the projective immersion $E \hookrightarrow |N\{O\}| \simeq \mathbf{P}^{N-1}$. We show that, by fixing a certain level structure on E , we can choose a unique coordinate system with prescribed properties. The situation differs depending on the parity of N . The odd case is well known, but we include it here for the comparison with the even case.

4.1. Odd case. Throughout this paragraph we assume N is an odd integer ≥ 3 . This section serves as a prototype for the even case, and all the material in this section is written in Vélú [13]. See also Fisher [3], [4] for geometric treatment.

Let (S, T) be a pair of points in $E[N]$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M_2(\mathbf{Z})$. Consider the right action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on (S, T) defined by

$$(S, T) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aS + cT, bS + dT).$$

Recall that if the pair (S, T) is a basis of $E[N]$ satisfying $e_N(S, T) = \zeta$, then the pair $(S', T') = (aS + cT, bS + dT)$ is once again a basis of $E[N]$ satisfying $e_N(S', T') = \zeta$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. The kernel of this action is given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

which is the principal congruence subgroup of level N in $SL_2(\mathbf{Z})$.

Definition 4.1 ($\Gamma(N)$ structure). Fix a primitive N th root of unity ζ . A $\Gamma(N)$ -structure on an elliptic curve E is a pair of N -torsion points (S, T) satisfying $e_N(S, T) = \zeta$.

From now on we fix a primitive N th root of unity ζ in K_s , and a $\Gamma(N)$ -structure (S, T) . Define the divisor D_T by

$$D_T = \sum_{n=0}^{N-1} \{nT\}.$$

Clearly, D_T is invariant under the translation by the cyclic group $C = \langle T \rangle$.

Since $\text{Sum}_{n=0}^{N-1} nT = O$, D_T is linearly equivalent to $N\{O\}$. Choose $f_{S,T}$ such that $\text{div } f_{S,T} = {}^S D_T - D_T$. Since $(S, f_{S,T})^N \equiv 1 \pmod{K^{\times N}}$ by Lemma 2.7 (1), by multiplying $f_{S,T}$ by a suitable constant if necessary, we may assume $(S, f_{S,T})^N = 1$.

Definition 4.2. Let $f_{S,T}$ be as above. Define functions $X_0, X_1, \dots, X_{N-1} \in K(E)$ indexed by the elements $k \in \mathbf{Z}/N\mathbf{Z}$ by

$$X_0 = 1, \quad X_k = \tau_1(S, f_{S,T})X_{k-1} = {}^S X_{k-1} \cdot f_{S,T} \quad (k = 1, \dots, N-1),$$

where $\tau_1(S, f_{S,T})$ is the automorphism of $\mathcal{L}(D)$ defined in Proposition 3.3.

Lemma 4.3. (1) X_k is well-defined, i.e., $\tau_1(S, f_{S,T})X_{N-1} = X_0$.

(2) $\text{div } X_k = {}^k S D_T - D_T$ for $k \in \mathbf{Z}/N\mathbf{Z}$.

(3) The action of $(T, 1) \in E[N](D_T)$ is given by

$$\tau_1(T, 1)X_k = {}^T X_k = e_N(S, T)^k X_k.$$

- (4) X_k is a basis of $\mathcal{L}^1(D_T, \chi_S^k)$ for $k \in \mathbf{Z}/N\mathbf{Z}$.
 (5) $X_k(-P) = X_{-k}(P)$ for any $P \in E$.

Proof. The assertion (1) follows from the fact that $(S, f_{S,T})^N = 1$. By definition we have $X_k = {}^S X_{k-1} \cdot f_{S,T}$, and then the assertion (2) follows by induction on k . By Proposition 2.5, we have $\langle T, S \rangle = e_N(S, T) = \chi_S(T)$, and thus

$$\tau_1(T, 1)X_1 = \tau_1(T, 1)\tau_1(S, f_{S,T})X_0 = \chi_S(T)\tau_1(S, f_{S,T})\tau_1(T, 1)X_0 = \chi_S(T)X_1.$$

The assertion (3) then follows by induction on k , and (4) follows immediately from (3) and Proposition 3.6 (2). To prove (5), it suffices to remark $[-1]({}^k S D_T) = {}^{-k} S D_T$. This implies that the functions $X_k \circ [-1]$ and X_{-k} has the same divisor and thus $X_{-k} = c_k X_k \circ [-1]$ for some constants c_k . It is clear that $c_0 = 1$. Since we have $\tau_1(S, f_{S,T}) \circ [-1] \circ \tau_1(S, f_{S,T}) = [-1]$, we can prove $c_k = 1$ by induction on k . \square

Interpreting the above proposition geometrically, we have

Proposition 4.4. *Let E be an elliptic curve, and $E \hookrightarrow \mathbf{P}^{N-1}$ be an immersion as an elliptic normal curve of odd degree N via the complete linear system $|N\{O\}|$. Choose a primitive N th root of unity ζ and a $\Gamma(N)$ -structure (S, T) . Then, there exists a unique coordinate system of \mathbf{P}^{N-1} such that the translation maps τ_S , τ_T , and the multiplication-by- (-1) map $[-1]$ are given by the following elements M_S , M_T and $M_{[-1]}$ in $PGL_N(K(\zeta))$, respectively.*

$$(4.1) \quad M_S = \left[\begin{array}{cccc|c} 0 & \cdots & 0 & 1 & 1 \\ 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & \\ & & & & 1 \\ 0 & & & & 0 \end{array} \right], \quad M_T = \left[\begin{array}{cccc|c} 1 & & & & \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ & & & & \zeta^{N-1} \end{array} \right],$$

$$M_{[-1]} = \left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & \\ \hline 0 & & & & 1 \\ \vdots & & & & \\ 0 & 1 & & & \end{array} \right]$$

Remark 4.5. When we use the square bracket $[\]$ for matrix, we mean it is a class in PGL or PSL .

Proposition 4.6. *The notation being as above, we have the following.*

$$\begin{aligned} \mathcal{I}^2(D_T) &= \bigoplus_{k \in \mathbf{Z}/N\mathbf{Z}} \mathcal{I}^2(D_T, \chi_S^k), \\ \mathcal{I}^2(D_T, \chi_S^k) &= \mathcal{I}^2(D_T) \cap \langle X_i X_j \mid i + j \equiv k \pmod{N} \rangle, \\ \dim \mathcal{I}^2(D_T, \chi_S^k) &= (N - 3)/2, \quad \dim \mathcal{I}^2(D_T) = N(N - 3)/2. \end{aligned}$$

Proof. These assertions are the contents of §3.2 of [13] expressed in our notation. Since N is odd, there is only one χ' such that $\chi'^2 = \chi$ for any character $\chi \in \text{Hom}(C, \mu_n)$. Thus we have $\dim \mathcal{I}^2(D_T, \chi_S^k) = (N - 3)/2$ by Corollaire 3.6 of [13]. \square

4.2. Even case. We now assume that N is a positive even integer. A critical difference between this case and the last is that we have $\text{Sum}_{n=0}^{N-1} nT = T_2 \neq O$ if N is even, and thus $D_T = \sum_{n=0}^{N-1} \{nT\}$ is *not* linearly equivalent to $N\{O\}$. To define a divisor linearly equivalent to $N\{O\}$, we use a $\tilde{T} \in E[2N]$ such that $2\tilde{T} = T$.

First we fix a primitive N th root of unity ζ and a $\Gamma(N)$ -structure (S, T) , then we consider $\Gamma(2N)$ -structures on E in relation to this fixed $\Gamma(N)$ -structure. To do so, we first choose a primitive $2N$ th root of unity $\tilde{\zeta}$ such that $\tilde{\zeta}^2 = \zeta$.

Definition 4.7. Suppose (S, T) is a $\Gamma(N)$ -structure on E with $e_N(S, T) = \zeta = \tilde{\zeta}^2$.

(1) We say that a $\Gamma(2N)$ -structure (\tilde{S}, \tilde{T}) on E is *above* (S, T) if

$$2\tilde{S} = S, \quad 2\tilde{T} = T, \quad \text{and} \quad e_{2N}(\tilde{S}, \tilde{T}) = \tilde{\zeta}.$$

(2) We say that two $\Gamma(2N)$ -structures (\tilde{S}, \tilde{T}) and (\tilde{S}', \tilde{T}') are *similar* if (\tilde{S}', \tilde{T}') equals either

$$(\tilde{S}, \tilde{T}) \quad \text{or} \quad (\tilde{S}, \tilde{T}) \begin{pmatrix} 1+N & 0 \\ 0 & 1+N \end{pmatrix}.$$

We write $(\tilde{S}', \tilde{T}') \sim (\tilde{S}, \tilde{T})$.

Notation 4.8. For simplicity, we denote by U_2 the 2-torsion point $\frac{N}{2}U \in E[2]$ for any $U \in E[N]$.

Note that if (S, T) is a $\Gamma(N)$ -structure, then (S_2, T_2) is a basis of $E[2]$, and $e_2(S_2, T_2) = e_N(S, T)^{N/2} = \zeta^{N/2} = -1$.

Lemma 4.9. *For a given $\Gamma(N)$ -structure (S, T) , there are eight $\Gamma(2N)$ -structures (\tilde{S}, \tilde{T}) above (S, T) . Up to similarity, these eight are classified into four classes. If (\tilde{S}, \tilde{T}) is one of them, then the following four represent the four different classes:*

$$(\tilde{S}, \tilde{T}), \quad (\tilde{S}, \tilde{T}) \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \quad (\tilde{S}, \tilde{T}) \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}, \quad (\tilde{S}, \tilde{T}) \begin{pmatrix} 1 & N \\ N & 1 \end{pmatrix},$$

or, using Notation 4.8,

$$(\tilde{S}, \tilde{T}), \quad (\tilde{S}, \tilde{T} + S_2), \quad (\tilde{S} + T_2, \tilde{T}), \quad (\tilde{S} + T_2, \tilde{T} + S_2).$$

Proof. There are sixteen pairs (\tilde{S}, \tilde{T}) satisfying $2\tilde{S} = S$ and $2\tilde{T} = T$. Choose one pair (\tilde{S}, \tilde{T}) . Replacing \tilde{S} by $\tilde{S} + S_2$ if necessary, we may assume $e_{2N}(\tilde{S}, \tilde{T}) = \tilde{\zeta}$. Thus, we obtain at least one $\Gamma(2N)$ -structure above (S, T) . All the sixteen pairs (\tilde{S}', \tilde{T}') satisfying $2\tilde{S}' = S$ and $2\tilde{T}' = T$ are

$$(\tilde{S}', \tilde{T}') = (\tilde{S}, \tilde{T}) + (S_2, T_2) \begin{pmatrix} \epsilon_1 & \epsilon_3 \\ \epsilon_2 & \epsilon_4 \end{pmatrix}, \quad \epsilon_j = 0, \text{ or } 1 \quad (j = 1, \dots, 4).$$

Now, using properties of the Weil pairing, we have

$$\begin{aligned} e_{2N}(\tilde{S}', \tilde{T}') &= e_{2N}(\tilde{S} + \epsilon_1 S_2 + \epsilon_2 T_2, \tilde{T} + \epsilon_3 S_2 + \epsilon_4 T_2) \\ &= e_{2N}(\tilde{S}, \tilde{T}) e_{2N}(\tilde{S}, T_2)^{\epsilon_4} e_{2N}(S_2, \tilde{T})^{\epsilon_1} = (-1)^{\epsilon_1 + \epsilon_4} e_{2N}(\tilde{S}, \tilde{T}). \end{aligned}$$

Thus, $e_{2N}(\tilde{S}', \tilde{T}') = \tilde{\zeta}$ if and only if $\epsilon_1 = \epsilon_4$. In other words, there are eight pairs of (\tilde{S}, \tilde{T}) above (S, T) . Since $(\tilde{S}', \tilde{T}') \sim (\tilde{S}, \tilde{T})$ if and only if $\epsilon_1 = \epsilon_4$ and $\epsilon_2 = \epsilon_3 = 0$ by definition, we see that there are four different classes up to similarity according to $(\epsilon_2, \epsilon_3) = (0, 0), (1, 0), (0, 1),$ or $(1, 1)$. \square

In view of Lemma 4.9, we define a subgroup between $\Gamma(N)$ and $\Gamma(2N)$ as follows.

Definition 4.10. Define the subgroup $\Gamma^{(N)}(2N)$ of $\Gamma(N)$ by

$$\Gamma^{(N)}(2N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{2N} \right\}.$$

It is easy to see that we have

$$\Gamma(N) \triangleright \Gamma^{(N)}(2N) \triangleright \Gamma(2N).$$

Since N is assumed even, we have

$$\begin{aligned} \Gamma(N)/\Gamma^{(N)}(2N) &= \left\langle \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \right\rangle \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \Gamma^{(N)}(2N)/\Gamma(2N) &= \left\langle \begin{pmatrix} 1+N & 0 \\ 0 & 1+N \end{pmatrix} \right\rangle \simeq \mathbf{Z}/2\mathbf{Z}. \end{aligned}$$

Remark 4.11. (1) If N is odd, $\Gamma^{(N)}(2N)$ coincide with $\Gamma(2N)$, and $[\Gamma(N) : \Gamma^{(N)}(2N)] = [\Gamma(N) : \Gamma(2N)] = 6$.

(2) The group $\Gamma^{(N)}(2N)$ coincides with the transformation group appears in Hurwitz [6] in the case of $N \equiv 0 \pmod{4}$. If $N \equiv 2 \pmod{4}$, the group in [6] is a different group. For comparison between our immersion and that of Hurwitz, see Appendix A.

Definition 4.12 ($\Gamma^{(N)}(2N)$ -structure). Fix a primitive N th root of unity ζ , and choose $\tilde{\zeta}$ such that $\tilde{\zeta}^2 = \zeta$. Let (S, T) be a $\Gamma(N)$ -structure. By a $\Gamma^{(N)}(2N)$ -structure above the given $\Gamma(N)$ -structure (S, T) we mean an equivalence class of $\Gamma(2N)$ -structure (\tilde{S}, \tilde{T}) above (S, T) modulo similarity.

Remark 4.13. Sometimes a $\Gamma(N)$ -structure is defined as a symplectic isomorphism $\mathbf{Z}/N\mathbf{Z} \times \mu_N \rightarrow E[N]$, where the anti-symmetric pairing $\langle \cdot, \cdot \rangle$ on $\mathbf{Z}/N\mathbf{Z} \times \mu_N$ is defined by

$$\langle (a_1, \zeta_1), (a_2, \zeta_2) \rangle = \zeta_2^{a_1} / \zeta_1^{a_2}, \quad a_i \in \mathbf{Z}/N\mathbf{Z}, \quad \zeta_i \in \mu_N \quad (i = 1, 2).$$

Define a map $d : \mathbf{Z}/2N\mathbf{Z} \times \mu_{2N} \rightarrow \mathbf{Z}/N\mathbf{Z} \times \mu_N$ by

$$d : (a \pmod{2N}, \zeta^b) \mapsto (a \pmod{N}, \zeta^{2b}).$$

Then a $\Gamma^{(N)}(2N)$ -structure above a $\Gamma(N)$ -structure is nothing but a pair of symplectic isomorphisms $\phi_N : \mathbf{Z}/N\mathbf{Z} \times \mu_N \rightarrow E[N]$ and $\phi_{2N} : \mathbf{Z}/2N\mathbf{Z} \times \mu_{2N} \rightarrow E[2N]$ that makes the following diagram commutative:

$$\begin{array}{ccc} \mathbf{Z}/2N\mathbf{Z} \times \mu_{2N} & \xrightarrow{\phi_{2N}} & E[2N] \\ d \downarrow & & \downarrow [2] \\ \mathbf{Z}/N\mathbf{Z} \times \mu_N & \xrightarrow{\phi_N} & E[N] \end{array}$$

We now fix a $\Gamma^{(N)}(2N)$ -structure (\tilde{S}, \tilde{T}) above (S, T) . Define

$$D_{\tilde{T}} = \sum_{n=0}^{N-1} \{\tilde{T} + nT\}.$$

Lemma 4.14. (1) *There is a function $\varphi_{\tilde{T}}$ such that $\operatorname{div} \varphi_{\tilde{T}} = D_{\tilde{T}} - N\{O\}$. As a consequence, the divisor $D_{\tilde{T}}$ is linearly equivalent to $N\{O\}$.*

(2) *If $\tilde{T}' = \tilde{T} + T_2$, then we have $D_{\tilde{T}'} = D_{\tilde{T}}$.*

Proof. (1) Since $\operatorname{Sum}_E D_{\tilde{T}} = N\tilde{T} + \frac{1}{2}N(N-1)T = O$, the assertion follows immediately from Abel's Theorem (Theorem 2.1). The assertion (2) follows immediately from the definition. \square

Remark 4.15. We may still use $D_T = \sum_{n=1}^{N-1} \{nT\}$ to obtain an immersion to \mathbf{P}^{N-1} even though $\operatorname{Sum}_E D_T \neq O$.

Lemma 4.16. *Let (\tilde{S}, \tilde{T}) be a $\Gamma^{(N)}(2N)$ -structure above (S, T) . Choose a function $F_{\tilde{T}}$ such that $\operatorname{div} F_{\tilde{T}} = D_{\tilde{T}} - N\{O\}$. Then, there is a unique function $f_{\tilde{S}, \tilde{T}}$ satisfying the condition*

$$(4.2) \quad \operatorname{div} f_{\tilde{S}, \tilde{T}} = {}^S D_{\tilde{T}} - D_{\tilde{T}}, \quad f_{\tilde{S}, \tilde{T}}(\tilde{S}) = {}^S F_{\tilde{T}}(\tilde{S}) / F_{\tilde{T}}(\tilde{S}).$$

Moreover $(S, f_{\tilde{S}, \tilde{T}}) \in E[N](D_{\tilde{T}})$ satisfies the following.

- (1) $(S, f_{\tilde{S}, \tilde{T}})^N = 1$.
- (2) ${}^{T_2} f_{\tilde{S}, \tilde{T}}(\tilde{S}) = -f_{\tilde{S}, \tilde{T}}(\tilde{S})$.

Proof. Choose an element $(S, \varphi) \in E[N](N\{O\})$ such that $\varphi(\tilde{S}) = 1$, and define $f_{\tilde{S}, \tilde{T}} = \varphi {}^S F_{\tilde{T}} / F_{\tilde{T}}$. Then, $f_{\tilde{S}, \tilde{T}}$ satisfies the condition (4.2). Since $f_{\tilde{S}, \tilde{T}}$ does not depend on the choice of $F_{\tilde{T}}$, $f_{\tilde{S}, \tilde{T}}$ is uniquely determined. By Lemma 2.7(1), we have $(S, f_{\tilde{S}, \tilde{T}})^N = (S, \varphi {}^S F_{\tilde{T}} / F_{\tilde{T}})^N = (S, \varphi)^N = \varphi(\tilde{S})^N = 1$, which proves (1). The assertion (2) is an immediate consequence of Lemma 2.7(2). \square

Definition 4.17. Let $f_{\tilde{S}, \tilde{T}}$ be the function defined in Lemma 4.16. Define $X_k^{(\tilde{S}, \tilde{T})} \in \mathcal{L}(D_{\tilde{T}})$ indexed by $k \in \mathbf{Z}/N\mathbf{Z}$ by

$$X_0^{(\tilde{S}, \tilde{T})} = 1, \quad X_k^{(\tilde{S}, \tilde{T})} = \tau_1(S, f_{\tilde{S}, \tilde{T}}) X_{k-1}^{(\tilde{S}, \tilde{T})} \quad (k = 1, \dots, N-1).$$

Lemma 4.18. (1) $X_k^{(\tilde{S}, \tilde{T})}$ is well-defined, i.e., $\tau_1(S, f_{\tilde{S}, \tilde{T}}) X_{N-1}^{(\tilde{S}, \tilde{T})} = X_0^{(\tilde{S}, \tilde{T})}$.

- (2) $\operatorname{div} X_k^{(\tilde{S}, \tilde{T})} = {}^k S D_{\tilde{T}} - D_{\tilde{T}}$ for $k \in \mathbf{Z}/N\mathbf{Z}$.
- (3) The action of $(T, 1) \in E[N](D_{\tilde{T}})$ is given by

$$\tau_1(T, 1) X_k^{(\tilde{S}, \tilde{T})} = \chi_S^k(T) X_k^{(\tilde{S}, \tilde{T})} = e_N(S, T)^k X_k^{(\tilde{S}, \tilde{T})}.$$

- (4) $X_k^{(\tilde{S}, \tilde{T})}$ is a basis of $\mathcal{L}^1(D_{\tilde{T}}, \chi_S^k)$ for $k \in \mathbf{Z}/N\mathbf{Z}$.
- (5) $X_k^{(\tilde{S}, \tilde{T})}(-P) = X_{-k}^{(\tilde{S}, \tilde{T})}(P)$ for any $P \in E$ and $k \in \mathbf{Z}/N\mathbf{Z}$.

Proof. The proof is the same as Lemma 4.3. \square

Lemma 4.19. (1) $X_k^{(\tilde{S}, \tilde{T})}$ is unchanged if (\tilde{S}, \tilde{T}) is replaced by $(\tilde{S} + S_2, \tilde{T} + T_2)$, i.e.,

$$X_k^{(\tilde{S}+S_2, \tilde{T}+T_2)} = X_k^{(\tilde{S}, \tilde{T})}, \quad k = 0, \dots, N-1.$$

In other words, the function $X_k^{(\tilde{S}, \tilde{T})}$ is uniquely determined by the $\Gamma^{(N)}(2N)$ -structure above (S, T) ,

(2) If (\tilde{S}, \tilde{T}) is replaced by $(\tilde{S} + T_2, \tilde{T})$, then

$$X_k^{(\tilde{S}+T_2, \tilde{T})} = (-1)^k X_k^{(\tilde{S}, \tilde{T})}, \quad k = 0, \dots, N-1.$$

(3) If (\tilde{S}, \tilde{T}) is replaced by $(\tilde{S}, \tilde{T} + S_2)$, then

$$X_k^{(\tilde{S}, \tilde{T}+S_2)} = X_{N/2+k}^{(\tilde{S}, \tilde{T})} / X_{N/2}^{(\tilde{S}, \tilde{T})}, \quad k = 0, \dots, N-1.$$

Proof. (1) It suffices to show that the function $f_{\tilde{S}, \tilde{T}}$ appearing in Lemma 4.16 is unchanged. First, we have $D_{\tilde{T}+T_2} = D_{\tilde{T}}$ by Lemma 4.14 (2). According to the proof of Lemma 4.16, $f_{\tilde{S}+S_2, \tilde{T}}$ is defined by choosing $(S, \varphi') \in E[N](N\{O\})$ such that $\varphi'(\tilde{S} + S_2) = 1$ and letting $f_{\tilde{S}+S_2} = \varphi' {}^S F_{\tilde{T}} / F_{\tilde{T}}$. But, by Lemma 2.7 (2), $\varphi'(\tilde{S} + S_2) = \varphi'(\tilde{S})$. This means we may use the same φ as the one we used to define $f_{\tilde{S}, \tilde{T}}$, and thus $f_{\tilde{S}+S_2, \tilde{T}} = f_{\tilde{S}, \tilde{T}}$.

(2) By Lemma 2.7 (2), we have ${}^{T_2} f_{\tilde{S}, \tilde{T}}(\tilde{S}) = e_2(T_2, S_2) f_{\tilde{S}, \tilde{T}}(\tilde{S}) = -f_{\tilde{S}, \tilde{T}}(\tilde{S})$. So, the function $f = -{}^{T_2} f_{\tilde{S}, \tilde{T}}$ satisfies $\operatorname{div} f = {}^{S+T_2} D_{\tilde{T}} - {}^{T_2} D_{\tilde{T}} = {}^S D_{\tilde{T}} - D_{\tilde{T}}$, and $f(\tilde{S} + T_2) = f_{\tilde{S}, \tilde{T}}(\tilde{S}) = {}^S F_{\tilde{T}}(\tilde{S}) / F_{\tilde{T}}(\tilde{S})$. This implies that $-{}^{T_2} f_{\tilde{S}, \tilde{T}}$ is nothing but the function $f_{\tilde{S}+T_2, \tilde{T}}$. The assertion follows from this immediately.

(3) Let $F_{\tilde{T}}$ be a function satisfying $\operatorname{div} F_{\tilde{T}} = D_{\tilde{T}} - N\{O\}$. Since $\operatorname{div} X_{N/2}^{(\tilde{S}, \tilde{T})} = {}^{S_2} D_{\tilde{T}} - D_{\tilde{T}} = D_{\tilde{T}+S_2} - D_{\tilde{T}}$, we have $\operatorname{div} F_{\tilde{T}} X_{N/2}^{(\tilde{S}, \tilde{T})} = D_{\tilde{T}+S_2} - N\{O\}$. So, $f_{\tilde{S}, \tilde{T}+S_2}$ is defined by $f_{\tilde{S}, \tilde{T}+S_2} = f_1 \cdot (F_{\tilde{T}} X_{N/2}^{(\tilde{S}, \tilde{T})}) / (F_{\tilde{T}} X_{N/2}^{(\tilde{S}, \tilde{T})})$ where f_1 is a function satisfying $\operatorname{div} f_1 = N\{S\} - N\{O\}$ and $f_1(\tilde{S}) = 1$. Then, we have

$$f_{\tilde{S}, \tilde{T}+S_2} = f_1 \cdot {}^S F_{\tilde{T}} / F_{\tilde{T}} \cdot {}^S X_{N/2}^{(\tilde{S}, \tilde{T})} / X_{N/2}^{(\tilde{S}, \tilde{T})} = f_{\tilde{S}, \tilde{T}} {}^S X_{N/2}^{(\tilde{S}, \tilde{T})} / X_{N/2}^{(\tilde{S}, \tilde{T})} = X_{N/2+1}^{(\tilde{S}, \tilde{T})} / X_{N/2}^{(\tilde{S}, \tilde{T})}.$$

This implies

$$X_1^{(\tilde{S}, \tilde{T}+S_2)} = f_{\tilde{S}, \tilde{T}+S_2} = X_{N/2+1}^{(\tilde{S}, \tilde{T})} / X_{N/2}^{(\tilde{S}, \tilde{T})},$$

and the assertion (2) follows by induction on k . \square

Theorem 4.20. Let E be an elliptic curve, and $E \hookrightarrow \mathbf{P}^{N-1}$ an immersion as an elliptic normal curve of even degree N via the complete linear system $|N\{O\}|$. Choose a primitive N th root of unity ζ , and then choose $\tilde{\zeta}$ such that $\tilde{\zeta}^2 = \zeta$. Let (\tilde{S}, \tilde{T}) be a $\Gamma^{(N)}(2N)$ -structure above a $\Gamma(N)$ -structure (S, T) . Then, (\tilde{S}, \tilde{T}) determines a unique coordinate system of \mathbf{P}^{N-1} such that the translation maps τ_S, τ_T , and the multiplication-by- (-1) map $[-1]$ are given by the matrices M_T, M_S and $M_{[-1]}$ in (4.1), respectively.

For a given $\Gamma(N)$ -structure (S, T) , there are four different choices of such coordinate systems related by the change of coordinates of \mathbf{P}^{N-1} given by

the transition matrices generated by the following two:

$$M_S^{N/2} = \left[\begin{array}{c|c} O & I_{N/2} \\ \hline I_{N/2} & O \end{array} \right], \quad M_T^{N/2} = \begin{bmatrix} 1 & 0 & & \\ 0 & -1 & & O \\ & & \ddots & \\ & O & & 1 & 0 \\ & & & 0 & -1 \end{bmatrix},$$

Proof. The existence of such a coordinate system follows from Lemma 4.18, and the uniqueness follows from Lemma 4.19 (1). The last part is a consequence of Lemma 4.19 (2) and (3). \square

When the choice of (\tilde{S}, \tilde{T}) is understood, we write $X_k^{(\tilde{S}, \tilde{T})} = X_k$ for simplicity.

Proposition 4.21. *The notation being as above, we have the following.*

$$\begin{aligned} \mathcal{S}^2(D_{\tilde{T}}) &= \bigoplus_{k \in \mathbf{Z}/N\mathbf{Z}} \mathcal{S}^2(D_{\tilde{T}}, \chi_S^k), \\ \mathcal{S}^2(D_{\tilde{T}}, \chi_S^k) &= \mathcal{S}^2(D_{\tilde{T}}) \cap \langle X_i X_j \mid i + j \equiv k \pmod{N} \rangle, \\ \dim \mathcal{S}^2(D_{\tilde{T}}, \chi_S^k) &= \begin{cases} (N-2)/2 & \text{if } k \equiv 0 \pmod{2}, \\ (N-4)/2 & \text{if } k \equiv 1 \pmod{2}, \end{cases} \\ \dim \mathcal{S}^2(D_{\tilde{T}}) &= N(N-3)/2. \end{aligned}$$

Proof. The contents of §3.2 of [13] are still valid even if N is even. In this case there are two $\chi' \in \text{Hom}(C, \mu_n)$ such that $\chi'^2 = \chi_S^k$ if $k \equiv 0 \pmod{2}$, and there is no such χ' if $k \equiv 1 \pmod{2}$. This shows $\dim \mathcal{S}^2(D_{\tilde{T}}, \chi_S^k)$ equals as stated by Corollaire 3.6 of [13]. \square

In order to find a basis of $\mathcal{S}^2(D_{\tilde{T}})$, it suffices to find a basis of $\mathcal{S}^2(D_{\tilde{T}}, \chi_0)$ and $\mathcal{S}^2(D_{\tilde{T}}, \chi_S)$, which we will do for the case $K = \mathbf{C}$, using theta functions.

5. THETA FUNCTIONS

From now on, we assume that the base field is the field of complex numbers \mathbf{C} . We construct functions X_0, X_1, \dots, X_{N-1} defined and described in §4 using theta functions. Then, we find quadratic relations among them from the classical relations among theta functions.

Definition 5.1. For a pair of real numbers (p, q) , we define the theta function $\theta_{(p,q)}(z, \tau)$ with characteristic (p, q) by

$$\theta_{(p,q)}(z, \tau) := \sum_{n \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}(n+p)^2\tau + (n+p)(z+q)\right),$$

where $z \in \mathbf{C}$ and $\tau \in \mathfrak{H} = \{\tau \in \mathbf{C} \mid \text{Im } \tau > 0\}$, and $\mathbf{e}(x) = e^{2\pi i x}$.

This conforms with the definition in Mumford [12, Ch. I. §3]. We have the following fundamental formulas.

Proposition 5.2. *Suppose $p, q, r, s \in \mathbf{R}$, and $l, m \in \mathbf{Z}$. Then, we have*

- (1) $\theta_{(p,q)}(z+s, \tau) = \theta_{(p,q+s)}(z, \tau)$.
- (2) $\theta_{(p,q)}(z+r\tau, \tau) = \mathbf{e}\left(-\frac{1}{2}r^2\tau - rz - rq\right)\theta_{(p+r,q)}(z, \tau)$.

$$(3) \theta_{(p+l, q+m)}(z, \tau) = \mathbf{e}(pm) \theta_{(p, q)}(z, \tau).$$

Let N be a positive integer. Although we are interested mainly in the case where N is even, we do not restrict ourselves to even N until §5.4.

Definition 5.3. Let N be a positive integer. For an integer or a half-integer k , define

$$\begin{aligned} \theta_k^{(N)}(z, \tau) &:= \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}(Nz, N\tau) \\ &= \sum_{n \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}N\left(n - \frac{k}{N} + \frac{1}{2}\right)^2 \tau + N\left(n - \frac{k}{N} + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right). \end{aligned}$$

It is easy to verify that $\theta_{k+N}^{(N)}(z, \tau) = \theta_k^{(N)}(z, \tau)$, and thus $\theta_k^{(N)}(z, \tau)$ depends only on the class $k \pmod N$.

5.1. Basic properties of $\theta_k^{(N)}(z, \tau)$ as a function of z . First, we fix a positive integer N and a point $\tau \in \mathfrak{H}$.

Proposition 5.4. For any positive integer $N \in \mathbf{N}$, and any integer or half-integer $k \in \frac{1}{2}\mathbf{Z}$, the following relations hold.

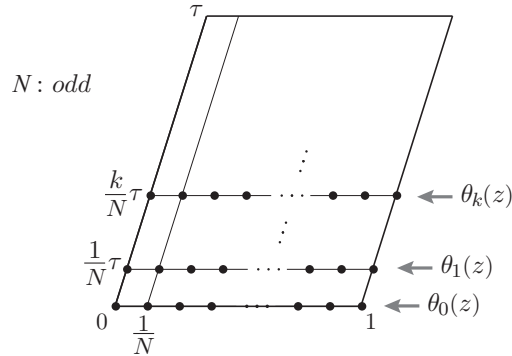
- (1) $\theta_k^{(N)}(z+1, \tau) = (-1)^{N+2k} \theta_k^{(N)}(z, \tau)$.
- (2) $\theta_k^{(N)}(z+\tau, \tau) = (-1)^N \mathbf{e}\left(-\frac{N}{2}\tau - Nz\right) \theta_k^{(N)}(z, \tau)$.
- (3) $\theta_k^{(N)}\left(z + \frac{1}{N}, \tau\right) = -\mathbf{e}\left(-\frac{k}{N}\right) \theta_k^{(N)}(z, \tau)$.
- (4) $\theta_k^{(N)}\left(z + \frac{\tau}{N}, \tau\right) = -\mathbf{e}\left(-\frac{\tau}{2N} - z\right) \theta_{k-1}^{(N)}(z, \tau)$.
- (5) $\theta_k^{(N)}\left(z + \frac{\tau}{2N}, \tau\right) = \mathbf{e}\left(-\frac{\tau}{8N} - \frac{z}{2} - \frac{1}{4}\right) \theta_{k-\frac{1}{2}}^{(N)}(z, \tau)$.
- (6) $\theta_k^{(N)}(-z, \tau) = (-1)^{N+2k} \theta_{-k}^{(N)}(z, \tau)$.

Proof. These formulas follow easily from the definition and Proposition 5.2. See also [12] for details. \square

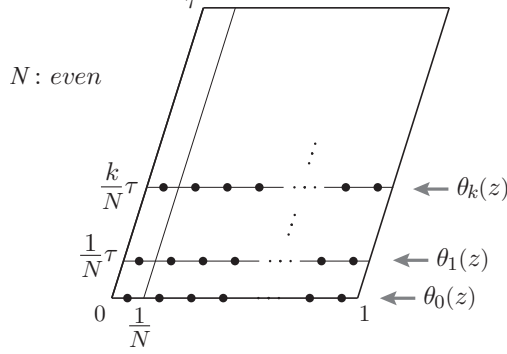
We denote $\theta_k^{(N)}(z, \tau)$ simply by $\theta_k(z)$ if no confusion arises.

Lemma 5.5. Let $Z_k^{(N)}$ be the set of zeros of $\theta_k(z)$.

- (1) If N is odd, then we have $Z_k^{(N)} = \left\{ \frac{m}{N} + \frac{k}{N}\tau + n\tau \mid m, n \in \mathbf{Z} \right\}$.



(2) If N is even, then $Z_k^{(N)} = \left\{ \frac{1}{2N} + \frac{m}{N} + \frac{k}{N}\tau + n\tau \mid m, n \in \mathbf{Z} \right\}$.



Proof. These follow easily from the fact that the set of zeros of $\theta_{(p,q)}(z, \tau)$ is given by $\left\{ (l + \frac{1}{2} - p)\tau + (m + \frac{1}{2} - q) \mid l, m \in \mathbf{Z} \right\}$. \square

Lemma 5.6. $\theta_0(z), \theta_1(z), \dots, \theta_{N-1}(z)$ are linearly independent over \mathbf{C} .

Proof. This is easily seen by looking at the series expansions of $\theta_i(z)$'s. \square

Definition 5.7. For $\tau \in \mathfrak{H}$, let $\Lambda_\tau = \langle 1, \tau \rangle$ be the lattice in \mathbf{C} spanned by 1 and τ , and let $E_\tau = \mathbf{C}/\Lambda_\tau$ be the elliptic curve with modulus τ . For $N \in \mathbf{N}$, define points $S, T \in E_\tau[N]$, and $\tilde{S}, \tilde{T} \in E_\tau[2N]$ by

$$\begin{aligned} S &= \frac{\tau}{N} \bmod \Lambda_\tau, & T &= \frac{1}{N} \bmod \Lambda_\tau, \\ \tilde{S} &= \frac{\tau}{2N} \bmod \Lambda_\tau, & \tilde{T} &= \frac{1}{2N} \bmod \Lambda_\tau. \end{aligned}$$

Lemma 5.8. Let $\zeta_N = \mathbf{e}(\frac{1}{N}) = e^{2\pi i/N}$ and $\tilde{\zeta}_N = \zeta_{2N} = \mathbf{e}(\frac{1}{2N}) = e^{2\pi i/2N}$. Then (S, T) is a level N structure with $e_N(S, T) = \zeta_N$ and (\tilde{S}, \tilde{T}) is a level $2N$ structure above (S, T) with $\tilde{\zeta}_N^2 = \zeta_N$.

Proof. Define a function $f(z)$ on \mathbf{C} by

$$f(z) = \begin{cases} \frac{\theta_1(z)}{\theta_0(z)} & \text{if } N \text{ is odd,} \\ \frac{\theta_1(z - \frac{1}{2N})}{\theta_0(z - \frac{1}{2N})} & \text{if } N \text{ is even.} \end{cases}$$

Then, the divisor of f is given by

$$\operatorname{div} f = \sum_{j=0}^{N-1} \left(\left\{ \frac{\tau}{N} + \frac{j}{N} \right\} - \left\{ \frac{j}{N} \right\} \right).$$

Thus, by Theorem 2.2 and Proposition 5.4 (3), we have

$$e_N(S, T) = e_N \left(\frac{\tau}{N} \bmod \Lambda_\tau, \frac{1}{N} \bmod \Lambda_\tau \right) = \frac{Tf(z)}{f(z)} = \frac{f(z - \frac{1}{N})}{f(z)} = \zeta_N.$$

By the same token we have $e_{2N}(\tilde{S}, \tilde{T}) = \zeta_{2N} = \tilde{\zeta}_N$, which implies (\tilde{S}, \tilde{T}) is above (S, T) . \square

5.2. Projective immersion via theta functions.

Theorem 5.9. *Let $E_\tau = \mathbf{C}/\Lambda_\tau$ be the elliptic curve with modulus $\tau \in \mathfrak{H}$, and let S, T, \tilde{S} , and \tilde{T} be as in Definition 5.7. Define functions $X_k(z)$ on \mathbf{C} indexed by $k \in \mathbf{Z}/N\mathbf{Z}$ by*

$$(5.1) \quad X_k(z) = \frac{\theta_k(z)}{\theta_0(z)} = \frac{\theta_k^{(N)}(z, \tau)}{\theta_0^{(N)}(z, \tau)} \quad k \in \mathbf{Z}/N\mathbf{Z}.$$

Then $X_k(z)$ satisfy the following.

- (1) X_k is a function on E_τ .
- (2) ${}^T X_k = \chi_S^k(T) X_k = \zeta_N^k X_k$.
- (3) ${}^S X_k = X_{k+1} \cdot (\theta_0/\theta_1)$.
- (4) $X_k(-z) = X_{-k}(z)$ for any $z \in E_\tau$.
- (5) If N is odd, then
 - (i) $\operatorname{div} X_k = {}^k S D_T - D_T$, where $D_T = \sum_{k=0}^{N-1} \{kT\}$.
 - (ii) X_0, \dots, X_{N-1} form a basis of $\mathcal{L}^1(D_T)$.
- (6) If N is even, then
 - (i) $\operatorname{div} X_k = {}^k S D_{\tilde{T}} - D_{\tilde{T}}$, where $D_{\tilde{T}} = \sum_{k=0}^{N-1} \{\tilde{T} + kT\}$.
 - (ii) X_0, \dots, X_{N-1} form a basis of $\mathcal{L}^1(D_{\tilde{T}})$.

Proof. We see that X_k are doubly periodic function with period 1 and τ by Proposition 5.4 (1) and (2). The assertions (2), (3) and (4) follow from Proposition 5.4 (3), (4) and (6), respectively. The assertions (5)(i) and (6)(i) follow from Lemma 5.5. From Lemma 5.6, the X_k 's are linearly independent. Thus, if N is odd, the functions X_0, X_1, \dots, X_{N-1} form a basis of $\mathcal{L}^1(D_T)$, and if N is even, they form a basis of $\mathcal{L}^1(D_{\tilde{T}})$. \square

Theorem 5.10. *Let $N \in \mathbf{N}$ and $\tau \in \mathfrak{H}$, and let $\Lambda_\tau = \langle 1, \tau \rangle$ be the lattice in \mathbf{C} spanned by 1 and τ .*

- (1) *The map $z \mapsto (\theta_0(z), \theta_1(z), \dots, \theta_{N-1}(z)) \in \mathbf{C}^N$ induces an immersion Θ_τ of the elliptic curve $E_\tau = \mathbf{C}/\Lambda_\tau$ into the projective space \mathbf{P}^{N-1} :*

$$\Theta_\tau : E_\tau = \mathbf{C}/\Lambda_\tau \longrightarrow \mathbf{P}^{N-1}$$

$$z \longmapsto (\theta_0(z) : \theta_1(z) : \dots : \theta_{N-1}(z)).$$

- (2) *The image $E_{\Theta_\tau} = \Theta_\tau(E_\tau)$ is a curve of degree N defined as the intersection of $N(N-3)/2$ quadrics in \mathbf{P}^{N-1} .*
- (3) *Let S, T, \tilde{S} , and \tilde{T} be as in Definition 5.7. The translations of E_{Θ_τ} by $\Theta_\tau(S)$ and $\Theta_\tau(T)$ can be extended to automorphisms of \mathbf{P}^{N-1} , and expressed by the matrices M_S and M_T in (4.1), respectively. The map $[-1]$ on E_{Θ_τ} is also extended to an automorphism of \mathbf{P}^{N-1} , and express by the matrix $M_{[-1]}$ in (4.1).*

Proof. The assertion (1) is a consequence of Theorem 5.9. The intersection between E_{Θ_τ} and the hyperplane $X_0 = 0$ in \mathbf{P}^{N-1} consists of N points since the number of the zeros of $\theta_0(0)$ equals N by Lemma 5.5. This means E_{Θ_τ} is a curve of degree N . If N is odd, the coordinate system of $E_{\Theta_\tau} \in \mathbf{P}^{N-1}$ given by Θ_τ is exactly the one described in Proposition 4.4 with $\Gamma(N)$ -structure $(\Theta_\tau(S), \Theta_\tau(T))$. If N is even, it is exactly the one described in Theorem 4.20 with $\Gamma^{(N)}(2N)$ structure $(\Theta_\tau(\tilde{S}), \Theta_\tau(\tilde{T}))$ above $(\Theta_\tau(S), \Theta_\tau(T))$. Thus, the

formulas for the translations by S and T , and the inversion $[-1]$ are as in (4.1), and the number of defining quadratic equations equals $N(N-3)/2$ by Proposition 4.6 or 4.21 depending on the parity of N . \square

In §6, we present explicit descriptions of the $N(N-3)/2$ quadrics in (2) (Theorems 6.4 and 6.8).

5.3. Transformation formula for $\theta_k^{(N)}(z, \tau)$. We now consider $\theta_k^{(N)}(z, \tau)$ as a function of τ . The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ acts on the upper half plane by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. We would like to know how this $SL_2(\mathbf{Z})$ action affects the immersion $\Theta_\tau : E_\tau \rightarrow \mathbf{P}^{N-1}$.

The following transformation formula for the theta function is classical and fundamental.

Lemma 5.11 (cf. Igusa [7, p.85]). *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $SL_2(\mathbf{Z})$. Then we have*

(5.2)

$$\theta_{(p,q)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \kappa(M) \mathbf{e}(\phi_M(p, q)) \mathbf{e}\left(\frac{cz^2}{2(cz+d)}\right) \sqrt{c\tau+d} \theta_{(p',q')}(z, \tau),$$

where $\kappa(M)$ is an eighth root of unity that depends on neither τ nor (p, q) ,

$$\begin{aligned} \phi_M(p, q) &= -\frac{1}{2}(abp^2 + 2bcpq + cdq^2 - bd(ap + cq)), \\ (p', q') &= (ap + cq - \frac{1}{2}ac, bp + dq - \frac{1}{2}bd), \end{aligned}$$

and $\sqrt{c\tau+d}$ is the principal value.

Lemma 5.12. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $\Gamma(2N)$. Then we have*

$$\theta_k^{(N)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \kappa'(M) \mathbf{e}\left(\frac{cNz^2}{2(cz+d)}\right) \sqrt{c\tau+d} \theta_k^{(N)}(z, \tau),$$

where $\kappa'(M)$ is an eighth root of unity that depends on neither τ nor k .

Proof. Since $\theta_k^{(N)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \theta_{\left(\frac{1}{2} - \frac{k}{N}, \frac{N}{2}\right)}\left(\frac{Nz}{(c/N)(N\tau)+d}, \frac{a(N\tau)+bN}{(c/N)(N\tau)+d}\right)$, we apply Lemma 5.11 with $M' = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix}$, and $(p, q) = \left(\frac{1}{2} - \frac{k}{N}, \frac{N}{2}\right)$. Then, by Proposition 5.2, we have

$$\begin{aligned} &\theta_k^{(N)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) \\ &= \kappa(M') \mathbf{e}(\phi_{M'}(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})) \mathbf{e}((\frac{1}{2} - \frac{k}{N})(q' - \frac{N}{2})) \mathbf{e}\left(\frac{cz^2}{2(cz+d)}\right) \sqrt{c\tau+d} \theta_k^{(N)}(z, \tau). \end{aligned}$$

We then see that $\kappa(M') \mathbf{e}(\phi_{M'}(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})) \mathbf{e}((\frac{1}{2} - \frac{k}{N})(q' - \frac{N}{2}))$ is an eighth root of unity that does not depend on k as long as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma(2N)$. \square

Proposition 5.13. *The following transformation formulas hold.*

- (1) $\theta_k^{(N)}(z, \tau + 1) = c \mathbf{e}\left(-\frac{k(N-k)}{2N}\right) \theta_k^{(N)}(z, \tau),$
- (2) $\theta_k^{(N)}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = c' \mathbf{e}\left(\frac{z}{2}\right) \sqrt{\frac{\tau}{N}} \sum_{j=0}^{N-1} \zeta_N^{-kj} \theta_j^{(N)}(z, \tau).$

Here, c and c' are constants independent of k , and $\zeta_N = \mathbf{e}\left(\frac{1}{N}\right)$.

Proof. (1) Since $\theta_k^{(N)}(z, \tau + 1) = \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}(Nz, N\tau + N)$, we use the transformation formula (5.2) with $M = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$. Then we have

$$\begin{aligned} \theta_k^{(N)}(z, \tau + 1) &= \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}(Nz, N\tau + N) \\ &= \kappa(M) \mathbf{e}\left(\frac{N}{2}\left(\frac{1}{4} - \frac{k^2}{N^2}\right)\right) \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2} - k)}(Nz, N\tau) \\ &= \kappa(M) \mathbf{e}\left(\frac{N}{2}\left(\frac{1}{4} - \frac{k^2}{N^2}\right)\right) \mathbf{e}\left(-\left(\frac{1}{2} - \frac{k}{N}\right)k\right) \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}(Nz, N\tau) \\ &= \kappa(M) \mathbf{e}\left(\frac{N}{8} - \frac{k(N-k)}{2N}\right) \theta_k^{(N)}(z, \tau) \\ &= c \mathbf{e}\left(-\frac{k(N-k)}{2N}\right) \theta_k^{(N)}(z, \tau). \end{aligned}$$

(2) Since $\theta_k^{(N)}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}\left(\frac{z}{\tau/N}, -\frac{1}{\tau/N}\right)$, we let $\tau' = \tau/N$ and we use the transformation formula (5.2) with $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, we have $\phi_M(p, q) = \frac{N}{4} - \frac{k}{2}$ and $(p', q') = \left(\frac{N}{2}, -\frac{1}{2} + \frac{k}{N}\right)$, and

$$\begin{aligned} \theta_k^{(N)}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) &= \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}\left(\frac{z}{\tau'}, -\frac{1}{\tau'}\right) \\ &= \kappa(M) \mathbf{e}\left(\frac{N}{4} - \frac{k}{2}\right) \mathbf{e}\left(\frac{z^2}{2\tau'}\right) \sqrt{\tau'} \theta_{(\frac{N}{2}, -\frac{1}{2} + \frac{k}{N})}(z, \tau'). \end{aligned}$$

We compute the series of $\theta_{(\frac{N}{2}, -\frac{1}{2} + \frac{k}{N})}(z, \tau')$ by splitting it into N parts:

$$\begin{aligned} \theta_{(\frac{N}{2}, -\frac{1}{2} + \frac{k}{N})}(z, \tau') &= \sum_{n \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}\left(n + \frac{N}{2}\right)^2 \frac{\tau'}{N} + \left(n + \frac{N}{2}\right)\left(z - \frac{1}{2} + \frac{k}{N}\right)\right) \\ &= \sum_{j=0}^{N-1} \left(\sum_{n+j \equiv 0 \pmod{N}} \mathbf{e}\left(\frac{1}{2}\left(n + \frac{N}{2}\right)^2 \frac{\tau'}{N} + \left(n + \frac{N}{2}\right)\left(z - \frac{1}{2} + \frac{k}{N}\right)\right) \right). \end{aligned}$$

Since $n + j \equiv 0 \pmod{N}$ if and only if $n = Nm - j$ for some $m \in \mathbf{Z}$, we have

$$\begin{aligned} &\sum_{n+j \equiv 0 \pmod{N}} \mathbf{e}\left(\frac{1}{2}\left(n + \frac{N}{2}\right)^2 \frac{\tau'}{N} + \left(n + \frac{N}{2}\right)\left(z - \frac{1}{2} + \frac{k}{N}\right)\right) \\ &= \sum_{m \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}\left((Nm - j) + \frac{N}{2}\right)^2 \frac{\tau'}{N} + \left((Nm - j) + \frac{N}{2}\right)\left(z - \frac{1}{2} + \frac{k}{N}\right)\right) \\ &= \sum_{m \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}N\left(m - \frac{j}{N} + \frac{1}{2}\right)^2 \tau + N\left(m - \frac{j}{N} + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right) \\ &\quad + \left(Nm - j + \frac{N}{2}\right)\left(\frac{k}{N} - 1\right) \\ &= \sum_{m \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}N\left(m - \frac{j}{N} + \frac{1}{2}\right)^2 \tau + N\left(m - \frac{j}{N} + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right) \mathbf{e}\left(-\frac{jk}{N} + \frac{k-N}{2}\right) \\ &= \mathbf{e}\left(\frac{k-N}{2}\right) \zeta_N^{-jk} \theta_j^{(N)}(z, \tau) \end{aligned}$$

Therefore,

$$\begin{aligned} \theta_k^{(N)}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) &= \kappa(M) \mathbf{e}\left(\frac{N}{4} - \frac{k}{2}\right) \mathbf{e}\left(\frac{Nz^2}{2\tau}\right) \sqrt{\frac{\tau}{N}} \theta_{(\frac{N}{2}, -\frac{1}{2} + \frac{k}{N})}(z, \tau') \\ &= \kappa(M) \mathbf{e}\left(-\frac{N}{4}\right) \mathbf{e}\left(\frac{Nz^2}{2\tau}\right) \sqrt{\frac{\tau}{N}} \sum_{j=0}^{N-1} \zeta_N^{-kj} \theta_j^{(N)}(z, \tau). \quad \square \end{aligned}$$

Corollary 5.14. *The action $\tau \mapsto \tau + 1$ induces the change of coordinates of \mathbf{P}^{N-1} given by the following transition matrices:*

$$P_A = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \zeta_{2N}^{-1(N-1)} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & \zeta_{2N}^{-k(N-k)} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \zeta_{2N}^{-(N-1)\cdot 1} \end{bmatrix} = \text{Diag}(\zeta_{2N}^{k(N-k)})_{0 \leq k \leq N-1}.$$

Similarly, $\tau \mapsto -1/\tau$ induces the change of coordinates given by:

$$P_B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_N^{-1} & \zeta_N^{-2} & \cdots & \zeta_N^{-(N-1)} \\ 1 & \zeta_N^{-2} & \zeta_N^{-4} & \cdots & \zeta_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_N^{-(N-1)} & \zeta_N^{-2(N-1)} & \cdots & \zeta_N^{-(N-1)^2} \end{bmatrix} = (\zeta_N^{-kj})_{0 \leq k, j \leq N-1}.$$

Here, we mean by the transition matrix from the old coordinates $(X_0 : X_1 : \cdots : X_{N-1})$ to the new one $(X'_0 : X'_1 : \cdots : X'_{N-1})$ the matrix P that satisfies ${}^t(X'_0, X'_1, \dots, X'_{N-1}) = P {}^t(X_0, X_1, \dots, X_{N-1})$.

Proof. Follows immediately from Proposition 5.13. \square

From this Corollary, we obtain the representation

$$\rho : SL_2(\mathbf{Z}) \rightarrow PGL_N(\mathbf{C})$$

determined by

$$\rho\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = P_A \quad \text{and} \quad \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = P_B.$$

For this representation, we have the following.

Theorem 5.15. *Let P_A and P_B be the matrices in Corollary 5.14, and let M_S , M_T and $M_{[-1]}$ be matrices in (4.1).*

- (1) *The representation ρ factors through $SL_2(\mathbf{Z})/\Gamma(N)$ if N is odd, and $SL_2(\mathbf{Z})/\Gamma^{(N)}(2N)$ if N is even.*
- (2) *We have the relations*

$$\begin{aligned} \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) M_T^m M_S^n \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} &\equiv M_T^{am+bn} M_S^{cm+dn}, \quad m, n \in \mathbf{Z}/N\mathbf{Z}, \\ \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) M_{[-1]} \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} &\equiv M_{[-1]}, \end{aligned}$$

where \equiv means two matrices are equivalent in $PGL_N(\mathbf{C})$.

Proof. (1) By Lemma 5.12, we see that $\ker \rho$ contains $\Gamma(2N)$. If N is odd, then $\Gamma(N)/\Gamma^{(N)}(2N) = \Gamma(N)/\Gamma(2N) \simeq \mathfrak{S}_3$ (the symmetric group of degree 3) is generated by $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$. Since $\rho\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = P_B^{-1} P_A^{-1} P_B$, we have $\rho\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \equiv \rho\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) \equiv I_N$ by straightforward calculations using Corollary 5.14. Hence we conclude that $\ker \rho$ contains $\Gamma(N)$.

If N is even, however, we have $\rho\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \equiv M_T^{\frac{N}{2}}$ and $\rho\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) \equiv M_S^{\frac{N}{2}}$, where $M_T^{\frac{N}{2}}$ and $M_S^{\frac{N}{2}}$ are matrices appearing in Theorem 4.20. By a simple calculation, we have

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1+N & 0 \\ 0 & 1+N \end{pmatrix} \pmod{2N},$$

if N is even. Again, by straightforward calculations, we have

$$P_A^N (P_B^{-1} P_A^{-1} P_B)^{N-1} P_A^N (P_B^{-1} P_A^{-1} P_B) \equiv I_N.$$

This proves $\rho\left(\begin{smallmatrix} 1+N & 0 \\ 0 & 1+N \end{smallmatrix}\right) \equiv I_N$, and we see that $\ker \rho$ contains $\Gamma^{(N)}(2N)$.

(2) By direct calculations, we have

$$\begin{aligned} P_A M_T P_A^{-1} &\equiv M_T, & P_A M_S P_A^{-1} &\equiv M_S M_T, \\ P_B M_T P_B^{-1} &\equiv M_S, & P_B M_S P_B^{-1} &\equiv M_T^{-1}, \end{aligned}$$

Since $M_T M_S \equiv M_S M_T$ in $PGL_N(\mathbf{C})$, we obtain the first relation in the statement. As for the second relation, it suffices to verify $P_A M_{[-1]} P_A^{-1} \equiv M_{[-1]}$ and $P_B M_{[-1]} P_B^{-1} \equiv M_{[-1]}$, which can be done easily. \square

5.4. Modular curves. In this paragraph we assume N is even. Let $Y^{(N)}(2N) = \Gamma^{(N)}(2N) \backslash \mathfrak{H}$ be the modular curve associated with $\Gamma^{(N)}(2N)$. Let $E^{(N)}(2N)$ be the universal elliptic curve over $Y^{(N)}(2N)$, that is, the fibration $E^{(N)}(2N) \rightarrow Y^{(N)}(2N)$ such that the fiber at $\tau \in Y^{(N)}(2N)$ is the elliptic curve $E_\tau = \mathbf{C}/\Lambda_\tau$. By Theorem 5.10, we have an immersion

$$\begin{aligned} \Theta : E^{(N)}(2N) &\hookrightarrow \mathbf{P}^{N-1} \times Y^{(N)}(2N) \\ (z \bmod \Lambda_\tau, \tau) &\longmapsto (\Theta_\tau(z), \tau). \end{aligned}$$

Let o be the 0-section $Y^{(N)}(2N) \rightarrow E^{(N)}(2N)$. We obtain a morphism $\alpha : Y^{(N)}(2N) \rightarrow \mathbf{P}^{N-1}$ by the following diagram

$$\begin{array}{ccc} E^{(N)}(2N) & \xrightarrow{\Theta} & \mathbf{P}^{N-1} \times Y^{(N)}(2N) \\ o \uparrow & & \downarrow \pi_1 \\ Y^{(N)}(2N) & \xrightarrow{\alpha} & \mathbf{P}^{N-1}. \end{array}$$

where $\pi_1 : \mathbf{P}^{N-1} \times Y^{(N)}(2N) \rightarrow \mathbf{P}^{N-1}$ is the projection onto the first factor. By taking its closure we also obtain $X^{(N)}(2N) = \overline{Y^{(N)}(2N)} \rightarrow \mathbf{P}^{N-1}$. In terms of coordinates, the morphism α is given by $\tau \mapsto (a_0^{(N)}(\tau), \dots, a_{N-1}^{(N)}(\tau))$, where $a_k^{(N)}(\tau)$ is the ‘‘Theta Null Werte’’ defined by

$$a_k^{(N)}(\tau) := \theta_k^{(N)}(0, \tau) = \theta_{\left(\frac{1}{2} - \frac{k}{N}, \frac{N}{2}\right)}(0, N\tau) \quad (k \in \mathbf{Z}).$$

We will see later in §6 that the image in \mathbf{P}^{N-1} satisfies a set of quartic equations. It is in general a difficult question whether these equations define $X^{(N)}(2N)$. We will discuss this question in each of examples $N = 4, 6$, and 8 later.

By using theta transformation formula (Lemma 5.11), we can describe the transformation properties of $a_k^{(N)}(\tau)$ under the action of the group $\Gamma(N)/\Gamma^{(N)}(2N)$. By Proposition 5.4 (6), we have

$$a_k^{(N)}(\tau) = (-1)^N a_{-k}^{(N)}(\tau) = a_{N-k}^{(N)}(\tau).$$

(Remember that N is even.) This implies that the image of α is of the form

$$(5.3) \quad (a_0(\tau) : a_1(\tau) : \cdots : a_{\frac{N}{2}-1}(\tau) : a_{\frac{N}{2}}(\tau) : a_{\frac{N}{2}+1}(\tau) : \cdots : a_1(\tau)).$$

In other words, it is contained in the linear subspace H in \mathbf{P}^{N-1} defined by the equations $X_k = X_{N-k}$, $k = 1, \dots, \frac{N}{2} - 1$. The linear space H is stable under the representation $\rho : SL_2(\mathbf{Z}) \rightarrow PGL_N(\mathbf{C})$ in Theorem 5.15. Moreover, since $\rho(-I_2) = M_{[-1]}$ acts trivially on H , and $\ker \rho \supset \Gamma^{(N)}(2N)$, ρ induces a representation

$$\bar{\rho} : SL_2(\mathbf{Z})/\Gamma^{(N)}(2N) \rightarrow PGL_{\frac{N}{2}+1}(\mathbf{C}).$$

Define the coordinates $(\bar{X}_0 : \bar{X}_1 : \cdots : \bar{X}_{\frac{N}{2}})$ of H by

$$\begin{cases} \bar{X}_0 = X_0, \\ \bar{X}_1 = X_1 + X_{N-1}, \\ \cdots \\ \bar{X}_{\frac{N}{2}-1} = X_{\frac{N}{2}-1} + X_{\frac{N}{2}+1}, \\ \bar{X}_{\frac{N}{2}} = X_{\frac{N}{2}}. \end{cases}$$

Then, $\bar{\rho}$ is given by

$$\bar{\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & \zeta + \zeta^{-1} & \zeta^2 + \zeta^{-2} & \cdots & -2 \\ 2 & \zeta^2 + \zeta^{-2} & \zeta^4 + \zeta^{-4} & \cdots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 1 & \cdots & (-1)^{\frac{N}{2}} \end{bmatrix},$$

$$\bar{\rho} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & & & & \\ \tilde{\zeta}^{1(N-1)} & & & & \\ & \tilde{\zeta}^{2(N-2)} & & & \\ & & \ddots & & \\ & & & \tilde{\zeta}^{N^2/4} & \end{bmatrix}.$$

In particular, we have

$$\bar{\rho} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & (-1)^{\frac{N}{2}} \end{bmatrix}, \quad \bar{\rho} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{bmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \\ & & & & \\ 1 & 1 & \cdots & & \end{bmatrix}.$$

Thus, we have the following.

Proposition 5.16. *Let N be an even positive integer. Write $a_k^{(N)}(\tau) = a_k$ for short. Then, the ratio $(a_0 : a_1 : \cdots : a_{\frac{N}{2}})$ is invariant under the action of $\Gamma^{(N)}(2N)$. Moreover, the following holds.*

(1) The action $\tau \mapsto \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \tau = \tau + N$ induces the action

$$(a_0 : a_1 : \cdots : a_k : \cdots : a_{\frac{N}{2}}) \mapsto (a_0 : -a_1 : \cdots : (-1)^k a_k : \cdots : (-1)^{\frac{N}{2}} a_{\frac{N}{2}}).$$

(2) The action $\tau \mapsto \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \tau = \frac{\tau}{N\tau+1}$ induces the action

$$(a_0 : a_1 : \cdots : a_k : \cdots : a_{\frac{N}{2}}) \mapsto (a_{\frac{N}{2}} : a_{\frac{N}{2}-1} : \cdots : a_{\frac{N}{2}-k} : \cdots : a_0).$$

6. QUADRATIC EQUATIONS SATISFIED BY THETA FUNCTIONS

Let E be an elliptic curve over a field K contained in \mathbf{C} . If we immerse E via the complete linear system $|N\{O_E\}|$ for some integer N , there exists a coordinate system $(X_0 : X_1 : \cdots : X_{N-1}) \in \mathbf{P}^{N-1}$ described in Proposition 4.4 or Theorem 4.20. Then, by a suitable choice of $\tau \in \mathfrak{H}$, we have a map $\mathbf{C}/\Lambda_\tau \rightarrow E \subset \mathbf{P}^{N-1}$ given by $X_k = \theta_k^{(N)}(z, \tau)$. In order to describe $N(N-3)/2$ quadratic equations satisfied by E explicitly, we would like to find relations satisfied among $\theta_k^{(N)}(z, \tau)$. It turns out that the situation is quite different depending on the parity of N .

We use the method of Jacobi [8], which is completely elementary and algebraic (i.e., no function theory is used).

Definition 6.1. Jacobi's basic theta functions $\vartheta_i(z)$ ($i = 0, 1, 2, 3$) are defined by the following formulas:

$$\begin{aligned} \vartheta_0(z) &= \theta_{(0, \frac{1}{2})}(z, \tau) = \sum_{n \in \mathbf{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + n(z + \frac{1}{2})), \\ \vartheta_1(z) &= \theta_{(\frac{1}{2}, \frac{1}{2})}(z, \tau) = \sum_{n \in \mathbf{Z}} \mathbf{e}(\frac{1}{2}(n + \frac{1}{2})^2\tau + (n + \frac{1}{2})(z + \frac{1}{2})), \\ \vartheta_2(z) &= \theta_{(\frac{1}{2}, 0)}(z, \tau) = \sum_{n \in \mathbf{Z}} \mathbf{e}(\frac{1}{2}(n + \frac{1}{2})^2\tau + (n + \frac{1}{2})z), \\ \vartheta_3(z) &= \theta_{(0, 0)}(z, \tau) = \sum_{n \in \mathbf{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz). \end{aligned}$$

Note that our definition differs slightly from Jacobi's original notation by some rescaling and sign. Although we should write $\vartheta_i(z, \tau)$ instead of $\vartheta_i(z)$, we omit τ for simplicity.

The function $\vartheta_1(z)$ is an odd function and the others are even functions:

$$(6.1) \quad \begin{aligned} \vartheta_0(-z) &= \vartheta_0(z), & \vartheta_1(-z) &= -\vartheta_1(z), \\ \vartheta_2(-z) &= \vartheta_2(z), & \vartheta_3(-z) &= \vartheta_3(z). \end{aligned}$$

The following formulas are immediate from the definition.

$$(6.2) \quad \begin{aligned} \vartheta_0(z + \frac{1}{2}) &= \vartheta_3(z), & \vartheta_1(z + \frac{1}{2}) &= -\vartheta_2(z), \\ \vartheta_2(z + \frac{1}{2}) &= \vartheta_1(z), & \vartheta_3(z + \frac{1}{2}) &= \vartheta_0(z). \end{aligned}$$

The starting point is the identity (A)-(1) in Jacobi [8, p.507].

For independent variables w, x, y, z , define variables w', x', y', z' by

$$(6.3) \quad \begin{cases} w' = \frac{1}{2}(w + x + y + z), \\ x' = \frac{1}{2}(w + x - y - z), \\ y' = \frac{1}{2}(w - x + y - z), \\ z' = \frac{1}{2}(w - x - y + z), \end{cases} \quad \text{or} \quad \begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}.$$

Denote by A the 4×4 matrix (including the factor $1/2$) in the second equation of (6.3). Then, we have $A \in O(4)$, $A^2 = I$, and $\det A = -1$. This shows that the transformation given by (6.3) induces an involution on the set

$$\{(w, x, y, z) \in \mathbf{Z}^4 \mid w \equiv x \equiv y \equiv z \pmod{2}\}$$

and preserving the norm $w^2 + x^2 + y^2 + z^2$.

Proposition 6.2. *Let w, x, y, z be independent variables, and let w', x', y', z' be the variables defined by (6.3). Then, the following identity holds:*

$$(6.4) \quad \begin{aligned} & \vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z) + \vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z) \\ &= \vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z') + \vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z'), \end{aligned}$$

Proof. Jacobi's identity (6.4) follows from the properties of (6.3). The reader is encouraged to consult the beautiful, original article of Jacobi [8]. \square

Theorem 6.3. *Let w, x, y, z be independent variables. As in Proposition 6.2, define w', x', y', z' by (6.3). Furthermore, define variables w'', x'', y'', z'' by*

$$(6.5) \quad {}^t(w'', x'', y'', z'') = A {}^t(w, x, y, -z).$$

Then, the following three-term identity holds:

$$(6.6) \quad \begin{aligned} & \vartheta_1(w)\vartheta_1(x)\vartheta_1(y)\vartheta_1(z) \\ &+ \vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z') \\ &- \vartheta_0(w'')\vartheta_0(x'')\vartheta_0(y'')\vartheta_0(z'') = 0. \end{aligned}$$

Proof. Replacing w by $w + 1$ in (6.4) and using (6.2), we obtain

$$(6.7) \quad \begin{aligned} & \vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z) - \vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z) \\ &= \vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z') + \vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z'). \end{aligned}$$

Adding (6.4) and (6.7), we obtain

$$\begin{aligned} & 2\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z) \\ &= \vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z') + \vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z') \\ &+ \vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z') + \vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z'). \end{aligned}$$

Replace w, x, y, z by $w + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}, z + \frac{1}{2}$ in this identity. Then w' becomes $w' + 1$, and x', y', z' unchanged. By (6.2) we obtain

$$(6.8) \quad \begin{aligned} & 2\vartheta_0(w)\vartheta_0(x)\vartheta_0(y)\vartheta_0(z) \\ &= \vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z') - \vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z') \\ &- \vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z') + \vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z'). \end{aligned}$$

Since the relation between w, x, y, z and w', x', y', z' are symmetric, we have

$$(6.9) \quad \begin{aligned} 2\vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z') \\ = \vartheta_0(w)\vartheta_0(x)\vartheta_0(y)\vartheta_0(z) - \vartheta_1(w)\vartheta_1(x)\vartheta_1(y)\vartheta_1(z) \\ - \vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z) + \vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z), \end{aligned}$$

By the definition of the transformation (6.5), the identity (6.9) can be translated to

$$(6.10) \quad \begin{aligned} 2\vartheta_0(w'')\vartheta_0(x'')\vartheta_0(y'')\vartheta_0(z'') \\ = \vartheta_0(w)\vartheta_0(x)\vartheta_0(y)\vartheta_0(-z) - \vartheta_1(w)\vartheta_1(x)\vartheta_1(y)\vartheta_1(-z) \\ - \vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(-z) + \vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(-z). \end{aligned}$$

Now, calculating (6.9) – (6.10), and using (6.1), we obtain the relation (6.6). \square

6.1. Odd case. Assume N is odd.

From our definition of $\theta_k^{(N)}(z, \tau)$ and Proposition 5.2, we have the relations

$$\theta_0^{(N)}(z, \tau) = (-1)^{\frac{N-1}{2}} \vartheta_1(Nz, N\tau) \quad \text{and} \quad \theta_{\frac{N}{2}}^{(N)}(z, \tau) = \vartheta_0(Nz, N\tau).$$

Dropping the superscript “ (N) ”, (6.6) is translated to

$$(6.11) \quad \begin{aligned} \theta_0(w)\theta_0(x)\theta_0(y)\theta_0(z) \\ + \theta_{\frac{N}{2}}(w')\theta_{\frac{N}{2}}(x')\theta_{\frac{N}{2}}(y')\theta_{\frac{N}{2}}(z') - \theta_{\frac{N}{2}}(w'')\theta_{\frac{N}{2}}(x'')\theta_{\frac{N}{2}}(y'')\theta_{\frac{N}{2}}(z'') = 0. \end{aligned}$$

We use this equation to obtain our quadratic equations.

Theorem 6.4. *Suppose N is an odd integer. Let E_{Θ_τ} be the image of Θ_τ in Theorem 5.10. Let $V = H^0(\mathbf{P}^{N-1}, \mathcal{I}_{E_{\Theta_\tau}}(2))$, and $V_k = V \cap \langle X_i X_j \mid i+j \equiv k \pmod{N} \rangle$, where X_i ($i = 0, \dots, N-1$) are the coordinate functions of \mathbf{P}^{N-1} in Proposition 4.4. Let $a_i = \theta_i^{(N)}(0, \tau)$. Then,*

- (1) *The vector space V_k is isomorphic to V_0 for all $k = 0, \dots, N-1$ by the map $X_i X_j \mapsto X_{i+k/2} X_{j+k/2}$, where indices are taken as elements of $\mathbf{Z}/N\mathbf{Z}$.*
- (2) *The quadratic forms*

$$(6.12) \quad \begin{aligned} a_{j+1}a_{N-j}X_0^2 - a_{\frac{N-1}{2}-j}a_{\frac{N+1}{2}+j}X_{\frac{N+1}{2}}X_{\frac{N-1}{2}} + a_{\frac{N-1}{2}}a_{\frac{N+1}{2}}X_{\frac{N-1}{2}-j}X_{\frac{N+1}{2}+j} \\ \text{with } j = 1, \dots, \frac{N-3}{2} \text{ form a basis of the vector space } V_0. \end{aligned}$$

Proof. (1) Since N is odd, 2 is invertible in $\mathbf{Z}/N\mathbf{Z}$ and thus $k/2$ in the indices makes sense. The isomorphism is nothing but the translation map $\tau_S^{k/2}$ in Proposition 4.4.

(2) First, we show that if we replace X_i by $\theta_i^{(N)}(z, \tau)$, the quadratic forms in (6.12) vanishes. In (6.11), let

$$(w, x, y, z) = \left(z, z, \frac{j\tau}{N}, -\frac{(j+1)\tau}{N} \right).$$

Then, we have

$$(w', x', y', z') = \left(z - \frac{\tau}{2N}, z + \frac{\tau}{2N}, \frac{(2j+1)\tau}{2N}, -\frac{(2j+1)\tau}{2N} \right),$$

$$(w'', x'', y'', z'') = \left(z + \frac{(2j+1)\tau}{2N}, z - \frac{(2j+1)\tau}{2N}, -\frac{\tau}{2N}, \frac{\tau}{2N} \right).$$

Then, use Proposition 5.4 (4) to see that (6.12) is satisfied by $X_i = \theta_i^{(N)}(z, \tau)$ and $a_i = \theta_i^{(N)}(0, \tau)$. Recall that a_k depends only on $k \bmod N$, and we have $a_0 = 0$ and $a_k = -a_{N-k}$. Thus, we may restrict the range of j to $1 \leq j \leq (N-3)/2$. ($j = (N-1)/2$ gives the trivial relation.)

Next we show that these $(N-3)/2$ quadratic forms are independent. Indeed, the coefficients of the terms $X_{\frac{N-1}{2}-j} X_{\frac{N+1}{2}+j}$ for $1 \leq j \leq (N-3)/2$ are all equal to $a_{\frac{N-1}{2}} a_{\frac{N+1}{2}}$, which is nonzero, and the terms $X_0^2, X_{\frac{N+1}{2}} X_{\frac{N-1}{2}}$ are different from $X_{\frac{N-1}{2}-j} X_{\frac{N+1}{2}+j}$ for $1 \leq j \leq (N-3)/2$. Therefore, these $(N-3)/2$ quadratic forms are independent. Since $\dim V_0 = (N-3)/2$ by Propositions 4.6, these $(N-3)/2$ quadratic forms are basis of V_0 . \square

Remark 6.5. By specializing X_i at $z = 0$ in (6.12), or sending $X_i \mapsto a_i$, we obtain quartic relations in a_j 's. These quartic equations are satisfied by the image of the morphism $\alpha : Y(N) \rightarrow \mathbf{P}^{N-1}$ in §5.4.

We briefly describe below the classical examples $N = 5$ and 7.

Example 6.6. When $N = 5$, (6.12) yields $(5-3)/2 = 1$ quadratic form, which is

$$a_1 a_2 X_0^2 - a_1^2 X_2 X_3 + a_2^2 X_1 X_4.$$

$V = H^0(\mathbf{P}^{N-1}, \mathcal{I}_{E_{\Theta\tau}}(2))$ is generated by this form and its various permutations by M_S in Proposition 4.4. If we define $\phi(\tau)$ by

$$\phi(\tau) = -\frac{a_1(\tau)}{a_2(\tau)} = q^{\frac{1}{5}} - q^{\frac{6}{5}} + q^{\frac{11}{5}} - q^{\frac{21}{5}} + q^{\frac{26}{5}} - q^{\frac{31}{5}} + \dots \quad (q = \mathbf{e}(\tau) = e^{2\pi i\tau}),$$

we have the equations

$$X_i^2 + \phi(\tau) X_{i+2} X_{i-2} - \frac{1}{\phi(\tau)} X_{i+1} X_{i-1} = 0 \quad (i = 0, \dots, 4).$$

This set of equations is the well-known Bianchi normal form [1].

Example 6.7. When $N = 7$, quadratic forms (6.12) give two equations

$$\begin{aligned} a_1 a_2 X_0^2 - a_2^2 X_3 X_4 + a_3^2 X_2 X_5 &= 0, \\ a_2 a_3 X_0^2 - a_1^2 X_3 X_4 + a_3^2 X_1 X_6 &= 0. \end{aligned}$$

By the specialization $z = 0$ in X_i after a suitable shift using Proposition 5.4 (4), that is, sending $X_i \mapsto a_{i+j}$ for some j , we obtain the unique relation

$$a_1^3 a_2 = a_2^3 a_3 + a_1 a_3^3.$$

This is the renowned Klein's quartic, which is a model of the modular curve $X(7)$ of genus 3.

6.2. Even case. Next suppose N is even. Again we start with Jacobi's (6.4), which in terms of our theta's is written as

$$\begin{aligned} (6.13) \quad & \theta_{\frac{N}{2}}(w) \theta_{\frac{N}{2}}(x) \theta_{\frac{N}{2}}(y) \theta_{\frac{N}{2}}(z) + \theta_0(w) \theta_0(x) \theta_0(y) \theta_0(z) \\ & = \theta_{\frac{N}{2}}(w') \theta_{\frac{N}{2}}(x') \theta_{\frac{N}{2}}(y') \theta_{\frac{N}{2}}(z') + \theta_0(w') \theta_0(x') \theta_0(y') \theta_0(z'). \end{aligned}$$

We can deduce our quadratic relations directly from this with the following specializations. For an integer j , consider the four substitutions

$$(w, x, y, z) = \left(z, -\frac{j\tau}{N}, -\frac{j\tau}{N}, z \right), \quad \left(-\frac{2j\tau}{N}, 0, z, z \right), \\ \left(z - \frac{\tau}{N}, -\frac{j\tau}{N}, -\frac{(j+1)\tau}{N}, z \right), \quad \left(-\frac{(2j+1)\tau}{N}, 0, z, z - \frac{\tau}{N} \right),$$

which, respectively, yield

$$(w', x', y', z') = \left(z - \frac{j\tau}{N}, 0, 0, z + \frac{j\tau}{N} \right), \quad \left(z - \frac{j\tau}{N}, -z - \frac{j\tau}{N}, -\frac{j\tau}{N}, -\frac{j\tau}{N} \right), \\ \left(z - \frac{(j+1)\tau}{N}, 0, -\frac{\tau}{N}, z + \frac{j\tau}{N} \right), \quad \left(z - \frac{(j+1)\tau}{N}, -z - \frac{j\tau}{N}, -\frac{j\tau}{N}, -\frac{(j+1)\tau}{N} \right).$$

Applying these to (6.13) and using Proposition 5.4, we obtain the following set of equations.

$$(6.14) \quad \begin{cases} a_j^2 X_0^2 + a_{\frac{N}{2}+j}^2 X_{\frac{N}{2}}^2 = a_0^2 X_j X_{N-j} + a_{\frac{N}{2}}^2 X_{\frac{N}{2}+j} X_{\frac{N}{2}-j}, \\ a_0 a_{2j} X_0^2 + a_{\frac{N}{2}+2j} X_{\frac{N}{2}}^2 = a_j^2 X_j X_{N-j} + a_{\frac{N}{2}+j}^2 X_{\frac{N}{2}+j} X_{\frac{N}{2}-j}, \end{cases}$$

$$(6.15) \quad \begin{cases} a_j a_{j+1} X_0 X_1 + a_{\frac{N}{2}+j} a_{\frac{N}{2}+j+1} X_{\frac{N}{2}} X_{\frac{N}{2}+1} \\ \quad = a_0 a_1 X_{j+1} X_{N-j} + a_{\frac{N}{2}} a_{\frac{N}{2}+1} X_{\frac{N}{2}+j+1} X_{\frac{N}{2}-j}, \\ a_0 a_{2j+1} X_0 X_1 + a_{\frac{N}{2}} a_{\frac{N}{2}+2j+1} X_{\frac{N}{2}} X_{\frac{N}{2}+1} \\ \quad = a_j a_{j+1} X_{j+1} X_{N-j} + a_{\frac{N}{2}+j} a_{\frac{N}{2}+j+1} X_{\frac{N}{2}+j+1} X_{\frac{N}{2}-j}. \end{cases}$$

Here, as before, the indices are considered modulo N . As in the odd case, let $V = H^0(\mathbf{P}^{N-1}, \mathcal{I}_{E_{\Theta_\tau}}(2))$, and $V_k = V \cap \langle X_i X_j \mid i + j \equiv k \pmod{N} \rangle$. Equations (6.14) (resp. (6.15)) give quadratic forms in the space V_0 (resp. V_1). Because of the relation $a_j = a_{N-j}$, it is easy to see that we may restrict ourselves to the case $0 \leq j \leq \frac{N}{2}$. Furthermore, letting $j = 0$ or $\frac{N}{2}$ in (6.14) gives trivial relations, and (6.15) becomes also trivial for $j = \frac{N}{2} - 1$. Finally, the second equation of (6.14) (resp. (6.15)) is unchanged if we replace j by $\frac{N}{2} - j$ (resp. $\frac{N}{2} - j - 1$).

Theorem 6.8. *Suppose N is an even integer. Let E_{Θ_τ} be the image of Θ_τ in Theorem 5.10. Let $V = H^0(\mathbf{P}^{N-1}, \mathcal{I}_{E_{\Theta_\tau}}(2))$, and $V_k = V \cap \langle X_i X_j \mid i + j \equiv k \pmod{N} \rangle$, where X_i ($i = 0, \dots, N-1$) are the coordinate functions of \mathbf{P}^{N-1} in Definition 4.17. Then,*

- (1) *The vector space V_{2k} is isomorphic to V_0 , and V_{2k+1} is isomorphic to V_1 for all $k = 0, \dots, \frac{N}{2} - 1$ by the map $X_i X_j \mapsto X_{i+k} X_{j+k}$, where indices are taken as elements of $\mathbf{Z}/N\mathbf{Z}$.*
- (2) *The quadratic forms*

$$(6.16) \quad a_j^2 X_0^2 + a_{\frac{N}{2}+j}^2 X_{\frac{N}{2}}^2 - a_0^2 X_j X_{N-j} - a_{\frac{N}{2}}^2 X_{\frac{N}{2}+j} X_{\frac{N}{2}-j}$$

with $j = 1, \dots, \frac{N}{2} - 1$ form a basis of the vector space V_0 , and the quadratic forms

$$(6.17) \quad \begin{aligned} & a_j a_{j+1} X_0 X_1 + a_{\frac{N}{2}+j} a_{\frac{N}{2}+j+1} X_{\frac{N}{2}} X_{\frac{N}{2}+1} \\ & - a_0 a_1 X_{j+1} X_{N-j} - a_{\frac{N}{2}} a_{\frac{N}{2}+1} X_{\frac{N}{2}+j+1} X_{\frac{N}{2}-j} \end{aligned}$$

with $j = 1, \dots, \frac{N}{2} - 2$ form a basis of the vector space V_1 .

Proof. (1) The isomorphism is nothing but the translation map τ_S^k in Proposition 4.4.

(2) By (6.14) and (6.15), we see that the quadratic forms in (6.16) and (6.17) belong to V_0 and V_1 respectively. Looking at the indices of $X_k X_l$, the only possible dependency occurring between the quadratic forms in (6.16) are between the ones $j = j_0$ and $j = \frac{N}{2} - j_0$. But then the determinant of the two by two matrix of coefficients of $X_{j_0} X_{-j_0}$ and $X_{\frac{N}{2}+j_0} X_{\frac{N}{2}-j_0}$ is $a_0^4 - a_{\frac{N}{2}}^4$, and it is non-zero, as is seen by looking at its Fourier series. So, these $\frac{N}{2} - 1$ forms are linearly independent. The same argument applies for the quadratic forms in (6.17), and the $\frac{N}{2} - 2$ forms are linearly independent. Since $\dim V = N(N-3)/2$, we have $\dim V_0 + \dim V_1 = N-3$ by (1). This implies that the $N-3$ independent forms in (6.16) and (6.17) form a basis of V_0 and V_1 respectively. \square

Remark 6.9. As before, by specializing X_i at $z = 0$ in (6.12), we obtain quartic relations in a_j 's. These quartic equations are satisfied by the image of the morphism $\alpha : Y^{(N)}(2N) \rightarrow \mathbf{P}^{N-1}$ in §5.4. We will study these equations in the examples in the following sections.

7. LEVEL 4

In this section we consider the case $N = 4$. As is well known, the congruence subgroups $\Gamma(4)$ and $\Gamma(8)$ are of genus 0 and 5, respectively, and according to the database [2], $\Gamma^{(4)}(8)$ is of genus 3. (In the notation of [2], $\Gamma^{(4)}(8)$ is denoted by “ $8B^3$ ”.) Theorem 6.8 shows that $V = H^0(\mathbf{P}^3, \mathcal{I}_{E_{\Theta_\tau}}(2)) = \bigoplus_{i=0}^3 V_i$ is given by

$$\begin{aligned} V_0 &= \langle a_1^2(X_0^2 + X_2^2) - (a_0^2 + a_2^2)X_1X_3 \rangle, \\ V_1 &= V_3 = \{0\}. \\ V_2 &= \langle a_1^2(X_1^2 + X_3^2) - (a_0^2 + a_2^2)X_0X_2 \rangle, \end{aligned}$$

Thus, the following two equations define the universal elliptic curve $E^{(4)}(8)$ over $X^{(4)}(8)$.

$$(7.1) \quad E^{(4)}(8) : \begin{cases} a_1^2(X_0^2 + X_2^2) = (a_0^2 + a_2^2)X_1X_3, \\ (a_0^2 + a_2^2)X_0X_2 = a_1^2(X_1^2 + X_3^2). \end{cases}$$

Letting $z = 0$, in other words, replacing X_i by a_i , we obtain two equations satisfied by a_i 's. However, considering the fact $a_1 = a_3$, we obtain only one nontrivial relation

$$(7.2) \quad a_0 a_2 (a_0^2 + a_2^2) = 2a_1^4.$$

This equation defines a nonsingular curve in \mathbf{P}^2 , and it is a curve of genus 3. On the other hand, as we mentioned above, the genus of $X^{(4)}(8)$ equals 3. Thus, the plane curve defined by (7.2) is nothing but $X^{(4)}(8)$.

Proposition 7.1. *The curve $E^{(4)}(8)$ is isomorphic to*

$$(7.3) \quad Y^2 = X(X - (a_0 - a_2)^4)(X - (a_0 + a_2)^4).$$

The points in $X^{(4)}(8)$ at which the fiber of $E^{(4)}(8)$ degenerates into a Néron polygon of four sides are

$$(a_0 : a_1 : a_2) = (0 : 0 : 1), (1 : 0 : 0), (\pm\zeta_8^2 : 0 : 1), (1 : \zeta_8^k : 1) \quad (k = 0, \dots, 7),$$

where ζ_8 is a primitive eighth root of unity.

Proof. From two quadrics (7.1), eliminate X_1 to obtain a quartic equation. Then, using the rational point $(X_0 : X_1 : X_2 : X_3) = (a_0 : a_1 : a_2 : a_1)$, we obtain the Weierstrass equation (7.3) after some simplifications. The change of coordinates is given by

$$(7.4) \quad \begin{cases} X = (a_0^2 - a_2^2)^2 \frac{(a_1^2 X_0 X_2 + a_0 a_2 X_1 X_3)}{(a_1^2 X_0 X_2 - a_0 a_2 X_1 X_3)}, \\ Y = 4a_1^2 (a_0^2 - a_2^2)^2 \frac{(X_1 + X_3)(X_0 + X_2)(a_0 a_2 X_0 X_2 - a_1^2 X_1 X_3)}{(X_1 - X_3)(X_0 - X_2)(a_1^2 X_0 X_2 - a_0 a_2 X_1 X_3)}. \end{cases}$$

The locus of degenerate fibers are where the right hand side of (7.3) has a multiple root. \square

The action of $SL_2(\mathbf{Z})/\Gamma^{(4)}(8)$ on $E^{(4)}(8)$ and $X^{(4)}(8)$ are as follows:

$$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta_8^2 & -1 & -\zeta_8^2 \\ 1 & -1 & 1 & -1 \\ 1 & -\zeta_8^2 & -1 & \zeta_8^2 \end{bmatrix}, \quad \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_8^3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \zeta_8^3 \end{bmatrix},$$

$$\bar{\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad \bar{\rho} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_8^3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In particular, we have

$$\begin{aligned} \rho \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} &: (X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : -X_1 : X_2 : -X_3), \\ \rho \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &: (X_0 : X_1 : X_2 : X_3) \mapsto (X_2 : X_3 : X_0 : X_1), \\ \bar{\rho} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} &: (a_0 : a_1 : a_2) \mapsto (a_0 : -a_1 : a_2), \\ \bar{\rho} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &: (a_0 : a_1 : a_2) \mapsto (a_2 : a_1 : a_0). \end{aligned}$$

Let $G = \Gamma(4)/\Gamma^{(4)}(8) = \langle (\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}) \Gamma^{(4)}(8), (\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}) \Gamma^{(4)}(8) \rangle \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. In order to obtain a model of the universal elliptic curve $E(4) \rightarrow X(4)$, we take the quotient of $E^{(4)}(8) \rightarrow X^{(4)}(8)$ by the action of G . To do so, we first dehomogenize the equations (7.1) and (7.2) by letting

$$x_0 = \frac{X_0}{X_3}, \quad x_1 = \frac{X_1}{X_3}, \quad x_2 = \frac{X_2}{X_3}, \quad \alpha_0 = \frac{a_0}{a_1}, \quad \alpha_2 = \frac{a_2}{a_1},$$

and consider the function field $K(\alpha_0, \alpha_2, x_0, x_1, x_2)$. Then, these variables satisfy the following relations:

$$(7.5) \quad \alpha_0 \alpha_2 (\alpha_0^2 + \alpha_2^2) = 2,$$

$$(7.6) \quad \begin{cases} \alpha_0 \alpha_2 (x_0^2 + x_2^2) = 2x_1, \\ 2x_0 x_2 = \alpha_0 \alpha_2 (x_1^2 + 1). \end{cases}$$

Let $\sigma_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \Gamma^{(4)}(8)$ and $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \Gamma^{(4)}(8)$ be the generators of G . Then, they act as follows:

$$\begin{aligned} \sigma_1 : (\alpha_0, \alpha_2, x_0, x_1, x_2) &\mapsto (-\alpha_0, -\alpha_2, -x_0, x_1, -x_2), \\ \sigma_2 : (\alpha_0, \alpha_2, x_0, x_1, x_2) &\mapsto \left(\alpha_2, \alpha_0, \frac{x_2}{x_1}, \frac{1}{x_1}, \frac{x_0}{x_1} \right). \end{aligned}$$

It is easy to see that $K(\alpha_0, \alpha_2)^G = K(\alpha_0 \alpha_2, \alpha_0^2 + \alpha_2^2)$. Considering (7.5), we see that $K(\alpha_0, \alpha_2)^G = K(\lambda)$ with $\lambda = \alpha_0 \alpha_2$. This shows that $X(4) \simeq \mathbf{P}_\lambda^1$. Further calculations show that the fixed field $K(\alpha_0, \alpha_2, x_0, x_1, x_2)^G$ is generated by

$$\lambda = \alpha_0 \alpha_2, \quad \xi_0 = \frac{x_0 x_2}{x_1}, \quad \xi_1 = x_1 + \frac{1}{x_1}, \quad \xi_2 = \frac{(x_1 + 1)(x_0 + x_2)}{(x_1 - 1)(x_0 - x_2)}.$$

Now, (7.4) can be written in terms of λ , ξ_0 and ξ_1 :

$$X = \frac{4(1 - \lambda^4)(\xi_0 + \lambda)}{\lambda^2(\xi_0 - \lambda)}, \quad Y = \frac{16(1 - \lambda^4)\xi_2(1 - \lambda\xi_0)}{\lambda^2(\xi_0 - \lambda)}.$$

If we let $X' = \lambda^2 X/4$ and $Y' = \lambda^2 Y/8$, then equation (7.3) becomes

$$E(4) : Y'^2 = X'(X' - (\lambda^2 - 1)^2)(X' - (\lambda^2 + 1)^2).$$

The representation $\varrho : SL_2(\mathbf{Z}/4\mathbf{Z}) \rightarrow \text{Aut}(E(4))$ is given by

$$\begin{aligned} \varrho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (\lambda, X', Y') &\mapsto \left(\frac{-\lambda + 1}{\lambda + 1}, -\frac{4(X' + (\lambda^2 + 1)^2)}{(\lambda + 1)^4}, -\frac{8\zeta_8^2 Y'}{(\lambda + 1)^6} \right), \\ \varrho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : (\lambda, X', Y') &\mapsto (-\zeta_8^2 \lambda, X', Y'). \end{aligned}$$

Note that we have

$$\begin{aligned} \lambda(\tau) &= \frac{a_0(\tau)a_2(\tau)}{a_1(\tau)^2} = 2 \left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3} \right)^2 \\ &= 2q^{\frac{1}{4}} - 4q^{\frac{5}{4}} + 10q^{\frac{9}{4}} - 20q^{\frac{13}{4}} + 36q^{\frac{17}{4}} + \dots \quad (q = e^{2\pi i\tau}), \end{aligned}$$

where $\eta(\tau)$ is the Dedekind eta function.

8. LEVEL 6

Next, we consider the case $N = 6$. The congruence subgroups $\Gamma(6)$ and $\Gamma(12)$ are of genus 1 and 25, respectively, and according to the database [2], $\Gamma^{(6)}(12)$ is denoted by “12B¹³”, and of genus 13. The basis of V_k obtained

by Theorem 6.8 is as follows:

$$(8.1) \quad \begin{aligned} V_0 &= \langle a_1^2 X_0^2 + a_2^2 X_3^2 - a_0^2 X_1 X_5 - a_3^2 X_2 X_4, a_2^2 X_0^2 + a_1^2 X_3^2 - a_3^2 X_1 X_5 - a_0^2 X_2 X_4 \rangle, \\ V_2 &= \langle a_1^2 X_1^2 + a_2^2 X_4^2 - a_0^2 X_2 X_0 - a_3^2 X_3 X_5, a_2^2 X_0^2 + a_1^2 X_3^2 - a_3^2 X_2 X_0 - a_0^2 X_3 X_5 \rangle, \\ V_4 &= \langle a_1^2 X_2^2 + a_2^2 X_5^2 - a_0^2 X_3 X_1 - a_3^2 X_2 X_4, a_2^2 X_0^2 + a_1^2 X_5^2 - a_3^2 X_3 X_1 - a_0^2 X_4 X_0 \rangle, \\ V_1 &= \langle a_1 a_2 (X_0 X_1 + X_3 X_4) - (a_0 a_1 + a_2 a_3) X_2 X_5 \rangle, \\ V_3 &= \langle a_1 a_2 (X_1 X_2 + X_4 X_5) - (a_0 a_1 + a_2 a_3) X_3 X_0 \rangle, \\ V_5 &= \langle a_1 a_2 (X_2 X_3 + X_5 X_0) - (a_0 a_1 + a_2 a_3) X_4 X_1 \rangle. \end{aligned}$$

By replacing X_i by a_i , we obtain quartic relations among a_i 's. Noting that $a_5 = a_1$ and $a_4 = a_2$, we obtain only two relations:

$$(8.2) \quad \begin{cases} a_1^4 + a_2^4 = a_0^3 a_2 + a_1 a_3^3, \\ a_0 a_3 (a_0 a_1 + a_2 a_3) = 2 a_1^2 a_2^2. \end{cases}$$

The curve in \mathbf{P}^3 defined by (8.2) turns out to be reducible. Computations using Groebner basis reveal that there are five irreducible components; four of them are lines

$$\begin{aligned} a_1 = a_2 = 0, \quad a_0 - a_2 = a_1 - a_3 = 0, \\ a_0 - \omega a_2 = \omega a_1 - a_3 = 0, \quad a_0 - \omega^2 a_2 = \omega^2 a_1 - a_3 = 0, \end{aligned}$$

where ω is a primitive third root of unity, and the other component is an irreducible curve of genus 13. Since the congruence subgroup $\Gamma^{(6)}(12)$ is of genus 13, the last irreducible component given by

$$a_2(a_1^4 - a_2^4)a_0^6 - (a_1^8 - a_2^8)a_0^3 + 8a_2^3 a_1^4(a_0^3 a_2 - a_1^4) = 0$$

is the modular curve $X^{(6)}(12)$ associated with the group $\Gamma^{(6)}(12)$.

The action of $SL_2(\mathbf{Z})/\Gamma^{(6)}(12)$ on $X^{(4)}(8)$ is as follows:

$$\bar{\rho} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}, \quad \bar{\rho} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_{12}^5 & 0 & 0 \\ 0 & 0 & \zeta_{12}^8 & 0 \\ 0 & 0 & 0 & \zeta_{12}^9 \end{bmatrix}.$$

In particular, we have

$$\begin{aligned} \bar{\rho} \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} : (a_0 : a_1 : a_2 : a_3) &\mapsto (a_0 : -a_1 : a_2 : -a_3), \\ \bar{\rho} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} : (a_0 : a_1 : a_2 : a_3) &\mapsto (a_3 : a_2 : a_1 : a_0). \end{aligned}$$

Let us find a model of $X(6)$ by taking the quotient of $X^{(6)}(12)$ by $\Gamma(6)/\Gamma^{(6)}(12) = \langle \bar{\rho} \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \bar{\rho} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \rangle \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Dehomogenizing the coordinates by $\alpha_1 = a_1/a_0$, $\alpha_2 = a_2/a_0$, $\alpha_3 = a_3/a_0$, the equation (8.2) becomes

$$(8.3) \quad \begin{cases} \alpha_1^4 + \alpha_2^4 = \alpha_2 + \alpha_1 \alpha_3^3, \\ \alpha_3(\alpha_1 + \alpha_2 \alpha_3) = 2\alpha_1^2 \alpha_2^2. \end{cases}$$

Let σ_1 and σ_2 denote the automorphisms induced on the function field $k(\alpha_1, \alpha_2, \alpha_3)$ by $\bar{\rho} \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ and $\bar{\rho} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$, respectively. We have

$$\sigma_1 : (\alpha_1, \alpha_2, \alpha_3) \mapsto (-\alpha_1, \alpha_2, -\alpha_3), \quad \sigma_2 : (\alpha_1, \alpha_2, \alpha_3) \mapsto \left(\frac{\alpha_2}{\alpha_3}, \frac{\alpha_1}{\alpha_3}, \frac{1}{\alpha_3} \right).$$

The fixed subfields by σ_1 and σ_2 are

$$k(\alpha_1, \alpha_2, \alpha_3)^{\sigma_1} = k\left(\alpha_1\alpha_3, \frac{\alpha_1}{\alpha_3}, \alpha_2\right),$$

$$k(\alpha_1, \alpha_2, \alpha_3)^{\sigma_2} = \left(\alpha_1 + \frac{\alpha_2}{\alpha_3}, \alpha_2 + \frac{\alpha_1}{\alpha_3}, \alpha_3 + \frac{1}{\alpha_3}\right).$$

Finally, define

$$(8.4) \quad \beta_1 = \frac{\alpha_1\alpha_2}{\alpha_3}, \quad \beta_2 = \alpha_1\alpha_3 + \frac{\alpha_2}{\alpha_3}.$$

Clearly, β_1 and β_2 are fixed by $G = \langle \sigma_1, \sigma_2 \rangle$. It is easy to show that the fixed field by the group G is given by

$$k(\alpha_1, \alpha_2, \alpha_3)^G = k(\beta_1, \beta_2).$$

Eliminating $\alpha_1, \alpha_2, \alpha_3$ from (8.3) and (8.4) by some calculations based on Groebner basis, we find that if we let

$$X = 2\beta_1, \quad Y = \frac{2\beta_1^2(4\beta_1^3 - 1)}{\beta_2},$$

they satisfy

$$Y^2 = X^3 + 1,$$

which is well-known model of the modular curve $X(6)$. Furthermore, we have

$$\beta_1 = \frac{X}{2}, \quad \beta_2 = \frac{X^2(X^3 - 2)}{4Y},$$

and thus $k(\beta_1, \beta_2) = k(X, Y)$.

On the curve $Y^2 = X^3 + 1$, the actions of $\bar{\rho}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\bar{\rho}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are given by

$$\bar{\rho}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) : (X, Y) \mapsto (2, -3) - (X, Y),$$

$$\bar{\rho}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : (X, Y) \mapsto [-\omega](X, Y) = (\omega X, -Y),$$

where the operation “ $-$ ” in the first map is the group operation of $Y^2 = X^3 + 1$, and the map $[\omega]$ is the complex multiplication of $Y^2 = X^3 + 1$. In terms of $a_i(\tau) = \theta_i^{(N)}(0, \tau)$, X and Y are expressed as follows:

$$X = \frac{2a_1(\tau)a_2(\tau)}{a_0(\tau)a_3(\tau)}, \quad Y = \frac{a_0(\tau)^2a_1(\tau)^2 - a_2(\tau)^2a_3(\tau)^2}{a_0(\tau)^2a_2(\tau)^2 - a_1(\tau)^2a_3(\tau)^2}.$$

Incidentally, X and Y , which are modular functions on $\Gamma(6)$, can also be written by using the Dedekind eta function as follows:

$$X = \frac{\eta(2\tau)\eta(3\tau)^3}{\eta(\tau)\eta(6\tau)^3} = q^{-\frac{1}{3}} + q^{\frac{2}{3}} + q^{\frac{5}{3}} - q^{\frac{8}{3}} - q^{\frac{11}{3}} + q^{\frac{17}{3}} + 2q^{\frac{20}{3}} - \dots,$$

$$Y = \frac{\eta(2\tau)^4\eta(3\tau)^2}{\eta(\tau)^2\eta(6\tau)^4} = q^{-\frac{1}{2}} + 2q^{\frac{1}{2}} + q^{\frac{3}{2}} - 2q^{\frac{7}{2}} - 2q^{\frac{9}{2}} + 2q^{\frac{11}{2}} + 4q^{\frac{13}{2}} + \dots,$$

where $q = e^{2\pi i\tau}$. The function Y has a simpler expression in terms of $a_i(\tau) = \theta_i^{(6)}(0, \tau)$ as

$$Y = \frac{a_0\left(\frac{\tau}{3}\right)a_3\left(\frac{\tau}{3}\right)}{a_0(\tau)a_3(\tau)}.$$

At the end of this section, we note the connection to the ‘‘Hesse cubic’’, that is the elliptic normal curve of degree 3. Let

$$\begin{aligned} 3\mu &= X^2 - \frac{2}{X} = \frac{Y^2 - 3}{X} \\ &= q^{-\frac{2}{3}} + 5q^{\frac{4}{3}} - 7q^{\frac{10}{3}} + 3q^{\frac{16}{3}} + 15q^{\frac{22}{3}} - 32q^{\frac{28}{3}} + 9q^{\frac{34}{3}} + \dots \end{aligned}$$

We can check that $\mu(\tau/2)$ is a modular function for the group $\Gamma(3)$, and we have the relation

$$X_0^3 + X_2^3 + X_4^3 = 3\mu(\tau)X_0X_2X_4.$$

Actually, our theta functions $\theta_k^{(3)}(z, \tau)$ ($k = 0, 1, 2$) for $N = 3$ can be obtained from those for $N = 6$ ($k = 0, 2, 4$) by changing the variables $z \rightarrow z/2 - 1/4$, $\tau \rightarrow \tau/2$, and thus the above equation gives the Hesse cubic family of elliptic curves with level 3 structure. We refer the reader to [9] for the derivation of the Hesse cubic in the same line of the current paper.

9. LEVEL 8

In the case $N = 8$, the basis of V_0 and V_1 obtained by Theorem 6.8 is as follows:

$$(9.1) \quad \begin{aligned} V_0 &= \langle a_1^2X_0^2 + a_3^2X_4^2 - a_0^2X_1X_7 && - a_4^2X_3X_5, \\ & a_2^2X_0^2 + a_2^2X_4^2 && - (a_0^2 + a_4^2)X_2X_6, \\ & a_3^2X_0^2 + a_1^2X_4^2 - a_4^2X_1X_7 && - a_0^2X_3X_5 \rangle, \\ V_1 &= \langle a_1a_2X_0X_1 - a_0a_1X_2X_7 - a_3a_4X_3X_6 + a_2a_3X_4X_5, \\ & a_2a_3X_0X_1 - a_3a_4X_2X_7 - a_0a_1X_3X_6 + a_1a_2X_4X_5 \rangle. \end{aligned}$$

The bases of V_{2k} and V_{2k+1} are obtained by replacing X_i by X_{i+k} in V_0 and V_1 respectively. By letting $z = j\tau/N$, $j = 0, 1, \dots$, in (9.1), we obtain the following relations among a_i :

$$(9.2) \quad \begin{cases} a_0a_4(a_0^2 + a_4^2) = 2a_2^4, \\ a_0a_4(a_1^2 + a_3^2) = 2a_1a_3a_2^2, \\ a_0a_2a_4(a_0 + a_4) = 2a_1^2a_3^2, \\ a_2^3(a_0 + a_4) = a_1a_3(a_1^2 + a_3^2), \\ a_1a_3(a_0^2 + a_4^2) = a_2^2(a_1^2 + a_3^2), \\ a_2(a_0^3 + a_4^3) = a_1^4 + a_3^4. \end{cases}$$

The curve in \mathbf{P}^4 defined by (9.2) turns out to be an irreducible curve of genus 41, and this is a model of $X^{(8)}(16)$.

Dehomogenize (9.2) by letting $\alpha_0 = a_0/a_2$, $\alpha_1 = a_1/a_2$, $\alpha_3 = a_3/a_2$, $\alpha_4 = a_4/a_2$. By some calculations of Groebner basis, it turns out that in the function field $k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$, the following three equations are enough to generate the ideal generated by the above six equations:

$$(9.3) \quad \alpha_0\alpha_4(\alpha_0^2 + \alpha_4^2) = 2, \quad \alpha_0 + \alpha_4 = \alpha_1\alpha_3(\alpha_1^2 + \alpha_3^2), \quad \alpha_1\alpha_3(\alpha_0^2 + \alpha_4^2) = \alpha_1^2 + \alpha_3^2.$$

Let σ_1 and σ_2 denote the automorphisms of $k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ induced by $\bar{\rho}\left(\begin{smallmatrix} 1 & 8 \\ 0 & 1 \end{smallmatrix}\right)$ and $\bar{\rho}\left(\begin{smallmatrix} 1 & 0 \\ 8 & 1 \end{smallmatrix}\right)$, respectively. We have

$$\begin{aligned}\sigma_1 &: (\alpha_0, \alpha_1, \alpha_3, \alpha_4) \mapsto (\alpha_0, -\alpha_1, -\alpha_3, \alpha_4), \\ \sigma_2 &: (\alpha_0, \alpha_1, \alpha_3, \alpha_4) \mapsto (\alpha_4, \alpha_3, \alpha_1, \alpha_0).\end{aligned}$$

It is easy to see that α and β commute, and thus the group of automorphism $\langle \alpha, \beta \rangle$ induced by α and β is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. The fixed subfields by α and β are

$$\begin{aligned}k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)^{\sigma_1} &= k\left(\alpha_0, \alpha_1\alpha_3, \frac{\alpha_1}{\alpha_3}, \alpha_4\right), \\ k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)^{\sigma_2} &= k\left(\alpha_0 + \alpha_4, \alpha_1 + \alpha_3, (\alpha_1 - \alpha_3)(\alpha_0 - \alpha_4), \frac{\alpha_1 - \alpha_3}{\alpha_0 - \alpha_4}\right).\end{aligned}$$

Finally, define

$$\beta_0 = \alpha_0 + \alpha_4, \quad \beta_1 = \alpha_1\alpha_3, \quad \beta_3 = \frac{\alpha_1}{\alpha_3} + \frac{\alpha_3}{\alpha_1}, \quad \beta_4 = \frac{\alpha_1\alpha_3(\alpha_0 - \alpha_4)}{(\alpha_1 + \alpha_3)(\alpha_1 - \alpha_3)}.$$

Then, $\beta_0, \beta_1, \beta_3$, and β_4 are all fixed by $G = \langle \sigma_1, \sigma_2 \rangle$. It is easy to show that the fixed field by the group G is given by

$$k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)^G = k(\beta_0, \beta_1, \beta_3, \beta_4).$$

Further calculations show that the relations (9.3) translates to the relations

$$\beta_0 = \beta_1\beta_3, \quad \beta_3 = \frac{1}{\beta_4^2}, \quad \beta_1^4 = 4\beta_4^6 + \beta_4^2.$$

This implies that

$$k(\alpha_0, \alpha_1, \alpha_3, \alpha_4)^G = k(\beta_1, \beta_4), \quad \text{with } \beta_1^4 = 4\beta_4^6 + \beta_4^2$$

The genus of the curve defined by $\beta_1^4 = 4\beta_4^6 + \beta_4^2$ is 5, which coincides with the genus of $X(8)$. Thus, we conclude that an affine model of $X(8)$ is given by

$$X(8) : \beta_1^2(1 + 4\beta_4^2) = \beta_4^4.$$

Recall that an equation of $X^{(4)}(8)$ is given by $a_0^{(4)}a_2^{(4)}((a_0^{(4)})^2 + (a_2^{(4)})^2) = 2(a_1^{(4)})^4$ (see (7.2)). We have a two-to-one map

$$X(8) \rightarrow X^{(4)}(8); \quad (\beta_1, \beta_4) \mapsto (a_0^{(4)} : a_2^{(4)} : a_4^{(4)}) = (1 : \beta_1 : 2\beta_4^2).$$

The modular functions β_1 and β_4 on $\Gamma(8)$ can be written in terms of the Dedekind eta function as follows:

$$\begin{aligned}\beta_1 &= \frac{\eta(2\tau)^4\eta(8\tau)^2}{\eta(\tau)\eta(4\tau)^5} = q^{\frac{1}{8}} + q^{\frac{9}{8}} - 2q^{\frac{17}{8}} - q^{\frac{25}{8}} + 4q^{\frac{33}{8}} + 2q^{\frac{41}{8}} - 7q^{\frac{49}{8}} - \dots, \\ \beta_4 &= -\frac{\eta(2\tau)\eta(8\tau)^2}{\eta(4\tau)^3} = q^{\frac{1}{4}} - q^{\frac{9}{4}} + 2q^{\frac{17}{4}} - 3q^{\frac{25}{4}} + 4q^{\frac{33}{4}} - 6q^{\frac{41}{4}} + 9q^{\frac{49}{4}} - \dots.\end{aligned}$$

They also have neat expressions in terms of our theta series:

$$\beta_1 = \frac{a_2(\frac{\tau}{2})a_2(2\tau)}{a_2(\tau)^2}, \quad \beta_4 = \frac{a_2(2\tau)}{a_2(\tau)}.$$

APPENDIX A. HURWITZ'S IMMERSION

Here, we describe the connections and differences between Hurwitz's σ -functions and our theta functions in detail.

Hurwitz [6] uses the Weierstrass σ -function to construct an immersion of an elliptic curve into projective space. The σ -function is defined with respect to a fundamental pair of periods $\omega_1, \omega_2 \in \mathbf{C}$. Replacing ω_2 by $-\omega_2$ if necessary, we may assume $\text{Im } \omega_2/\omega_1 > 0$. So, if we define $\tau = \omega_2/\omega_1$, τ is a point in the upper half plane \mathfrak{H} .

Let Λ be the lattice $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbf{Z}\}$, and Λ^* denotes the set of nonzero elements of Λ . The Weierstrass sigma function $\sigma(u)$ associated with $\Lambda \subset \mathbf{C}$ is defined by

$$\sigma(u) = u \prod_{w \in \Lambda^*} \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2},$$

The function σ is a quasiperiodic holomorphic function having a simple zero at each of the points in Λ . Here, "quasiperiodic" means that $\sigma(u)$ satisfies the identities $\sigma(u + \omega_k) = \exp(\eta_k(u + \frac{\omega_k}{2}) + \pi i)\sigma(u)$ ($k = 1, 2$), where η_1, η_2 satisfies the identity $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. We sometimes write $\sigma(u \mid \omega_1, \omega_2)$ instead of $\sigma(u)$ to make clear that it depends on ω_1 and ω_2 .

A "shift" of σ by $a\omega_1 + b\omega_2$ is defined by

$$(A.1) \quad \sigma_{a,b}(u \mid \omega_1, \omega_2) \\ = \exp\left((a\eta_1 + b\eta_2)\left(u - \frac{a\omega_1 + b\omega_2}{2}\right)\right) \cdot \sigma(u - a\omega_1 - b\omega_2 \mid \omega_1, \omega_2).$$

Clearly, $\sigma_{a,b}(u)$ has simple zeros at $u \equiv a\omega_1 + b\omega_2 \pmod{\Lambda}$.

Let N be an integer, and consider the overlattice $\Lambda_N = \langle \frac{\omega_1}{N}, \omega_2 \rangle \supset \Lambda$. Hurwitz defines, for any integer k

$$X_k(u) = \mu_k \cdot \exp(-G_1 u^2) \cdot \sigma_{\epsilon, \epsilon + \frac{k}{N}}\left(u \mid \frac{\omega_1}{N}, \omega_2\right),$$

where $\epsilon = 0$ if N odd, and $\epsilon = \frac{1}{2}$ if N even. The quantity G_1 is given by the formula $G_1 = \frac{N\bar{\eta}_1 - N\eta_1}{2\omega_1} = \frac{\bar{\eta}_2 - N\eta_2}{2\omega_2}$, where $\bar{\eta}_1$ and $\bar{\eta}_2$ denote the values that η_1 and η_2 take when ω_1 is replaced by ω_1/N . Finally, Hurwitz defines μ_k by $\mu_k = e^{-\frac{\pi i}{2N}k} \cdot e^{-\frac{5\pi i}{4}} \cdot \mu$ if N even, and $(-1)^k \cdot \mu$ if N odd, where μ is an arbitrary constant.

The function $X_k(u)$ has N simple zeros within the fundamental domain $\{r\omega_1 + s\omega_2 \mid 0 \leq r, s < 1\}$ of Λ . They are

$$(A.2) \quad \begin{aligned} & \frac{m}{N}\omega_1 + \frac{k}{N}\omega_2, & m = 0, 1, \dots, N-1, & \text{if } N \text{ is odd,} \\ & \left(\frac{1}{2N} + \frac{m}{N}\right)\omega_1 + \left(\frac{1}{2} + \frac{k}{N}\right)\omega_2, & m = 0, 1, \dots, N-1, & \text{if } N \text{ is even.} \end{aligned}$$

Hurwitz then shows, among other things, the following relation for $\lambda_1, \lambda_2 \in \mathbf{Z}$.

$$\begin{aligned} X_k\left(u + \frac{\lambda_1\omega_1 + \lambda_2\omega_2}{N}\right) &= (-1)^{N(\lambda_1 + \lambda_2)} \cdot \exp\left(-2\pi i\left(\frac{k\lambda_1}{N} + \frac{\lambda_1\lambda_2}{2N}\right)\right) \\ &\quad \times \exp\left((\lambda_1\eta_1 + \lambda_2\eta_2)\left(u + \frac{\lambda_1\omega_1 + \lambda_2\omega_2}{2N}\right)\right) \cdot X_{k-\lambda_2}(u) \end{aligned}$$

In particular, if $\lambda_1 = 0$ and $\lambda_2 = k$, we have

$$(A.3) \quad X_k(u) = (-1)^{Nk} \cdot \exp\left(k\eta_2\left(u - \frac{k\omega_2}{2N}\right)\right) \cdot X_0\left(u - \frac{k\omega_2}{N}\right).$$

On the other hand, we can show using Proposition 5.4 (4) that our function $\theta_k(z) = \theta_{(\frac{1}{2} - \frac{k}{N}, \frac{N}{2})}(Nz, N\tau)$ satisfies

$$(A.4) \quad \theta_k(z) = (-1)^k \cdot \exp\left(-2\pi i k \left(z - \frac{k\tau}{2N}\right)\right) \cdot \theta_0\left(z - \frac{k\tau}{N}\right).$$

Using (A.3) and (A.4), we can deduce the relation between $X_k(u)$ and $\theta_k(z)$ for any k once we establish the relation between Hurwitz's $X_0(u)$ and $\theta_0(z)$.

Define $\varphi_{\epsilon, \epsilon}(u)$ by

$$\varphi_{\epsilon, \epsilon}(u) = \exp\left(\frac{-N\eta_1}{2\omega_1} u^2\right) \cdot \sigma_{\epsilon, \epsilon}\left(u \mid \frac{\omega_1}{N}, \omega_2\right).$$

Then, we have $X_0(u) = \mu_0 \cdot \exp\left(\frac{N\eta_1}{2\omega_1} u^2\right) \cdot \varphi_{\epsilon, \epsilon}(u)$. We then compute $\varphi_{\epsilon, \epsilon}(u + \omega_1/N)$ and $\varphi_{\epsilon, \epsilon}(u + \omega_2)$ using the formula (A.1),

$$\begin{aligned} \varphi_{\epsilon, \epsilon}\left(u + \frac{\omega_1}{N}\right) &= -\exp(-2\pi i \epsilon) \cdot \varphi_{\epsilon, \epsilon}(u), \\ \varphi_{\epsilon, \epsilon}(u + \omega_2) &= -\exp\left(-2\pi i \left(\frac{Nu}{\omega_1} + \frac{N\omega_2}{2\omega_1}\right) + 2\pi i \epsilon\right) \cdot \varphi_{\epsilon, \epsilon}(u). \end{aligned}$$

Let $\tau = \omega_2/\omega_1$ and $u = \omega_1 z$, and define $\Phi_{\epsilon_1, \epsilon_2}(z) = \varphi_{\epsilon, \epsilon}(\omega_1 z)$. Then, we have

$$(A.5) \quad \begin{aligned} \Phi_{\epsilon, \epsilon}\left(z + \frac{1}{N}\right) &= -\exp(-2\pi i \epsilon) \cdot \Phi_{\epsilon, \epsilon}(z), \\ \Phi_{\epsilon, \epsilon}(z + \tau) &= -\exp\left(-2\pi i \left(Nz + \frac{N\tau}{2}\right) + 2\pi i \epsilon\right) \cdot \Phi_{\epsilon, \epsilon}(z). \end{aligned}$$

A.1. Odd case. By Proposition 5.4 (2) and (3), $\theta_0(z)$ satisfies

$$\begin{aligned} \theta_0\left(z + \frac{1}{N}\right) &= -\theta_0(z), \\ \theta_0(z + \tau) &= -\exp\left(-2\pi i \left(Nz + \frac{N\tau}{2}\right)\right) \cdot \theta_0(z), \end{aligned}$$

which is exactly the same transformation rules (A.5) for $\epsilon = 0$. Thus, if N is odd, the ratio $\Phi_{0,0}(z)/\theta_0(z)$ is a doubly periodic function of periods $1/N$ and τ . Since the function $\Phi_{0,0}(z)/\theta_0(z)$ has at most one simple pole in the fundamental domain, it must be a constant C . The constant C is obtained by L'Hôpital's rule as follows

$$C = \lim_{z \rightarrow 0} \frac{\Phi_{0,0}(z)}{\theta_0(z)} = \frac{\omega_1 \sigma'_{0,0}(0)}{N \theta'_{(\frac{1}{2}, \frac{N}{2})}(0, N\tau)} = \frac{\omega_1}{N \theta'_{(\frac{1}{2}, \frac{N}{2})}(0, N\tau)}.$$

Thus, if N is odd, then $X_0(\omega_1 z) = C \cdot \mu_0 \cdot \exp\left(\frac{N\eta_1 \omega_1}{2} z^2\right) \cdot \theta_0(z)$. The ratio of $X_k(u)$ and $\theta_k(z)$ can be calculated using (A.3) and Proposition 5.4 (4),

$$\frac{X_k(\omega_1 z)}{\theta_k(z)} = \frac{\exp\left(k\eta_2 \left(\omega_1 z - \frac{k\omega_2}{2N}\right)\right)}{\exp\left(-2\pi i k \left(z - \frac{k\tau}{2N}\right)\right)} \cdot \frac{X_0\left(\omega_1 z - \frac{k\omega_2}{N}\right)}{\theta_0\left(z - \frac{k\tau}{N}\right)} = C \cdot \mu_0 \cdot \exp\left(\frac{N\eta_1 \omega_1}{2} z^2\right).$$

This shows that two immersions by $X_k(u)$ and $\theta_k(z)$ coincide with each other.

A.2. **Even case.** If N is even, the zeros of Hurwitz's $X_0(u)$ and the zeros of $\theta_0(u/\omega_1)$ are off by $\frac{1}{2}\omega_2$, and so the two immersions differ. We thus try to modify the definition of $\theta_k^{(N)}$ to shift the zeros. It turns out that if we define an immersion using the function

$$\theta_k^*(z) = (-1)^k \theta_{\left(\frac{k}{N}, \frac{1}{2}\right)}(Nz, N\tau)$$

instead of $\theta_k(z) = \theta_{\left(\frac{1}{2} - \frac{k}{N}, \frac{N}{2}\right)}(Nz, N\tau)$, the immersion coincides with that of Hurwitz's. To see this, compare $X_0(u)$ and $\theta_0^*(z)$. In this case $\Phi_{\frac{1}{2}, \frac{1}{2}}(z) = \varphi_{\frac{1}{2}, \frac{1}{2}}(\omega_1 z)$ satisfy

$$\begin{aligned} \Phi_{\frac{1}{2}, \frac{1}{2}}\left(z + \frac{1}{N}\right) &= -\exp(\pi i) \cdot \Phi_{\frac{1}{2}, \frac{1}{2}}(z) = \Phi_{\frac{1}{2}, \frac{1}{2}}(z), \\ \Phi_{\frac{1}{2}, \frac{1}{2}}(z + \tau) &= \exp\left(-2\pi i\left(Nz + \frac{N\tau}{2}\right)\right) \cdot \Phi_{\frac{1}{2}, \frac{1}{2}}(z). \end{aligned}$$

On the other hand, by Proposition 5.2, we can show

$$\begin{aligned} \theta_0^*\left(z + \frac{1}{N}\right) &= \theta_{\left(0, \frac{N}{2}\right)}(Nz + 1, N\tau) = \theta_0^*(z), \\ \theta_0^*(z + \tau) &= \theta_{\left(0, \frac{N}{2}\right)}(Nz + N\tau, N\tau) = \exp\left(-2\pi i\left(Nz + \frac{N\tau}{2}\right)\right) \cdot \theta_0^*(z). \end{aligned}$$

These identities imply that if N is even, the ratio $\Phi(z)/\theta_0^*(z)$ is a doubly periodic function of periods $1/N$ and τ . Since this function has at most one simple pole in the fundamental domain, it must be a constant C^* . Thus, if N is even, then $X_0(\omega_1 z) = C^* \cdot \mu_0 \cdot \exp\left(\frac{N\eta_1\omega_1}{2} z^2\right) \cdot \theta_0^*(z)$.

By Proposition 6.2 (2) with values $p = 0$, $q = N/2$, $r = -k/N$, we can show $\theta_k^*(z) = \exp\left(-2\pi i k\left(z - \frac{k\tau}{2N}\right)\right) \cdot \theta_0^*\left(z - \frac{k\tau}{N}\right)$. Therefore, the ratio of the two functions becomes

$$\frac{X_k(\omega_1 z)}{\theta_k^*(z)} = \frac{\exp\left(k\eta_2\left(\omega_1 z - \frac{k\omega_2}{2N}\right)\right)}{\exp\left(-2\pi i k\left(z - \frac{k\tau}{2N}\right)\right)} \cdot \frac{X_0\left(\omega_1 z - \frac{k\omega_2}{N}\right)}{\theta_0^*\left(z - \frac{k\tau}{N}\right)} = C^* \cdot \mu_0 \cdot \exp\left(\frac{N\eta_1\omega_1}{2} z^2\right).$$

This implies that two immersions by $X_k(u)$ and $\theta_k^*(z)$ coincide with each other.

Alternatively, we may modify Hurwitz's definition to make it coincides with ours. To do so, we use $\sigma_{\frac{1}{2}, 0}(u \mid \frac{\omega_1}{N}, \omega_2)$, which has the same set of simple zeros with $\theta_0^{(N)}(z, \tau)$.

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