On a kind of duality of multiple zeta-star values

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1 Main result

In this note, we prove a certain dulality-type result for height 1 *multiple zeta-star values* and discuss its possible generalization.

For an index set $(k_1, k_2, \ldots, k_n)$ of positive integers with $k_1 > 1$, the multiple zeta-star value $\zeta^\star(k_1, k_2, \ldots, k_n)$ is defined by

$$\zeta^\star(k_1, k_2, \ldots, k_n) := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

If we remove the equality signs in the summation, we obtain the usual *multiple zeta value*:

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

The *height* of the multiple zeta or zeta-star value is the number of k_i in the index set which is greater than 1. The following theorem can be regarded as a kind of duality for multiple zeta-star values of height 1.

Theorem 1 *For any integers $k, n \geq 1$, we have*

$$(-1)^k \zeta^\star(k+1, \underbrace{1, \ldots, 1}_{n}) - (-1)^n \zeta^\star(n+1, \underbrace{1, \ldots, 1}_{k}) \in \mathbf{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots],$$

the right-hand side being the algebra over $\mathbf{Q}$ generated by the values of the Riemann zeta function at positive integer arguments (> 1).

Remark For multiple zeta values, there is a well-known duality formula [9], and the height 1 case of the formula reads as

$$\zeta(k+1, \underbrace{1, \ldots, 1}_{n-1}) = \zeta(n+1, \underbrace{1, \ldots, 1}_{k-1})$$

for $k, n \geq 1$. No such simple formula has been known for multiple zeta-star values. It should be noted that the pair of indices

$$(k+1, \underbrace{1, \ldots, 1}_{n}) \longleftrightarrow (n+1, \underbrace{1, \ldots, 1}_{k})$$

in Theorem 1 is different from that in the duality formula for multiple zeta values above.

We can also compute the generating function of the quantity

$$(-1)^k\zeta^\star(k+1,\underbrace{1,\ldots,1}_{n})-(-1)^n\zeta^\star(n+1,\underbrace{1,\ldots,1}_{k})$$

in Theorem 1.

Theorem 2 *We have*

$$\sum_{k,n\geq 1}\big((-1)^k\zeta^\star(k+1,\underbrace{1,\ldots,1}_{n})-(-1)^n\zeta^\star(n+1,\underbrace{1,\ldots,1}_{k})\big)x^ky^n$$
$$=\psi(x)-\psi(y)+\pi\left(\cot(\pi x)-\cot(\pi y)\right)\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}.$$

Here, $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function, the logarithmic derivative of the gamma function.

2 Proof of Theorems

We prove the following basic identity, from which follow both Theorem 1 and Theorem 2. [1]

Proposition *For $k,n\geq 1$, we have*

$$\begin{aligned} &(-1)^k\zeta^\star(k+1,\underbrace{1,\ldots,1}_{n})-(-1)^n\zeta^\star(n+1,\underbrace{1,\ldots,1}_{k})\\ =\;& k\zeta(k+2,\underbrace{1,\ldots,1}_{n-1})-n\zeta(n+2,\underbrace{1,\ldots,1}_{k-1})\\ &+(-1)^k\sum_{j=0}^{k-2}(-1)^j\zeta(k-j)\zeta(n+1,\underbrace{1,\ldots,1}_{j})\\ &-(-1)^n\sum_{j=0}^{n-2}(-1)^j\zeta(n-j)\zeta(k+1,\underbrace{1,\ldots,1}_{j}), \end{aligned}$$

where we understand an empty sum to be 0.

Proof. We use two formulas for the special value of the function $\xi_k(s)$ defined for $k\geq 1$ by

$$\xi_k(s):=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}}{e^t-1}Li_k(1-e^{-t})dt. \tag{1}$$

[1]Recently, C. Yamazaki ([8]) gave another proof of them. It uses a generating function of certain sums of multiple zeta-star values which was introduced in [1].

In [3], we studied this function and obtained among others the formula

$$\begin{aligned}\xi_k(n+1) &= (-1)^{k-1}\big[\zeta(n+1,\underbrace{2,1,\dots,1}_{k-1})+\zeta(n+1,\underbrace{1,2,1,\dots,1}_{k-1})+\cdots\\ &\quad \cdots+\zeta(n+1,\underbrace{1,\dots,1,2}_{k-1})+(n+1)\cdot\zeta(n+2,\underbrace{1,\dots,1}_{k-1})\big]\\ &\quad +\sum_{j=0}^{k-2}(-1)^j\zeta(k-j)\cdot\zeta(n+1,\underbrace{1,\dots,1}_{j}),\end{aligned} \tag{2}$$

where k,n are integers ≥ 1.

On the other hand, we showed in [6] that the value $\xi_k(n)$ is nothing but the multiple zeta-star value of hetight 1, i.e., we have the formula

$$\xi_k(n+1)=\zeta^\star(k+1,\underbrace{1,\dots,1}_{n}). \tag{3}$$

Since the index sets $(k+1,\underbrace{1,\dots,1}_{n-1})$ and $(n+1,\underbrace{1,\dots,1}_{k-1})$ are dual (in the context of multiple zeta values) with each other, the main theorem in [6] applied to these index sets with $l=1$ gives the identity

$$\begin{aligned}&\zeta(k+2,\underbrace{1,\dots,1}_{n-1})+\zeta(k+1,\underbrace{2,1,\dots,1}_{n-1})+\zeta(k+1,\underbrace{1,2,1,\dots,1}_{n-1})+\cdots\\ &\cdots+\zeta(k+1,\underbrace{1,\dots,1,2}_{n-1})\\ = \; &\zeta(n+2,\underbrace{1,\dots,1}_{k-1})+\zeta(n+1,\underbrace{2,1,\dots,1}_{k-1})+\zeta(n+1,\underbrace{1,2,1,\dots,1}_{k-1})+\cdots\\ &\cdots+\zeta(n+1,\underbrace{1,\dots,1,2}_{k-1}).\end{aligned} \tag{4}$$

Combining (2), (3) and (4), we obtain the proposition. □

Proof of Theorems 1 and 2. Recall the formula of Aomoto [2] and Drinfeld [4]

$$\sum_{k,n\geq 1}\zeta(k+1,\underbrace{1,\dots,1}_{n-1})x^ky^n = 1-\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}. \tag{5}$$

This together with the standard Taylor expansion of the (logarithm of) gamma function

$$\Gamma(1+x)=\exp\left(-\gamma x+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}x^n\right) \quad (|x|<1, \gamma: \text{Euler's constant}) \tag{6}$$

shows that all multiple zeta values of height 1 (= of type $\zeta(m,1,\dots,1)$) can be expressed as polynomials over $\mathbf{Q}$ in the Riemann zeta values. Theorem 1 therefore follows from the formula in Proposition.

As for the generating series, we start with the formula (5). Replace k with $k+1$ in (5) and divide the both-hand sides out by xy, and then differentiate with respect to x and multiply xy. Then we obtain

$$\begin{aligned}&\sum_{k,n\geq 1} k\zeta(k+2,\underbrace{1,\ldots,1}_{n-1})x^k y^n\\ &\qquad = -\frac{1}{x}+\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\left(\frac{1}{x}+\psi(1-x)-\psi(1-x-y)\right),\end{aligned}$$

and hence by interchanging x and y and subtracting, we have

$$\begin{aligned}&\sum_{k,n\geq 1}\left(k\zeta(k+2,\underbrace{1,\ldots,1}_{n-1})-n\zeta(n+2,\underbrace{1,\ldots,1}_{k-1})\right)x^k y^n\\ =\;& -\frac{1}{x}+\frac{1}{y}+\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\left(\frac{1}{x}+\psi(1-x)-\frac{1}{y}-\psi(1-y)\right). \qquad (7)\end{aligned}$$

Next, by the formula

$$\sum_{i=2}^{\infty}(-1)^i\zeta(i)x^{i-1}=\psi(1+x)+\gamma$$

(take the logarithmic derivative of (6)) and by (5), we have

$$\begin{aligned}&\sum_{k,n\geq 1}(-1)^k\sum_{j=0}^{k-2}(-1)^j\zeta(k-j)\zeta(n+1,\underbrace{1,\ldots,1}_{j})x^k y^n\\ =\;&\sum_{i\geq 2,j,n\geq 1}(-1)^i\zeta(i)\zeta(n+1,\underbrace{1,\ldots,1}_{j-1})x^{i+j-1}y^n\\ =\;&\left(\sum_{i\geq 2}(-1)^i\zeta(i)x^{i-1}\right)\left(\sum_{j,n\geq 1}\zeta(n+1,\underbrace{1,\ldots,1}_{j-1})x^j y^n\right)\\ =\;&(\psi(1+x)+\gamma)\left(1-\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\right),\end{aligned}$$

and thus we obtain

$$\begin{aligned}&\sum_{k,n\geq 1}\Bigg((-1)^k\sum_{j=0}^{k-2}(-1)^j\zeta(k-j)\zeta(n+1,\underbrace{1,\ldots,1}_{j})\\ &\qquad\qquad -(-1)^n\sum_{j=0}^{n-2}(-1)^j\zeta(n-j)\zeta(k+1,\underbrace{1,\ldots,1}_{j})\Bigg)x^k y^n\\ &=\left(1-\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\right)(\psi(1+x)-\psi(1+y)). \qquad (8)\end{aligned}$$

By Proposition, Theorem 2 follows from (7), (8), and the standard identities

$$\psi(1+x) = \frac{1}{x} + \psi(x) \quad \text{and} \quad \pi\cot(\pi x) = \frac{1}{x} + \psi(1-x) - \psi(1+x).$$

3 Possible generalization

In this section, we propose a possible generalization of Theorem 1 for arbitrary heights.

First, we recall a few notations which are used in [1]. The *weight* and the *depth* of multiple zeta-star values $\zeta^\star(k_1, k_2, \ldots, k_n)$ are the sum $k_1+k_2+\cdots+k_n$ and the length n of its index, respectively. We denote by $X_0(k, n, s)$ the sum of all multiple zeta-star values of weight k, depth n and height s, for $k \geq n+s$ and $n \geq s \geq 1$.

Based on the numerical experiments up to weight 11, we conjecture the following.

Conjecture *For any integers $k, n \geq s \geq 1$, we have*

$$(-1)^k X_0(k+n+1, n+1, s) - (-1)^n X_0(k+n+1, k+1, s) \in \mathbf{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots].$$

Remark Theorem 1 is nothing but the case when $s = 1$ of the above conjecture.

Examples When the weight is 8 and the height is 2 or 3, we can show (using the double shuffle relations of multiple zeta values) the following identities, which are in favor of the conjecture.

$$\begin{aligned}
X_0(8,3,2) + X_0(8,6,2) &= \frac{876}{175}\zeta(2)^4 - \zeta(2)\zeta(3)^2 - 3\zeta(3)\zeta(5) \\
X_0(8,4,2) + X_0(8,5,2) &= \frac{1083}{280}\zeta(2)^4 + \zeta(2)\zeta(3)^2 + 2\zeta(3)\zeta(5) \\
X_0(8,4,3) + X_0(8,5,3) &= \frac{1349}{280}\zeta(2)^4 - \frac{1}{2}\zeta(2)\zeta(3)^2 - \zeta(3)\zeta(5)
\end{aligned}$$

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