The third order modular linear differential equations

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Abstract

We propose a third order generalization of the Kaneko-Zagier modular differential equation, which has two parameters. We describe modular and quasimodular solutions of integral weight in the case where one of the exponents at infinity is a multiple root of the indicial equation. We also classify solutions of "character type", which are the ones that are expected to relate to characters of simple modules of vertex operator algebras and one-point functions of two-dimensional conformal field theories. Several connections to generalized hypergeometric series are also discussed.

AMS Subject Classification 2010: Primary 11F11, 81T40, Secondary 17B69

Key words: Vertex operator algebra, Modular invariance, Modular linear differential equation, n-dimensional conformal field theory

1 Introduction

This paper studies a third order generalization of the Kaneko-Zagier equation (K-Z equation for short), which is called the *third order K-Z equation* here. The K-Z equation first appeared in [11] in connection with supersingular j-invariants of elliptic curves, and subsequently, various modular and quasimodular solutions of the K-Z equation were found and studied in [5]–[8], etc. One of the characteristic properties of the K-Z equation is the invariance of the space of solutions under the standard slash action of the modular group $\Gamma_1 = SL_2(\mathbb{Z})$, and indeed, our generalization has the same property.

[1]This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 23340010.

[2]This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Challenging Exploratory Research 26610007. The second author was also partially supported by Max-Planck Institute für Mathematik.

Under mild conditions on the coefficient functions, we determine in §2 the form of what we call the third order *modular linear differential equation* as

$$f''' - \frac{k+2}{4}E_2(\tau)f'' + \left\{\frac{(k+1)(k+2)}{4}E_2'(\tau) + \alpha E_4(\tau)\right\} f'(\tau)$$
$$- \left\{\frac{k(k+1)(k+2)}{24}E_2''(\tau) + \frac{k\alpha}{4}E_4'(\tau) - \beta E_6(\tau)\right\} f(\tau) \;=\; 0\,, \quad (1)$$

where τ is a variable in the complex upper half-plane $\mathbb{H}$, and $'$ is the Euler operator of $q\,(= e^{2\pi\sqrt{-1}\tau})$ (see Theorem 1 in §2). The $E_k(\tau)$ is the *normalized* Eisenstein series of weight k given by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where B_k is the kth Bernoulli number and $\sigma_m(n)$ is the sum of mth powers of positive divisors of n. The parameter k is expected to stand for the weight of f, and α, β are complex parameters.

As shown in [8], the K-Z equation is closely related to two-dimensional conformal field theory (2DCFT for short). The papers [1] and [3], which may be viewed as companion papers of the present one, study affine 2DCFT with at most 20 simple modules or 5 independent pseudo-characters and the minimal models with at most three simple modules, respectively. The (formal) characters of such 2DCFT were expected to satisfy one of the third order K-Z equations.

One of the main results in this paper (given in §3) is an almost complete description of modular and quasimodular solutions in the case where the indicial equation of (1) with $\beta = 0$ at $q = 0$ has a multiple root and k is an integer (Theorem 2 in §3.1, Theorem 4 in §3.2). The other is the determination of solutions of *character type*, which is characterized by integrality and positivity of Fourier coefficients of an associated weight 0 function (Proposition 6 and Theorem 7 in §4). As motivated by [10], we discuss in §5 a relation between the third order K-Z equation and hypergeometric series (Theorem 8 in §5).

2 Modular linear differential equations of third order

Let $\mathbb{H}$ be the complex upper half-plane and $\mathcal{F}$ the space of holomorphic functions on $\mathbb{H}$. The slash operator of weight k on $\mathcal{F}$ is defined as usual by

$$(f\,|_k\gamma)(\tau) \;=\; (c\tau+d)^{-k}f(\gamma(\tau)) \text{ for each } \gamma \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

for each real number k. In this section, we determine the general form of third order linear differential equations

$$f'''(\tau) + A(\tau)f''(\tau) + B(\tau)f'(\tau) + C(\tau)f(\tau) = 0\,, \quad ' \;=\; q\frac{d}{dq} \;=\; \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau} \qquad (2)$$

under the conditions that each coefficient $A(\tau)$, $B(\tau)$ and $C(\tau)$ are holomorphic on $\mathbb{H}$, bounded as $\mathrm{Im}(\tau) \to \infty$, and the space of solutions is invariant under the slash action $|_k\gamma$ for every $\gamma \in \Gamma_1 = SL_2(\mathbb{Z})$. In general a linear ordinary differential equation on $\mathbb{H}$ with meromorphic coefficients is called a *modular linear differential equation* (MLDE) of weight k on a discrete subgroup Γ of $SL_2(\mathbb{R})$ if the space of solutions is invariant under the slash action $|_k\gamma$ of a fixed weight k for each $\gamma \in \Gamma$.

Theorem 1. *Let $f'''(\tau) + A(\tau)f''(\tau) + B(\tau)f'(\tau) + C(\tau)f(\tau) = 0$ be a third order linear differential equation for f such that the coefficient functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ are holomorphic on $\mathbb{H}$ and are bounded as $\mathrm{Im}(\tau) \to \infty$. Then this is a modular linear differential equation of weight k on $SL_2(\mathbb{Z})$ if and only if the equation is given in the form*

$$\begin{aligned} f''' - \frac{k+2}{4}E_2(\tau)f'' + \left\{\frac{(k+1)(k+2)}{4}E_2'(\tau) + \alpha E_4(\tau)\right\} f'(\tau) \\ - \left\{\frac{k(k+1)(k+2)}{24}E_2''(\tau) + \frac{k\alpha}{4}E_4'(\tau) - \beta E_6(\tau)\right\} f(\tau) = 0 \,, \end{aligned} \quad (3)$$

where α and β are complex numbers and $' = q\dfrac{d}{dq}$.

Proof. The requirement that the equation is invariant under $|_k\gamma$ ($\gamma \in \Gamma_1$) implies (after a little complicated but similar calculations which were given in [6, Section 5])

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 A(\tau) - 3\mu(k+2)(c\tau+d)\,, \quad (4)$$

$$B\left(\frac{a\tau+b}{c\tau+d}\right) = \mu(c\tau+d)^4 B(\tau) - 2(k+1)(c\tau+d)^3 A(\tau) + 3\mu^2(k+1)(k+2)(c\tau+d)^2\,, \quad (5)$$

$$\begin{aligned} C\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^6 C(\tau) - k\mu(c\tau+d)^5 B(\tau) + \mu^2 k(k+1)(c\tau+d)^4 A(\tau) \\ - \mu^3 k(k+1)(k+2)(c\tau+d)^3\,, \end{aligned} \quad (6)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ and $\mu = c/2\pi i$. By (4)–(6) and the transformation formula of the weight two (quasimodular) Eisenstein series

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + 12\mu(c\tau+d) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\,,$$

it follows that the three functions

$$A(\tau) + \frac{(k+2)}{4}E_2(\tau)\,, \quad B(\tau) + (k+1)A'(\tau)\,, \quad C(\tau) + \frac{k}{4}B'(\tau) + \frac{k(k+1)}{12}A''(\tau) \quad (7)$$

are invariant under slash operators $|_2$, $|_4$ and $|_6$ on $\gamma \in \Gamma_1$, respectively. Because $A(\tau)$, $B(\tau)$ and $C(\tau)$ are holomorphic and are bounded as $\mathrm{Im}(\tau) \to \infty$, these functions are holomorphic modular forms of weights 2, 4 and 6, respectively. By the well-known fact that there does not exist a non-zero holomorphic modular form of weight 2 on Γ_1 and that the space of holomorphic modular forms of weights 4 and 6 are both one-dimensional and are spanned by $E_4(\tau)$ and $E_6(\tau)$, respectively, we conclude that

$$A(\tau) = -\frac{(k+2)}{4}E_2(\tau)\,, \tag{8}$$

$$B(\tau) = \frac{(k+1)(k+2)}{4}E_2'(\tau) + \alpha E_4(\tau)\,, \tag{9}$$

$$C(\tau) = -\frac{k(k+1)(k+2)}{24}E_2''(\tau) - \frac{k\alpha}{4}E_4'(\tau) + \beta E_6(\tau) \tag{10}$$

with complex numbers α and β.

Conversely, the discussions above show that for any complex numbers α and β, eq. (3) is a modular linear differential equation of weight k on Γ_1. □

We call (3) the **third order Kaneko-Zagier equation** (of weight k) or the *third order K-Z equation* (of weight k) for short.

Remarks. (1) Eq. (3) can be rewritten with the help of the Ramanujan relations $12E_2' = E_2^2 - E_4$, $3E_4' = E_2E_4 - E_6$ and $2E_6' = E_2E_6 - E_4^2$ as

$$\vartheta_k^3(f) + \hat{\alpha}E_4\vartheta_k(f) + \hat{\beta}E_6(\tau)(f) = 0 \tag{11}$$

with the values of $\hat{\alpha} = \alpha - (3k^2+12k+8)/144$ and $\hat{\beta} = \beta + k\alpha/12 - k^2(k+3)/864$. Here $\vartheta_k(f)$ is the Serre derivative $\vartheta_k(f) := f' - \frac{k}{12}E_2 f$ and the iterated Serre derivations are defined by $\vartheta_k^2 = \vartheta_{k+2}\circ\vartheta_k$ and $\vartheta_k^3 = \vartheta_{k+4}\circ\vartheta_{k+2}\circ\vartheta_k$.
(2) The third order modular differential equation obtained by applying the Serre derivation to the second order K-Z equation $\vartheta_k^2(f) - \frac{k(k+2)}{144}E_4(f) = 0$ is a special case of (11) with the values $\alpha = (k+1)(k+4)/72$ and $\beta = 0$.

In the rest of the paper, we consider the special case $\beta = 0$ of (3), that is, the differential equation

$$f'''(\tau) - \frac{k+2}{4}E_2(\tau)f''(\tau) + \left\{\frac{(k+1)(k+2)}{4}E_2'(\tau) + \alpha E_4(\tau)\right\} f'(\tau) - \left\{\frac{k(k+1)(k+2)}{24}E_2''(\tau) + \frac{k\alpha}{4}E_4'(\tau)\right\} f(\tau) = 0\,. \tag{12}$$

The reason why we restrict ourselves to this case is first to reduce the number of parameters from 3 to more manageable 2, but why we choose $\beta = 0$ among other specializations comes from our interest in 2DCFT or VOA. There, characters often take the form f/η^{2k} where f is a modular form of weight k = half of the central charge and is $1 + O(q)$ satisfying a modular differential equation, and $\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. A typical example is the character of a lattice vertex operator algebra, which is written as Θ_L/η^{2d}, where L is a d-dimensional lattice and Θ_L is the theta series associated to L. If f is a solution of (3) and $f = 1 + O(q)$, then β should necessarily equal 0.

Our aim in the following two sections is to give a fairly complete description of both general and "character type" solutions of (3) under the assumption that the indicial equation

$$\lambda^3 - \frac{k+2}{4}\lambda^2 + \alpha\lambda = 0 \tag{13}$$

has a multiple root and the weight k is an integer.

3 Third order Kaneko-Zagier equations with multiple exponents

In this section we sutudy modular or quasimodular solutions of (12) when equation (13) has a multiple root and the weight parameter k is an integer. We are not able to establish any general statement when k is not an integer. However, computer experiments suggest that there are no (quasi)modular solutions (whose Fourier coefficients have bounded denominators) to (12) when equation (13) has a multiple root and k is not an integer.

Equation (13) has a triple root if and only if $k = -2$ and $\alpha = 0$ (the root is 0). This case is trivial because (12) becomes $f''' = 0$ and a fundamental system of solutions of this equation is $\{1, \tau, \tau^2\}$. We therefore assume in the remaining sections that $(k, \alpha) \neq (-2, 0)$.

Clearly, a possible double root of (13) is either $\lambda = 0$ or $\lambda = (k+2)/8$. If $\lambda = 0$ is a double root, then $\alpha = 0$ (and $k \neq -2$) and the corresponding MLDE is

$$f''' - \frac{k+2}{4}E_2 f'' + \frac{(k+1)(k+2)}{4}E_2' f' - \frac{k(k+1)(k+2)}{24}E_2'' f = 0\,. \tag{14}$$

If $\lambda = (k+2)/8$ is a double root ($k \neq -2$), then $\alpha = (k+2)^2/64$ and the corresponding MLDE is

$$f''' - \frac{k+2}{4}E_2 f'' + \left\{\frac{(k+1)(k+2)}{4}E_2' + \frac{(k+2)^2}{64}E_4\right\} f' - \left\{\frac{k(k+1)(k+2)}{24}E_2'' + \frac{k(k+2)^2}{256}E_4'\right\} f = 0\,. \tag{15}$$

Convention. We sometimes use equation numbers in the text such as $(14)_k$ to make the dependence on the parameter k explicit.

In fact, solutions of one of (14) and (15) are obtained from those of the other. More precisely, as is checked easily by direct computations, if f is a solution of $(15)_{-2k-6}$ (resp. of $(14)_{-(k+6)/2}$), then $f\Delta^{(k+2)/4}$ (resp. $f\Delta^{(k+2)/8}$) is a solution of $(14)_k$ (resp. of $(15)_k$), and this correspondences is a bijection of the sets of solutions of (14) and (15). Therefore, we only need to consider either of equations (14) and (15). Or alternatively, by the equivalence

$$k > -2 \Leftrightarrow -2k-6 < -2 \Leftrightarrow -(k+6)/2 < -2,$$

it suffices to consider both (14) and (15) under the assumption $k > -2$. Since we are mainly interested in (quasi)modular forms of positive weights, we hereafter assume $k > -2$ and study (14) and (15) separately in the following subsections.

3.1 Case $\lambda = 0$ is a double root

We describe solutions of $(14)_k$ when k is an integer > -2. If $k = -1$, $(14)_k$ becomes $f''' - \frac{1}{4}E_2 f'' = 0$, and a set of fundamental solutions is given by $1, \log q = 2\pi\sqrt{-1}\tau$, and the double Eichler integral of η^6 (a solution of $f'' = \eta^6$).

We define a sequence of functions f_k by the four-term recursion formula

$$f_{k+4} = a_k E_4 f_k - b_k E_4^2 f_{k-4} + c_k \Delta f_{k-8} \quad (k \geq 4),$$

where coefficients and initial values are given by

$$a_k = \begin{cases} \dfrac{2(k^3+k^2-6k-12)}{(k-3)(k+2)(k+4)} & \text{if } k \not\equiv 2 \pmod 4, \\ \dfrac{(k+6)^2(k^3+k^2-6k-12)}{128(k-3)(k+3)(k+4)^2(k+5)} & \text{if } k \equiv 2 \pmod 4, \end{cases}$$

$$b_k = \begin{cases} \dfrac{(k-2)k(k+1)}{(k-3)(k+2)(k+4)} & \text{if } k \not\equiv 2 \pmod 4, \\ \dfrac{(k-2)^2(k+2)^2(k+6)^2}{65536(k-3)(k-1)(k+3)(k+4)^2(k+5)} & \text{if } k \equiv 2 \pmod 4, \end{cases}$$

$$c_k = \begin{cases} \dfrac{256(k-5)(k-4)k(k+1)}{(k-6)(k-2)(k+2)(k+4)} & \text{if } k \not\equiv 2 \pmod 4, \\ b_k \quad \text{if } k \equiv 2 \pmod 4, \end{cases}$$

and $f_{-4} = 0, f_{-3} = \eta^{-6}, f_{-2} = f_{-1} = f_0 = 1, f_1 = E_2, f_2 = -E_2'/24, f_3 = f_4 = E_4, f_5 = E_6 + 49E_4'/15, f_6 = -(21E_4'' + 10E_6')/151200, f_7 = E_4^2 + 16E_6'/105$.

When $k \equiv 0, 2 \pmod 4$, these f_k s are same (up to normalization constants) as functions already given in [10, Theorem 3.1] and [7, Theorem 3.1]. We now have:

Theorem 2. *Let k be a non-negative integer. The function f_k is a solution of* $(14)_k$. *Its q-expansion is $f_k = 1 + O(q)$ if $k \not\equiv 2 \pmod 4$ and $f_k = q^{(k+2)/4} + O(q^{(k+6)/4})$ if $k \equiv 2 \pmod 4$. Moreover,*

(1) *if $k \equiv 0 \pmod 4$, f_k is a modular form of weight k on $SL_2(\mathbb{Z})$,*

(2) *if $k \equiv 2 \pmod 4$, f_k is a quasimodular form of weight $k+2$ and depth 2 on $SL_2(\mathbb{Z})$,*

(3) *if k is odd, f_k is a quasimodular form of weight $k+1$ and depth at most 1 on $SL_2(\mathbb{Z})$.*

We can prove the theorem in a similar manner as in the proof of [7, Theorem 3.1] by using Lemma 3 below.

Let $[f,g]_n^{(k,\ell)}$ $(n \geq 0)$ be the **Rankin-Cohen bracket** which is defined by

$$[f,g]_n^{(k,\ell)} = \sum_{\substack{r,s \geq 0 \\ r+s=n}} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} g^{(s)}, \quad f^{(r)} = \left(q\frac{d}{dq}\right)^r f$$

for modular forms f and g of weight k and ℓ on Γ_1 (indeed, the Rankin-Cohen bracket is well defined for any holomorphic functions f and g), respectively. Then it is known that $[f,g]_n^{(k,\ell)}$ is a modular form of weight $k+\ell+2n$ on Γ_1. An easy calculation shows that

$$[f,g]_1^{(k,\ell)} = kf\vartheta_\ell(g) - \ell\vartheta_k(f)g,$$

$$[f,E_4]_2^{(k,4)} = 10E_4\,\vartheta_k^2(f) + \frac{5}{3}(k+1)E_6\,\vartheta_k(f) + \frac{5}{72}k^2E_4^2\,f, \tag{16}$$

$$[f,E_6]_2^{(k,6)} = 21E_6\,\vartheta_k^2(f) + \frac{7}{2}(k+1)E_4^2\,\vartheta_k(f) + \frac{7}{48}k^2E_4E_6\,f. \tag{17}$$

Lemma 3. *Let k be a rational number.*
(1) *Suppose that f is is a solution of $(14)_k$. Then we have*

$$\begin{aligned} \vartheta_{k+8}\left([f,E_4]_2^{(k,4)}\right) &= \frac{5}{63}(k-1)[f,E_6]_2^{(k,6)}\,, \\ \vartheta_{k+10}\left([f,E_6]_2^{(k,6)}\right) &= \frac{7}{20}(k-2)E_4[f,E_4]_2^{(k,4)} - 42k^2(k+1)\Delta f\,, \end{aligned}$$

and $[f,E_4]_2^{(k,4)}/\Delta$ is a solution of $(14)_{k-4}$.

(2) *Suppose that F_k, F_{k-4} and F_{k-8} are solutions of $(14)_k$, $(14)_{k-4}$ and $(14)_{k-8}$, respectively. Then $F_{k+4} = E_4F_k + E_4^2F_{k-4} + \Delta F_{k-8}$ is a solution of $(14)_{k+4}$ if and only if*

$$\begin{aligned} &[F_k,E_6]_2^{(k,6)} + 2E_4[F_{k-4},E_6]_2^{(k-4,6)} \\ &\qquad = -\frac{7}{8}\Delta\left(27648\vartheta_{k-4}(F_{k-4}) + 12kE_4\vartheta_{k-8}(F_{k-8}) + (k^2-2k+12)E_6F_{k-8}\right)\,. \end{aligned} \tag{18}$$

Proof. We give a sketch of a proof. A fairly direct calculation of both sides by using equation $(11)_k$ (with $\alpha = \beta = 0$)

$$\vartheta_k^3(f)(\tau) - \frac{3k^2+12k+8}{144}E_4(\tau)\vartheta_k(f)(\tau) - \frac{k^2(k+3)}{864}E_6(\tau)f(\tau) \;=\; 0 \tag{19}$$

together with (16) and (17) provides the equalities in (1).

That the $[f,E_4]_2^{(k,4)}/\Delta$ is a solution of $(14)_{k-4}$ can be seen by substituting $[f,E_4]_2^{(k,4)}/\Delta$ into $(19)_{k-4}$ and using the relation $\vartheta_{k-4}\big([f,E_4]_2^{(k,4)}/\Delta\big) = \vartheta_{k+8}\big([f,E_4]_2^{(k,4)}\big)/\Delta$ with (16).

Finally, substituting F_{k+4} into $(19)_{k+4}$, we see that the left-hand side of $(19)_{k+4}$ coincides with

$$\begin{aligned} -\frac{1}{21}\Big(&[F_k,E_6]_2^{(k,6)} + 2E_4[F_{k-4},E_6]_2^{(k-4,6)} \\ &+ \frac{7}{8}\Delta\left(27648\vartheta_{k-4}(F_{k-4}) + 12kE_4\vartheta_{k-8}(F_{k-8}) + (k^2-2k+12)E_6F_{k-8}\right)\Big)\,, \end{aligned}$$

which proves (3). □

The condition (18) in (2) of the lemma is effectively used in proving that the recursively defined function f_k satisfies the differential equation $(14)_k$.

3.2 Case $\lambda = (k+2)/8$ is a double root

Solutions of $(15)_k$ are described similarly as in the previous subsection. However, we need modular forms of level 2 here, and computer experiments suggest that there exist modular or quasimodular solutions only when k is even.

Let $H_2(\tau) = 2E_2(2\tau) - E_2(\tau)$ and $\Delta_2(\tau) = \eta(2\tau)^8/\eta(\tau)^4$, which are modular forms of weight 2 on $\Gamma_0(2)$ and $\Gamma(2)$, respectively. The groups $\Gamma_0(2)$ and $\Gamma(2)$ are the standard congruence subgroups of $SL_2(\mathbb{Z})$ of level 2;

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod 2 \right\},$$
$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}.$$

For even integers $k \not\equiv 6 \pmod 8$, we define g_k by the four-term recursion formula

$$g_{k+8} = E_4^2 g_k - b_k E_4 \Delta g_{k-8} + c_k \Delta^2 g_{k-16} \quad (k \geq 8)$$

with coefficients

$$b_k = \frac{512k(k^3 - 8k^2 - 4k + 128)}{(k-6)^2(k+2)^2}, \quad c_k = \frac{65536(k-12)(k-10)^2(k-8)k(k+4)}{(k-14)^2(k-6)^2(k+2)^2}$$

and with initial values $g_{-8} = g_{-6} = g_{-4} = 0, g_0 = 1, g_2 = H_2, g_4 = E_4, g_8 = E_4^2, g_{10} = H_2(H_2^4 + 61440\Delta_2^4), g_{12} = (3E_4^3 - 2048\Delta)/3$.

We define another series of functions h_k for even $k \equiv 2 \pmod 4$ by the similar recursion

$$h_{k+8} = a_k E_4^2 h_k - b_k E_4 \Delta h_{k-8} + c_k \Delta^2 h_{k-16} \quad (k \geq 10)$$

with coefficients

$$a_k = \frac{(k+2)^2(k+10)}{256(k+4)(k+6)(k+8)}, \quad b_k = \frac{(k+2)(k+10)(k^3 - 8k^2 - 4k + 128)}{128(k-4)(k-2)(k+4)(k+6)(k+8)},$$
$$c_k = \frac{(k-10)(k-6)(k+2)(k+10)}{256(k-4)(k-2)(k+6)(k+8)}$$

and initial values $h_{-6} = \eta^{-12}, h_{-2} = 1, h_2 = \Delta_2, h_6 = E_4''/240, h_{10} = \Delta_2^3(H_2^2 + 192\Delta_2^2/5)$, $h_{14} = (E_4^3 - 720\Delta)''/786240$. Then we have:

Theorem 4. *Let k be a non-negative even integer. The functions g_k for $k \not\equiv 6 \pmod 8$ and h_k for $k \equiv 2 \pmod 4$ are solutions of $(15)_k$ with Fourier expansions of the form $g_k = 1 + O(q)$ and $h_k = q^{(k+2)/8} + O(q^{(k+10)/8})$. Moreover,*
(1) if $k \equiv 0 \pmod 4$, the function g_k is a modular form of weight k on $SL_2(\mathbb{Z})$,
(2) if $k \equiv 2 \pmod 8$, the function g_k and h_k are modular forms of weight k on $\Gamma_0(2)$ and $\Gamma(2)$, respectively,
(3) if $k \equiv 6 \pmod 8$, the function h_k is a quasimodular form of weight $k+2$ and depth at most 2 on $SL_2(\mathbb{Z})$.

Remarks. (1) Since $c_8 = c_{10} = c_{12} = 0$, we may choose any functions as the initial values g_{-8}, g_{-6}, and g_{-4}. The following (non-modular) functions are solutions of $(15)_{-8}$, $(15)_{-6}$, and $(15)_{-4}$ with $1 + O(q)$ respectively.

$$g_{-8} = \frac{9}{16\eta^{18}} \left(\big(\log(q)E_2 + 12\big) \int_0^q \frac{E_2 \Delta^{3/4}}{E_4^2} \frac{dq}{q} - E_2 \int_0^q \frac{\big(\log(q)E_2 + 12\big)\Delta^{3/4}}{E_4^2} \frac{dq}{q} \right),$$
$$g_{-6} = \frac{1}{2\,\eta^{12}} \int_0^q \Delta_2 \frac{dq}{q}, \quad g_{-4} = \frac{1}{16\,\eta^6} \int_0^q \int_0^q \eta^6 \left(\frac{dq}{q}\right)^2.$$

(2) From the general theory of ordinary differential equations, we know that if f and g are two independent solutions of $(3)_k$, then the other (meromorphic) solution is given by

$$g\int_0^q \frac{\Delta^{\frac{k+2}{4}}}{f^3\left\{(g/f)'\right\}^2}\frac{dq}{q} - f\int_0^q \frac{g\,\Delta^{\frac{k+2}{4}}}{f^4\left\{(g/f)'\right\}^2}\frac{dq}{q}\,.$$

The proof of Theorem 4 goes similarly to that of Theorem 2 if we replace Lemma 3 by the lemma below.

Lemma 5. *Let k be a rational number.*
(1) *Suppose that f is a solution of $(15)_k$. Then we have*

$$\begin{aligned}\vartheta_{k+8}\big([f,E_4]_2^{(k,4)}\big) &= \frac{5}{63}(k-1)[f,E_6]_2^{(k,6)} + \frac{5}{128}(k+2)^2E_4[f,E_4]_1^{(k,4)},\\ \vartheta_{k+10}\big([f,E_6]_2^{(k,6)}\big) &= \frac{7}{20}(k-2)E_4[f,E_4]_2^{(k,4)} + \frac{21}{64}(k+2)^2E_6[f,E_4]_1^{(k,4)} - 42k^2(1+k)\Delta f\,,\end{aligned}$$

and $\big([f,E_4]_2^{(k,4)} + \frac{5}{18}(k+1)[f,E_6]_1^{(k,6)}\big)/E_4\Delta$ *is a solution of* $(15)_{k-8}$.
(2) *Suppose that F_k, F_{k-8} and F_{k-16} are solutions of $(15)_k$, $(15)_{k-8}$ and $(15)_{k-16}$, respectively. Then $F_{k+8} = E_4^2F_k + E_4\Delta F_{k-8} + \Delta^2F_{k-16}$ is a solution of $(15)_{k+8}$ if and only if*

$$\begin{aligned}&\frac{1}{5}E_6[F_k,E_4]_2^{(k,4)} + \frac{k+6}{48}[F_k,E_4^3]_1^{(k,12)} + 576(k+3)\Delta\vartheta_k(F_k)\\ &\quad + \Delta\left(\frac{1}{21}[F_{k-8},E_6]_2^{(k-8,6)} + \frac{1}{24}[F_{k-8},E_4^2]_1^{(k-8,8)} - \frac{9k^2+36k+292}{576}E_4E_6F_{k-8}\right)\\ &= \Delta^2\left(-\frac{k-2}{4}E_4\vartheta_{k-16}(F_{k-16}) + \frac{k^2+44}{96}E_6F_{k-16}\right)\,.\end{aligned}$$

3.3 Quasimodular forms and solutions with logarithmic terms

We found solutions of quasimodular forms of the third order K-Z equations in Theorems 2 and 4. A simple observation shows that, because of the modular invariance of the space of solutions, if MLDEs of weight k have solutions of quasimodular forms of weight $k+r$ and depth $r>0$, then there exist solutions with logarithmic terms. We briefly illustrate this in the cases of depth 1 and 2.

Suppose that a quasimodular form $J_k := A_kE_2 + B_k$ of weight $k+1$ and depth 1 is a solution of a K-Z equation of order 3 and weight k, where A_k and B_k are modular forms on Γ_1 of weight $k-1$ and $k+1$, respectively. Then the function $G_k := \tau^{-k}J_k(-1/\tau) = (2\pi\sqrt{-1})^{-1}(J_k\log q + 12A_k)$ is a solution because of the modular invariance. Moreover, it follows from the transformation formula of E_2 that

$$\begin{pmatrix}J_k\\G_k\end{pmatrix}\Bigg|_k\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}d&c\\b&a\end{pmatrix}\begin{pmatrix}J_k\\G_k\end{pmatrix}\ \text{for each}\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1\,.$$

For example, eqs. $(14)_1$ and $(14)_5$ have solutions $J_1 = E_2$ and $J_5 = (49E_2E_4 - 4E_6)/45$, respectively. Then $G_1 = (2\pi\sqrt{-1})^{-1}(J_1\log q + 12)$ and $G_5 = (2\pi\sqrt{-1})^{-1}(F_5\log q + 196E_4/15)$ are solutions of $(14)_1$ and $(14)_5$ with logarithmic terms, respectively.

Next suppose that a quasimodular form $K_k := A_k E_2^2 + B_k E_2 + C_k$ of weight $k+2$ and depth 2 is a solution of a third order K-Z equation of weight k, where A_k, B_k and C_k are modular forms on Γ_1 of weight $k-2$, k and $k+2$, respectively. Then we have solutions with logarithmic terms

$$\begin{aligned} I_k &= (2\pi\sqrt{-1})^{-2} \left(K_k (\log q)^2 + 12(2A_k E_2 + B_k) \log q + 144 A_k \right) , \\ G_k &= (2\pi\sqrt{-1})^{-1} \left(K_k \log q + 6(2A_k E_2 + B_k) \right) , \end{aligned}$$

which are obtained by $I_k = K_k|_k \gamma_1$ and $G_k = (K_k|_k\gamma_2 - K_k - I_k)/2$ ($\gamma_1 : \tau \mapsto -1/\tau,\ \gamma_2 : \tau \mapsto -1/(\tau+1)$), respectively. It then follows that

$$\left.\begin{pmatrix} K_k \\ G_k \\ I_k \end{pmatrix}\right|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^2 & 2cd & c^2 \\ bd & ad+bc & ac \\ b^2 & 2ab & a^2 \end{pmatrix} \begin{pmatrix} K_k \\ G_k \\ I_k \end{pmatrix} \text{ for each } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 .$$

For instance, eq. $(15)_6$ has a solution $K_6 = (E_2^2 E_4 - 2E_2E_6 + E_4^2)/1728$. Consequently, there exist solutions $I_6 = (2\pi\sqrt{-1})^{-2} \left(K_6 (\log q)^2 + (E_2E_4 - E_6)\log q/72 + E_4/12 \right)$ and $G_6 = (2\pi\sqrt{-1})^{-1} \left(K_6 \log q + (E_2E_4 - E_6)/144 \right)$.

This kind of phenomena that if a MLDE has a quasimodular solution of depth r then associated vector-valued modular form corresponds to the symmetric tensor representation with degree $r+1$ is studied for instance in [4] and [9].

4 Solutions of character type

As mentioned in the introduction, we are interested in special type of solutions of MLDEs in connection with 2DCFT and VOAs, namely, solutions of "character type" which is defined as follows.

Definition. A solution f of the third order K-Z equation (of weight k) is said to be of **character type** if it is a (quasi)modular form and all Fourier coefficients of f/η^{2k} are non-negative integers. Furthermore, if its leading Fourier coefficient is 1, we call it of **vacuum character type**.

In this section we shall give a fairly complete description of solutions of character type of equation (14). As explained in the paragraph before §3.1, equations (14) and (15) are equivalent in the sense that solutions of either of these equations are obtained from the other by multiplying a suitable power of η. And under this equivalence, the property of a solution being of character type is clearly unchanged because we look at the Fourier coefficients of the associated weight 0 function obtained by multiplying a power of η. Hence, we shall exclusively look at equation (14), but for any (positive and non-positive) integer k.

Let f be a solution of $(14)_k$ and set $g = f/\eta^{2k}$. Then $(14)_k$ is rewritten in terms of g as

$$g''' - \frac{1}{2}E_2\, g'' + \left\{ \frac{1}{2}E_2' - \frac{k(k+4)}{48} E_4 \right\} g' - \frac{k^2(k+3)}{864} E_6\, g = 0 . \tag{20}$$

Our aim in this section is to determine all solutions g of $(20)_k$ which have the form

$$g \,=\, q^{\nu}\Big(1+\sum_{n=1}^{\infty} a_n q^n\Big), \tag{21}$$

where each a_n is a *non-negative integer* and $\nu \in \mathbb{R}$. Since the characteristic equation of (20) is

$$\nu^3 - \frac{1}{2}\nu^2 - \frac{k(k+4)}{48}\nu - \frac{k^2(k+3)}{864} \;=\; \left(\nu + \frac{k}{12}\right)^2 \left(\nu - \frac{k+3}{6}\right) \;=\; 0\,,$$

the indicial roots of (20) are $\{-k/12, -k/12, (k+3)/6\,\}$. The indicial roots $\nu = -k/12$ and $(k+3)/6$ of g correspond to the exponents $\mu = 0$ and $(k+2)/4$ of f, respectively. We compute coefficients a_n of a solution (21) of (20) by the Frobenius method, and seek for conditions that a_n are non-negative integers. (Even if an index is a double root, we can obtain a solution when a pair of indices does not have a integral difference.)

Substituting (21) into (20), we have

$$\begin{aligned} 2\left(n+\nu+\frac{k}{12}\right)^2\left(n+\nu-\frac{k+3}{6}\right)a_n \;=\; & \\ \sum_{i=1}^{n}\left((n+\nu-i)^2\,e_{2,i} - \left(i\cdot e_{2,i} - \frac{k(k+4)}{24}e_{4,i}\right)(n+\nu-i) + \frac{k^2(k+3)}{432}e_{6,i}\right)a_{n-i} & \end{aligned} \tag{22}$$

with $a_0 = 1$, where $E_2(\tau) = \sum_{i=0}^{\infty} e_{2,i}\,q^i$, $E_4(\tau) = \sum_{i=0}^{\infty} e_{4,i}\,q^i$ and $E_6(\tau) = \sum_{i=0}^{\infty} e_{6,i}\,q^i$, respectively.

We consider two cases, that is, $\nu = -k/12$ and $\nu = (k+3)/6$, separately in the following subsections.

4.1 Case $\nu = -k/12$

We now study the solutions of $(20)_k$ with an indicial root $\nu = -k/12$, which corresponds to the solutions of $(14)_k$ with an exponent $\mu = 0$. Though we are not able to prove the positivity of Fourier coefficients for all weights, we can list all possible solutions of vacuum character type when the weights are integers.

Theorem 6. *Let k be an integer.*
(1) *Suppose that the equation $(14)_k$ has a solution of vacuum character type with the exponent* 0. *Then k is one of the values in the set*

$$\{\,-30,\ -22,\ -14,\ -10,\ -6,\ -4,\ 0,\ 3,\ 4,\ 8\,\}\,. \tag{23}$$

(2) *For each $k \in \{-10,\ -6,\ -4,\ 0,\ 3,\ 4,\ 8\}$, the function v_k given below is a solution of $(14)_k$*

of vacuum character type with the exponent 0;

$$\begin{aligned}
v_0 &= 1, \\
v_3 &= v_4 = E_4 = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + \cdots, \\
v_8 &= E_4^2 = 1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + 37500480q^5 + \cdots, \\
v_{-4} &= \frac{\Delta_2}{\sqrt{\Delta}} = 1 + 16q + 144q^2 + 960q^3 + 5264q^4 + 25056q^5 + \cdots, \\
v_{-6} &= \frac{h_6}{\Delta} = 1 + 60q + 1440q^2 + 22080q^3 + 253680q^4 + 2369160q^5 + \cdots, \\
v_{-10} &= \frac{h_{14}}{\Delta^2} = 1 + 240q + 18540q^2 + 792960q^3 + 23080560q^4 + 508465440q^5 + \cdots,
\end{aligned}$$

where the subscripts indicate the weights of the solutions.
(3) *For each* $k = -14, -22, -30$*, we have the following solution* v_k *of* $(14)_k$*, which is possibly of vacuum character type*;

$$\begin{aligned}
v_{-14} &= \frac{h_{22}}{\Delta^3} = 1 + 546q + 88452q^2 + 7440888q^3 + 405394080q^4 + 16071109236q^5 + \cdots, \\
v_{-22} &= \frac{h_{38}}{\Delta^5} = 1 + 1540q + 657360q^2 + 137466120q^3 + 17723389420q^4 + \cdots, \\
v_{-30} &= \frac{h_{54}}{\Delta^7} = 1 + 3045q + 2494870q^2 + 974923740q^3 + 229294066260q^4 + \cdots,
\end{aligned}$$

where $\Delta_2(\tau) = \eta(2\tau)^8/\eta(\tau)^4$ *and* h_k *is a quasimodular form defined in* §3.2.
(4) *The functions* v_k $(k = 0, 3, 4, 8)$ *are modular forms of weight* k (*except that* v_3 *has weight* 4) *and* v_k $(k = -6, -10, -14, -22, -30)$ *are quasimodular forms of weight* $k+2$ *and depth* 2 *on* $SL_2(\mathbb{Z})$*. The function* v_{-4} *is a meromorphic modular form on* $\Gamma_0(2)$ *of weight* -4 *with the pole at the cusp* 0.

Proof. Using the recursive formula (22) with $\nu = -k/12$

$$2n^2\left(n - \frac{k+2}{4}\right)a_n = \sum_{i=1}^{n}\left((n - \frac{k}{12} - i)^2\, e_{2,i} - \left(i\, e_{2,i} - \frac{k(k+4)}{24}e_{4,i}\right)\left(n - \frac{k}{12} - i\right) + \frac{k^2(k+3)}{432}e_{6,i}\right)a_{n-i}\,, \tag{24}$$

we can determine the Fourier coefficients a_n of a solution g of (20) of the form (21).

The case $n = 1$ of (24) gives $(1 - k/2)a_1 = -2k^3 - 7k^2 - 2k$. Then it follows that $a_1 = 4k^2 + 22k + 48 + 96/(k-2)$. Suppose that a_1 is an integer. Then the condition $(k-2)|96$ gives $k-2 = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24, \pm32, \pm48, \pm96$. The another condition $a_1 \geq 0$ reduces the values of k to

$$\{-94, -46, -30, -22, -14, -10, -6, -4, 0, 3, 4, 5, 6, 8, 10, 14, 18, 26, 34, 50, 98\}\,. \tag{25}$$

We pick up each value of k from the list (25) and determine every a_n by (24) recursively (up to $n = 30$). Then the list of k such that all a_n are non-negative integers for $0 < n \leq 30$ reduces the values of k to $\{-30, -22, -14, -10, -6, -4, 0, 3, 4, 8\}$, which coincides with (23). For each k in (23), it follows from Theorem 2 and Theorem 4 that the functions v_k are solutions of $(14)_k$.

For $k = 0, 3, 4$ and 8, the function v_k is clearly of vacuum character type because E_4 and $1/\eta^{2k}$ have positive integral Fourier coefficients. Since $v_{-4}\eta^8 = q^{1/3}\prod_{n=1}^{\infty}(1+q^n)^8$ and $v_{-6}\eta^{12} = E_4''/(240\eta^{12})$, the functions v_{-4} and v_{-6} are also of vacuum character type.

For v_{-10}, we use the identity

$$\begin{aligned} v_{-10}\,\eta^{20} &= \frac{1}{12\eta^{28}}\left(5E_4\left(\frac{E_4'}{240}\right)^2 + 7\left(\frac{E_6'}{504}\right)^2\right) \\ &= \frac{1}{12\eta^{28}}\left(5\Big(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)\Big(\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)^2 + 7\Big(\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big)^2\right) \end{aligned}$$

which shows the positivity of Fourier coefficients. Moreover, using the congruence $\sigma_3(n) \equiv \sigma_5(n) \pmod{12}$, we have

$$\begin{aligned} &5\Big(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)\Big(\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)^2 + 7\Big(\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big)^2 \\ &\quad \equiv 5\Big(\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)^2 + 7\Big(\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big)^2 \equiv 12\Big(\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)^2 \equiv 0 \pmod{12} \end{aligned}$$

and thus conclude that the Fourier coefficients of $v_{-10}\,\eta^{20}$ are integers.

Because $\Delta_2/\sqrt{\Delta} = (H_2^2 - 64\Delta_2^2)^{-1}$ and Δ_2^2 is a modular form on $\Gamma_0(2)$, the function v_{-4} is a modular form on $\Gamma_0(2)$ of weight -4. Since

$$\frac{\Delta_2(\tau)}{\sqrt{\Delta(\tau)}}\bigg|_{-4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \frac{\eta(\tau/2)^8}{\eta(\tau)^{16}} = \frac{1}{q^{1/2}} - 8 + 36q^{1/2} + O(q)\,, \tag{26}$$

v_{-4} has a pole at 0. The rest of assertions are clear from the definition of the functions v_k. □

Remark. It is worth noting that the solution v_{-10} has an another expression $v_{-10} = (E_4^3 - 720\Delta)''/(786240\Delta^2)$, and the modular form $E_4^3 - 720\Delta$ of weight 12 is the theta series of the 24-dimensional Leech lattice. Let N_m be the number of vectors of the Leech lattice whose norms are m. The identity

$$E_4^3 - 720\Delta \;=\; E_{12} - \frac{65520}{691}\Delta \;=\; 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\big(\sigma_{11}(n) - \tau(n)\big)q^n,$$

where $\tau(n)$ is the nth Fourier coefficient of $\Delta(\tau)$, shows the well-known formula

$$N_{2n} = \frac{65520}{691}(\sigma_{11}(n) - \tau(n))\,. \tag{27}$$

By the integrality of coefficients of $(E_4^3 - 720\Delta)''/786240$ proved in the previous paragraph, we derive the fact that (i) N_{2n} is divisible by $N_4 = 196560$ if n is even, (ii) N_{2n} is divisible by $4N_4 = 786240$ if n is odd and not divisible by 3 and (iii) $3N_{2n}$ is divisible by $4N_4$ if n is odd and divisible by 3 since $786240 = 65520 \times 12 = 196560 \times 3$. (The facts (i)–(iii) can also be proved by using (27) and classical congruences of $\tau(n)$ given in [13].)

Example. The function $v_8/\eta^{16} = j^{2/3}$ is the character of the lattice VOA associated with the unimodular lattice of rank 8. Here $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ is the elliptic modular function.

At the present time, we do not know if v_k is of character type except a few k. However, computer experiments suggest that any solution of $(14)_k$ with the exponent 0 is of character type after multiplied by a suitable integer. More specifically, we give the following conjecture.

Conjecture. Let k be either a positive integer or a negative even integer. Define a natural number $p_1(k)$ by

$$p_1(k) \;=\; \begin{cases} \displaystyle\prod_{i=1}^{\lfloor (k+1)/8 \rfloor} (k-4i+2) & \text{if } k \text{ is positive}, \\ \displaystyle\prod_{i=1}^{\lfloor (|k|-4)/4 \rfloor} (|k|/2+2i-1) & \text{if } k \text{ is negative}, \end{cases}$$

where an empty product is regarded as 1. If $v_k = 1 + O(q)$ is a solution of $(14)_k$, then $p_1(k)\, v_k$ is of character type.

For example, we have $p_1(7) = 5$ and

$$5v_7 = 5E_4^2 + 16E_6'/21 = 5 + 96 \sum_{n=1}^{\infty} (25\sigma_7(n) - n\sigma_5(n))\, q^n.$$

Since $25\sigma_7(n) - 4n\sigma_5(n) > 0$ $(n \geq 1)$ (this is because $\sigma_7(n) \geq n^7 \geq n(1^5 + 2^5 + \cdots + n^5) \geq n\sigma_5(n)$), we see that $p_1(7) \cdot v_7$ is a solution of $(15)_7$ of character type.

4.2 Case $\nu = (k+3)/6$

We next study the solutions of (20) with index $\nu = (k+3)/6$, which corresponds to those of (14) with exponent $\mu = (k+2)/4$.

Theorem 7. *Let k be an integer.*
(a) *Suppose that $(14)_k$ has a solution of vacuum character type with the exponent $(k+2)/4$. Then k is one of the values in the list*

$$\{-8,\ -7,\ -5,\ -4,\ 2,\ 6\}. \tag{28}$$

(b) *For each k in (28), the following functions w_k are of vacuum character type with the* *exponent* $(k+2)/4$*:*

$$w_2 \;=\; f_2 \;=\; q+6q^2+12q^3+28q^4+30q^5+72q^6+56q^7+\cdots\,,$$
$$w_6 \;=\; f_6 \;=\; q^2+16q^3+102q^4+416q^5+1308q^6+3360q^7+\cdots\,,$$
$$w_{-4} \;=\; H_2/\sqrt{\Delta} \;=\; \frac{1}{q^{1/2}}+36q^{1/2}+402q^{3/2}+3064q^{5/2}+18351q^{7/2}+93300q^{9/2}+\cdots\,,$$
$$w_{-5} \;=\; \frac{E_4}{\Delta^{3/4}} \;=\; \frac{1}{q^{3/4}}+258q^{1/4}+6669q^{5/4}+92442q^{9/4}+911976q^{13/4}+7168716q^{17/4}+\cdots\,,$$
$$w_{-7} \;=\; \frac{E_4^2}{\Delta^{5/4}} \;=\; \frac{1}{q^{5/4}}+\frac{510}{q^{1/4}}+76815q^{3/4}+3151330q^{7/4}+72967305q^{11/4}+\cdots\,,$$
$$w_{-8} \;=\; \frac{H_2(H_2^4+61440\Delta_2^4)}{\Delta^{3/2}} \;=\; \frac{1}{q^{3/2}}+\frac{156}{q^{1/2}}+72342q^{1/2}+5125368q^{3/2}+176987145q^{5/2}+\cdots\,,$$

where f_2, f_6 *are defined in* §3.1 *and* H_2, Δ_2 *are defined in* §3.2.
(c) *The functions* w_2 *and* w_6 *are quasimodular forms on* $SL_2(\mathbb{Z})$ *of depth* 2 *of weights* 4 *and* 8*, respectively. The functions* w_{-4} *and* w_{-8} *are meromorphic modular forms of weights* -4 *and* -8 *of level* 2*. The functions* w_{-5} *and* w_{-7} *are meromorphic modular of weights* -5 *and* -8 *of level* 4*.*

Proof. Suppose that there exists a Fourier series solution (21) with $\nu=(k+3)/6$ of (20). Then recursion (22) gives

$$2n\left(n+\frac{k+2}{4}\right)^2 a_n \;=\;$$
$$\sum_{i=1}^{n}\Big\{e_{2,i}\Big(n+\frac{k+3}{6}-i\Big)^2-\Big(i\,e_{2,i}-\frac{k(k+4)}{24}e_{4,i}\Big)\Big(n+\frac{k+3}{6}-i\Big)+\frac{k^2(k+3)}{3\cdot 12^2}e_{6,i}\Big\}a_{n-i}$$

for every $n>0$. The case $n=1$ gives $a_1=4k+12-128(k+3)/(k+6)^2$. Suppose that a_1 is an integer. Then we have $128(k+3)/(k+6)^2\in\mathbb{Z}$, and in particular $128(k+3)/(k+6)=128-384/(k+6)\in\mathbb{Z}$. Therefore, we have $(k+6)|384=2^7\cdot 3$ and then we obtain the possible list $\{-8,\,-7,\,-5,\,-4,\,2,\,6\}$ of k in the same manner as in the case $\mu=0$.

It follows from Theorem 2 and Theorem 4 that w_k is a solution of $(14)_k$. The Fourier coefficients of the functions w_k/η^{2k} for $k=-4,-5,-7$ and -8 are non-negative integers because $H_2=1+24\sum_{n=1}^{\infty}(\sigma_1(n)-2\sigma_1(n/2))q^n$, $\Delta_2=\eta(2\tau)^8/\eta(\tau)^4=(\sum_{n=1}^{\infty}q^{(n-1/2)^2/2})^4$ and $E_4=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n$ have positive integral Fourier coefficients.

It is obvious that w_2 is of vacuum character type since $w_2=-E_2'/24=\sum_{n=1}^{\infty}n\sigma_1(n)q^n$ ($\sigma_1(1)=1$). Finally, we show that w_6 has positive integral Fourier coefficients, which also shows w_6 is of vacuum character type. By the definition of the Eisenstein series, we have

$$w_6 \;=\; \frac{1}{30}\left(\frac{E_6'}{-504}-\frac{E_4''}{240}\right) \;=\; \frac{1}{30}\sum_{n=1}^{\infty}n(\sigma_5(n)-n\sigma_3(n))q^n\,.$$

We prove that the coefficient $n\,(\sigma_5(n)-n\sigma_3(n))$ is positive for $n>1$ and is divisible by 30. Since $\sigma_5(n)\geq n^5$ and $\sigma_3(n)\leq 1+2^3+\cdots+n^3\leq n\cdot n^3=n^4$, it follows that $\sigma_5(n)-n\sigma_3(n)\geq 0$ (equality only holds when $n=1$).

To prove that $n(\sigma_5(n) - n\sigma_3(n))$ is divisible by 30, it is enough to show the divisibility by 6 because the other expression of w_6 by $(E_4''/240 - E_2 \cdot E_4'/240)/42$ shows that 5 does not appears in the denominators. Because of the congruence $n\sigma_5(n) \equiv n^2\sigma_3(n) \pmod 6$ $(n \geq 1)$, it follows that $n(\sigma_5(n) - n\sigma_3(n))$ is divisible by 6. Therefore, all coefficients of w_6 are positive integers. (The positivity of the coefficients of w_6 can also be seen from the identity $w_6' = (-E_2'/24) \cdot (E_4'/240)$.)

For the statements in (c) that w_{-4} and w_{-8} are meromorphic modular forms of level 2 and that w_{-5}, w_{-7} are of level 4, we only mention that the former is on the principal congruence subgroup $\Gamma(2)$ and the latter on the group

$$\Gamma_0^0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, b \equiv c \equiv 0 \mod 4\,, \right\}$$

because the functions $\Delta^{1/2}$ and $\Delta^{1/4}$ are modular forms of weight 6 and 3 on $\Gamma(2)$ and $\Gamma_0^0(4)$, respectively.

□

Taking account of [7, Conjecture 2] together with computer experiences, we propose the second conjecture below. Compare this conjecture to the one in the previous subsection.

Conjecture. Let k be a positive integer satisfying $k \equiv 2 \pmod 4$. Define an integer $p_2(k)$ by $p_2(k) = \prod_{i=1}^{\lfloor (k+2)/8 \rfloor} \big((k+4-4i)^2/2\big)$. Suppose that $(14)_k$ has a solution $w_k = q^{(k+2)/4} + O(q^{(k+6)/4})$ with the index $(k+2)/4$. Then $p_2(k)w_k$ is a (quasi)modular form of character type.

Remarks. (1) From the relation $a_1 = 4k + 12 - 128(k+3)/(k+6)^2$ obtained in the proof of the theorem, it follows that there do not exist (quasi)modular forms of character type if the weight k satisfies $k < -2(3+2\sqrt{2}) = -11.6569\cdots$ or $-3 < k < 2(-3+2\sqrt{2}) = -0.3431\cdots$, because a_1 becomes negative for such an integer k.

Let k be an integer between -11 and -3. We do not know the exact form of solutions at the moment except $k = -9$ and -11. For $k = -9$, the solution $w_{-9} = (E_4^3 - 2048\Delta/3)/\eta^{42}$ is of character type as

$$w_{-9}\,\eta^{18} = j(\tau) - \frac{2048}{3} \left(= \frac{1}{q} + \frac{184}{3} + 196884q + 21493760q^2 + 864299970q^3 + \cdots. \right)$$

For $k = -11$, we have

$$\begin{aligned} w_{-11}\,\eta^{22} &= \frac{E_4}{\Delta^{4/3}} \left(E_4^3 - \frac{24576}{25}\Delta \right) \\ &= j(q)^{1/3}\,(j(q) - 24576/25) \\ &\left(= \frac{1}{q^{4/3}} + \frac{224}{25q^{1/3}} + \frac{3543152}{25}q^{2/3} + \frac{1734248576}{25}q^{5/3} + \frac{174965201848}{25}q^{8/3} + \cdots \right) \end{aligned}$$

and we can verify that these coefficients are all positive and the denominators are at most 25 up to q^{1023}.

(2) The functions w_2 and w_6 are extremal quasimodular forms (see [7] for the definition).

5 Solutions of hypergeometric type and symmetries

In [10, Theorem 3.1], one of the authors and his collaborator obtained a solution of (12) under the condition $k+2\pm\sqrt{(k+2)^2-64\alpha}\in 8\mathbb{Z}$, which has the form $f=1+O(q)$ and is expressed in terms of a hypergeometric polynomial. Since the modular group $SL_2(\mathbb{Z})$ is one of the arithmetic triangle groups, it is natural and interesting to seek for conditions under which solutions of the third order K-Z equations can be written by using hypergeometric series. In this section we generalize Theorem 3.1 of [10] and write down solutions of (12) by using the **generalized hypergeometric series** under several restrictions on k. At the end of the section, we mention that the modified K-Z equation

$$g'''-\frac{1}{2}E_2g''+\left\{\frac{1}{2}E_2'+\left(\alpha-\frac{k}{12}-\frac{k^2}{48}\right)E_4\right\}g'+\frac{k}{12}\left(\alpha-\frac{k}{24}-\frac{k^2}{72}\right)E_6\,g\;=0 \tag{29}$$

(as well as its solutions) admits a symmetry of S_3.

It is well known that the generalized hypergeometric series of degree 3 (cf. [14]), which is defined by

$$ {}_3F_2(\alpha_1,\,\alpha_2,\,\alpha_3\,;\beta_1,\,\beta_2\,;x)\;=\;\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n}\cdot\frac{x^n}{n!}\,,$$

satisfies the differential equation ($F={}_3F_2$)

$$\begin{aligned}x^2(1-x)\,\frac{d^3F}{dx^3}&+x\big((\beta_1+\beta_2+1)-(\alpha_1+\alpha_2+\alpha_3+3)\big)\,\frac{d^2F}{dx^2}\\&+\big(\beta_1\beta_2-(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1+\alpha_1+\alpha_2+\alpha_3+1)x\big)\frac{dF}{dx}-\alpha_1\alpha_2\alpha_3\,F\;=\;0\,,\end{aligned} \tag{30}$$

where $(\alpha)_n$ is the (ascending) Pochhammer symbol $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n>0$ and $(\alpha)_0=1$. The series ${}_3F_2$ is absolutely convergent inside the unit circle if the real part of $\sum_{i=1}^{2}\beta_i-\sum_{i=1}^{3}\alpha_i$ is positive. Set $F=E_4^{-k/4}\cdot f$ and denote a local coordinate in a neighborhood of the cusp ∞ by and $x=x(\tau)=12^3/j(\tau)$. Then we have

$$\begin{aligned}x^2(1-x)\frac{d^3F}{dx^3}&+x\left(\frac{10-k}{4}-\frac{16-k}{4}x\right)\frac{d^2F}{dx^2}\\&+\left\{\left(\alpha-\frac{k-2}{4}\right)-\frac{3k^2-60k+320}{12^2}x\right\}\frac{dF}{dx}+\frac{k(k-4)(k-8)}{12^3}F\;=\;0\end{aligned} \tag{31}$$

which is equivalent to (12) by the Ramanujan relations and the property $\left(2\pi\sqrt{-1}\right)^{-1}dx/d\tau=x(1-x)E_4^2/E_6$. The comparison of (30) and (31) shows that $\alpha_1=-k/12$, $\alpha_2=-(k-4)/12$, $\alpha_3=-(k-8)/12$, $\beta_1=(-k+6+\xi)/8$ and $\beta_2=(-k+6-\xi)/8$, where $\xi=\sqrt{(k+2)^2-64\alpha}$ and α is a complex number such that $k+2\pm\xi\notin 8\mathbb{Z}$. Hence (12) has a solution

$$f_1(\tau)\;=\;E_4(\tau)^{k/4}{}_3F_2\left(-\frac{k}{12},-\frac{k-4}{12},-\frac{k-8}{12};\frac{-k+6+\xi}{8},\frac{-k+6-\xi}{8};\frac{1728}{j(\tau)}\right)\,.$$

Now suppose that $1-\beta_1$, $1-\beta_2$, $\beta_1-\beta_1 \notin \mathbb{Z}$, or equivalently, $k+2\pm\xi \notin 8\mathbb{Z}$ and $\xi \notin 4\mathbb{Z}$. Then the functions defined by

$$x^{1-\beta_1}{}_3F_2(\alpha_1-\beta_1+1, \alpha_2-\beta_1+1, \alpha_3-\beta_1+1; \beta_2-\beta_1, 2-\beta_1; x),$$
$$x^{1-\beta_2}{}_3F_2(\alpha_1-\beta_2+1, \alpha_2-\beta_2+1, \alpha_3-\beta_2+1; \beta_1-\beta_2, 2-\beta_2; x)$$

are solutions of (30) (see [14, §2.1]). Therefore, we conclude that

$$f_2 = b_2 \cdot {}_3F_2\left(\frac{k+6-3\xi}{24}, \frac{k+14-3\xi}{24}, \frac{k+22-3\xi}{24}; -\frac{1}{4}, \frac{k+10-\xi}{8}; \frac{1728}{j}\right),$$
$$f_3 = b_3 \cdot {}_3F_2\left(\frac{k+6+3\xi}{24}, \frac{k+14+3\xi}{24}, \frac{k+22+3\xi}{24}; \frac{1}{4}, \frac{k+10+\xi}{8}; \frac{1728}{j}\right)$$

are solutions of (12), where $b_2 = \Delta^{(k+2-\xi)/8} E_4^{(-k+3\xi-6)/8}$ and $b_3 = \Delta^{(k+2+\xi)/8} E_4^{(-k-3\xi-6)/8}$. The f_2 and f_3 absolutely converge on the domain $|j| > 1728$ if $\mathrm{Re}(\xi) > 3$ and $\mathrm{Re}(\xi) < -1$, respectively.

Theorem 8. *Suppose that* $\xi = \sqrt{(k+2)^2 - 64\alpha} \notin 4\mathbb{Z}$ *and* $k+2\pm\xi \notin 8\mathbb{Z}$. *Then* f_2 *and* f_3 *are solutions of* (12) *if* $\mathrm{Re}(\xi) > 3$ *and* $\mathrm{Re}(\xi) < -1$, *respectively.*

Remark. The modified third order K-Z equation (29) admits an action of the symmetric group S_3. Let σ and ρ be affine transformations on $\mathbb{R}^2$ defined by

$$\sigma(k,\xi) = \left(\frac{-k+6+3\xi}{2}, \frac{k+2-\xi}{2}\right), \quad \rho(k,\xi) = \left(\frac{-k+6+3\xi}{2}, \frac{k-2+\xi}{2}\right),$$

which have orders 3 and 2, respectively. It is obvious that these two affine transformations form the symmetric group S_3 and that (29) is invariant under this action of S_3 if $\xi = \sqrt{(k+2)^2 - 64\alpha}$. Moreover, $h_1 = f_1/\eta^{2k}$, $h_2 = f_2/\eta^{2k}$ and $h_3 = f_3/\eta^{2k}$ are solutions of (29), which have relations $\sigma(h_1,h_2,h_3) = (h_2,h_3,h_1)$ and $\rho(h_1,h_2,h_3) = (h_2,h_1,h_3)$.

Acknowledgment We thank the anonymous referee for his/her useful comments, which helped greatly to improve and clarify several arguments in the original manuscript.

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