A NOTE

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Certain Derivations

Following Hoffman [2], we let $\mathfrak{h} = \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial algebra in two variables x and y, and $\mathfrak{h}^0$ its subalgebra $\mathbb{Q}\cdot 1 + x\mathfrak{h}y$. In $\mathfrak{h}$ are defined three kinds of product, $\cdot$ (concatenation), $*$ (Hoffman's harmonic product), and $\circ$ (shuffle product). Let $\widetilde{\zeta} : \mathfrak{h}^0 \to \mathbb{R}$ be the $\mathbb{Q}$-linear "evaluation- by-iterated-integral" map ($x \leftrightarrow \frac{dt}{t}, y \leftrightarrow \frac{dt}{1-t}$). The map $\widetilde{\zeta}$ is not only a $\mathbb{Q}$-linear map but also a $\mathbb{Q}$-algebra homomorphism w.r.t. both $*$ and $\circ$. Fundamental problem is then

Problem. *What is $Ker\widetilde{\zeta}$?*

Let $\tau : \mathfrak{h}^0 \to \mathfrak{h}^0$ be the "dual" map, i.e., the map interchanges x and y and reverse the order of product. Now standard duality theorem for multiple zeta values states

Theorem (Duality). $(1-\tau)\mathfrak{h}^0 \subset Ker\widetilde{\zeta}$.

Inspired by [2], we define certain derivations on $\mathfrak{h}^0$ which conjecturally would produce a portion of elements of $Ker\widetilde{\zeta}$.

Definition. *For $n \geq 1$, let ∂_n be the derivation on $\mathfrak{h}^0$ w.r.t. concatenation product defined by $\partial_n(x) = s_n$ and $\partial_n(y) = -s_n$ where s_n is the sum of all monomials in $\mathfrak{h}^0$ of degree $n+1$ ($s_1 = xy, s_2 = x^2y + xy^2, s_3 = x^3y + x^2y^2 + xyxy + xy^3$ and so on).*

Hoffman's reformulation [2] of his previous theorem [1] can also be stated as

Theorem (Hoffman's relation). $\partial_1\mathfrak{h}^0 \subset Ker\widetilde{\zeta}$.

A possible generalization of this would be

Conjecture. $\forall n \geq 1, \partial_n\mathfrak{h}^0 \subset Ker\widetilde{\zeta}$.

It seems that the relations produced by these derivations are equivalent (under the duality) to those given by Y. Ohno. I have checked this up to weight 12 by computer (i.e., $\partial_n\mathfrak{h}^0_m \subset Ker\widetilde{\zeta}$ for $n+m \leq 12$ where $\mathfrak{h}^0_m$ is the degree m part of $\mathfrak{h}^0$). It is not hard to deduce the case $n = 2$ from Ohno's relation:

Proposition. *The conjecture is valid for $n = 2$.*

Question. *Is there any derivation ∂ other than ∂_n having the property $\partial\mathfrak{h}^0 \subset Ker\widetilde{\zeta}$?*

Now, derivations on $\mathfrak{h}^0$ naturally form a Lie algebra (over $\mathbb{Q}$) by $[\partial, \partial'] := \partial\partial' - \partial'\partial$. Let $\mathcal{D}_\zeta$ denote the sub Lie algebra of homogeneous derivations satisfying $\partial\mathfrak{h}^0 \subset Ker\widetilde{\zeta}$ (we may call them "special derivations").

Question. *What is $\mathcal{D}_\zeta$?*

It seems $[\partial_m, \partial_n] = 0, \forall m, n \geq 1$. (I lazily haven't checked this.) What is the center of $\mathcal{D}_\zeta$?

About double shuffle relations

Definition. $\langle w_1, w_2 \rangle := w_1 \circ w_2 - w_1 * w_2$ $(w_1, w_2 \in \mathfrak{h})$.

Because of the formula $\widetilde{\zeta}(w_1 \circ w_2) = \widetilde{\zeta}(w_1 * w_2) = \widetilde{\zeta}(w_1)\widetilde{\zeta}(w_2)$ for any $w_1, w_2 \in \mathfrak{h}^0$, we have

Double shuffle relation. $\langle w_1, w_2 \rangle \in Ker\widetilde{\zeta}$, $\forall w_1, w_2 \in \mathfrak{h}^0$.

This is also valid when we replace w_1 by y (which does not belong to $\mathfrak{h}^0$), and in fact it gives again the Hoffman's relation.

Proposition. $\partial_1(w) = \langle y, w \rangle$, $\forall w \in \mathfrak{h}^0$. *Hence* $\langle y, w \rangle \in Ker\widetilde{\zeta}$, $\forall w \in \mathfrak{h}^0$.

The proof is a simple calculation.

Question. *Is $Ker\widetilde{\zeta}$ generated (as $\mathbb{Q}$-vector space) by $\langle y, \mathfrak{h}^0 \rangle$, $\langle \mathfrak{h}^0, \mathfrak{h}^0 \rangle$ and $(1-\tau)\mathfrak{h}^0$?*

I have no idea as to what extent the affirmative answer of this is convincing. I only have checked this up to weight 10 by computer. Perhaps I should remark that, up to weight 6, the relations $\langle y, \mathfrak{h}^0 \rangle$ and $\langle \mathfrak{h}^0, \mathfrak{h}^0 \rangle$ are enough to reduce dimensions to the conjectured ones, but from weight 7 on (to 10 at least), we need the duality.

Question. *Let $\mathfrak{h}^0_m$ be the degree m part of $\mathfrak{h}^0$. What is the dimension (number of independent relations) of $\langle \mathfrak{h}^0_m, \mathfrak{h}^0_n \rangle$?*

As far as I calculated, no reductions (other than obvious ones) occur within $\langle \mathfrak{h}^0_m, \mathfrak{h}^0_n \rangle$.

References

1. M. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), 275–290.
2. ———, *The algebra of multiple harmonic series*, J. of Algebra **194** (1997), 477–495.