Kyushu J. Math. **63** (2009), 353–364
doi:10.2206/kyushujm.63.353

OBSERVATIONS ON THE 'VALUES' OF THE ELLIPTIC MODULAR FUNCTION $j(\tau)$ AT REAL QUADRATICS

Masanobu KANEKO

(Received 18 May 2009)

Abstract. We define 'values' of the elliptic modular j-function at real quadratic irrationalities by using Hecke's hyperbolic Fourier expansions, and present some observations based on numerical experiments.

1. Introduction

> '... von dem Studium des Verhaltens der elliptischen Modulfunktionen in der Nähe der nicht-rationalen Randpunkte noch sehr bemerkenswerte Ergebnisse erwartet werden können, die sowohl für die Funktionentheorie wie die Arithmetik wichtig sein dürften.' (Hecke, Werke S.417)

We define the 'value', written val(w),† of the elliptic modular function $j(\tau)$ at each *real* quadratic irrationality w as the constant term of a hyperbolic Fourier expansion‡ at w. The map $w \mapsto \text{val}(w)$ is $\text{PSL}_2(\mathbf{Z})$-invariant and hence assigns to each $\text{PSL}_2(\mathbf{Z})$-equivalence class of real quadratic numbers a certain (real or complex) number. We conducted numerical experiments on the numbers val(w) and observed the following phenomena, which we find quite remarkable, though no precise formulation (especially for (ii) and (iii)) nor proofs have yet been established.

Observations

(i) The minimum among all real values of val(w) is realized at $w = (1+\sqrt{5})/2$ (the golden ratio), with $\text{val}((1+\sqrt{5})/2) = 706.324\,813\,540\ldots$. Also, all real values of val(w) lie in the interval $[706.324\,813\,540\ \ldots, 744]$, where 744 is the constant term in the Fourier expansion of $j(\tau)$ at the cusp (which is the $\text{PSL}_2(\mathbf{Z})$-equivalence class of rational numbers and $i\infty$).

(ii) As the rational approximation of w improves, val(w) increases (see Tables 1–5).

2000 Mathematics Subject Classification: Primary 11F03, 11Y70.
Keywords: elliptic modular function; values at real quadratic numbers.

†Dedekind, in his seminal paper [**1**] on $j(\tau)$, used the symbol val(ω) for $j(\tau)$ (where ω is a variable in the upper half-plane) and called it the 'Valenz'. We borrow his notation.

‡Hecke considered this type of expansion for modular forms of positive weight [**3**].

(iii) The imaginary part of any $\mathrm{val}(w)$ lies in the interval $(-1, 1)$. Also, the distribution of the imaginary parts of $\mathrm{val}(w)$, with the discriminants of w bounded, seems to be peaked at 0 and symmetric about this peak. Furthermore, the phenomena described in (i) and (ii) also hold for the absolute value (or real part) of $\mathrm{val}(w)$.

In this paper, we give a precise definition of $\mathrm{val}(w)$ and then establish its basic properties, which follow almost immediately from the definition. We then describe experiments related to Markoff numbers. This also seems to support the existence of certain 'Diophantine continuity' of val suggested (but not yet well formulated) above.

2. Definition and basic properties

Let w be a real quadratic number with discriminant $\mathrm{disc}(w) = D > 0$. Denote by Γ_w the stabilizer of w in $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ (with the action being the standard linear fractional transformation):

$$\Gamma_w := \{\gamma \in \Gamma \mid \gamma w = w\}.$$

Let U_D be the group of units of norm one in the quadratic order O_D of the discriminant D and $\varepsilon = \varepsilon_D^{(1)}$ be a generator of the infinite cyclic part of U_D. Then, if

$$\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_w,$$

we have $cw^2 + (d - a)w - b = 0$, and thus the number cw is an algebraic integer, and

$$(a - cw)(a - cw') = a^2 - ac(w + w') + c^2 ww' = 1,$$

that is, $a - cw \in U_D$. Here, w' is the algebraic conjugate of w. It is known that the map

$$\Gamma_w \ni \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - cw)^2 \in U_D^2$$

gives an isomorphism from the group Γ_w to U_D^2, which is an infinite cyclic group generated by ε^2. Let γ_ε be the element in Γ_w that corresponds to ε^2 under this isomorphism. For $\gamma \in \Gamma_w$, a straightforward computation shows that

$$\frac{\gamma\tau - w}{\gamma\tau - w'} = (a - cw)^2 \frac{\tau - w}{\tau - w'},$$

and, in particular, that

$$\frac{\gamma_\varepsilon\tau - w}{\gamma_\varepsilon\tau - w'} = \varepsilon^2 \frac{\tau - w}{\tau - w'}.$$

Denote by $\delta(w)$ the sign of $w - w'$. Then, if τ is a variable in the upper half-plane $\mathfrak{H}$, we have

$$z := \delta(w) \frac{\tau - w}{\tau - w'} \in \mathfrak{H}$$

and

$$\tau = \frac{w - \delta(w)w'z}{1 - \delta(w)z}.$$

Let

$$j(\tau) = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots \quad (q = e^{2\pi i\tau})$$

be *the* classical elliptic modular function. It is Γ-invariant, and hence, by the relations

$$\frac{\gamma_w\tau - w}{\gamma_w\tau - w'} = \varepsilon^2\delta(w)z$$

and

$$\gamma_w\tau = \frac{w - \varepsilon^2\delta(w)w'z}{1 - \varepsilon^2\delta(w)z},$$

the function

$$j(\tau) = j\left(\frac{w - \delta(w)w'z}{1 - \delta(w)z}\right) \quad (z \in \mathfrak{H})$$

is invariant under $z \mapsto \varepsilon^2 z$. Thus, if we set $z = e^u$, the function

$$j\left(\frac{w - \delta(w)w'e^u}{1 - \delta(w)e^u}\right),$$

which is holomorphic in the domain $0 < \mathrm{Im(u)} < \pi$, is invariant under the translation $u \mapsto u + 2\log\varepsilon$. It therefore has a Fourier expansion of the form

$$j\left(\frac{w - \delta(w)w'e^u}{1 - \delta(w)e^u}\right) = \sum_{n=-\infty}^{\infty} a_n \exp\left(2\pi i n \frac{u}{2\log\varepsilon}\right). \tag{1}$$

Definition. We define the 'value', $\mathrm{val}(w)$, of $j(\tau)$ at w as the constant term of the series (1):

$$\mathrm{val}(w) := a_0 = \frac{1}{2\log\varepsilon}\int_{\sigma_0}^{\sigma_0+2\log\varepsilon} j\left(\frac{w - \delta(w)w'e^u}{1 - \delta(w)e^u}\right) du, \tag{2}$$

where σ_0 is any complex number satisfying $0 < \mathrm{Im}(\sigma_0) < \pi$.

If we set $\sigma_0 = \pi i/2 - \log\varepsilon$ and make the change of variable $u \mapsto u + \pi i/2$, we have

$$\mathrm{val}(w) = \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{w - \delta(w)w'ie^u}{1 - \delta(w)ie^u}\right) du. \tag{3}$$

Note that $\mathrm{val}(w)$ is a complex-valued function defined *only* on the real quadratic irrationalities.

PROPOSITION *The 'value' function* $\mathrm{val}(w)$ *possesses the following properties:*

(i) *if* w *and* w_1 *are* Γ*-equivalent, then* $\mathrm{val}(w) = \mathrm{val}(w_1)$*;*

(ii) $\mathrm{val}(w) = \mathrm{val}(w')$*; and*

(iii) $\overline{\mathrm{val}(w)} = \mathrm{val}(-w')$.

Proof. (i) Let

$$w_1 = (aw + b)/(cw + d), \quad \text{with } \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Because $j(\tau)$ is Γ-invariant, we have

$$\begin{aligned} j\left(\frac{w-\delta(w)w'e^u}{1-\delta(w)e^u}\right) &= j\left(\frac{a[w-\delta(w)w'e^u]/[1-\delta(w)e^u]+b}{c[w-\delta(w)w'e^u]/[1-\delta(w)e^u]+d}\right) \\ &= j\left(\frac{aw+b-\delta(w)(aw'+b)e^u}{cw+d-\delta(w)(cw'+d)e^u}\right) \\ &= j\left(\frac{(aw+b)/(cw+d)-\delta(w)[(aw'+b)/(cw+d)]e^u}{1-\delta(w)[(cw'+d)/(cw+d)]e^u}\right) \\ &= j\left(\frac{w_1-\delta(w)w_1'\eta e^u}{1-\delta(w)\eta e^u}\right) \\ &= j\left(\frac{w_1-\delta(w_1)w_1'\operatorname{sgn}(\eta)\eta e^u}{1-\delta(w_1)\operatorname{sgn}(\eta)\eta e^u}\right), \end{aligned}$$

where $\eta=(cw'+d)/(cw+d)$ and we have used $\delta(w)=\delta(w_1)\operatorname{sgn}(\eta)$, because $w_1-w_1'=(w-w')/[(cw+d)(cw'+d)]=(w-w')\eta/(cw'+d)^2$. Therefore, from (2), we obtain

$$\operatorname{val}(w)=\frac{1}{2\log\varepsilon}\int_{\sigma_0}^{\sigma_0+2\log\varepsilon} j\left(\frac{w_1-\delta(w_1)w_1\operatorname{sgn}(\eta)\eta e^u}{1-\delta(w_1)\operatorname{sgn}(\eta)\eta e^u}\right)du.$$

Then, because $\operatorname{sgn}(\eta)\eta>0$, we can make the change of variable $u\mapsto u-\log(\operatorname{sgn}(\eta)\eta)$, and we conclude that $\operatorname{val}(w)=\operatorname{val}(w_1)$.

(ii) Changing u to $-u$ in (3) and using the relation $\delta(w')=-\delta(w)$, we have

$$\begin{aligned} \operatorname{val}(w) &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{w-\delta(w)w'ie^{-u}}{1-\delta(w)ie^{-u}}\right)du \\ &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{w'+\delta(w)wie^{u}}{1+\delta(w)ie^{u}}\right)du \\ &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{w'-\delta(w')(w')'ie^{u}}{1-\delta(w')ie^{u}}\right)du \\ &= \operatorname{val}(w'). \end{aligned}$$

(iii) By (3), we have

$$\begin{aligned} \overline{\operatorname{val}(w)} &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} \overline{j\left(\frac{w-\delta(w)w'ie^{u}}{1-\delta(w)ie^{u}}\right)}\,du \\ &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(-\frac{w+\delta(w)w'ie^{u}}{1+\delta(w)ie^{u}}\right)du \\ &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{-w'+\delta(w)wie^{-u}}{1-\delta(w)ie^{-u}}\right)du \\ &= \frac{1}{2\log\varepsilon}\int_{-\log\varepsilon}^{\log\varepsilon} j\left(\frac{-w'-\delta(-w')(-w')'ie^{-u}}{1-\delta(-w')ie^{-u}}\right)du \\ &= \operatorname{val}(-w'). \end{aligned}$$

□

Remark. The invariance in (i) does not hold in general for other coefficients $a_n = a_n(w)$ ($n \neq 0$). The general transformation formula is similarly deduced and reads

$$a_n\left(\frac{aw+b}{cw+d}\right) = \left|\frac{cw'+d}{cw+d}\right|^{-\pi i n/\log\varepsilon} a_n(w).$$

COROLLARY (i) *Suppose* $\mathrm{disc}(w) = D$ *and let* ε_D *be the fundamental unit of the order* O_D. *Then, if* $N(\varepsilon_D) := \varepsilon_D\varepsilon'_D = -1$, *we always have* $\mathrm{val}(w) \in \mathbf{R}$.

(ii) *If* w *and* $-w'$ *are* Γ*-equivalent, then* $\mathrm{val}(w) \in \mathbf{R}$.

Proof. (i) In this case, w and $-w$ are Γ-equivalent, and thus, by applying (iii), (ii) and (i) of the Proposition in turn, we obtain

$$\overline{\mathrm{val}(w)} = \mathrm{val}(-w') = \mathrm{val}(-w) = \mathrm{val}(w).$$

(ii) This follows from (iii) and (i) of the Proposition. □

We denote by $\mathcal{A}$ the class in the narrow ideal class group $Cl^+(D)$ to which the ideal corresponding to w belongs. By the Proposition, $\mathrm{val}(w)$ depends only on the class $\mathcal{A}$. (With this in mind, we may write $\mathrm{val}(\mathcal{A})$.) The class corresponding to $-w'$ is $\mathcal{A}^{-1}$, and hence the Γ-equivalence of w and $-w'$ implies that $\mathcal{A}^2 = 1$ and vice versa. Hence, the assertion (ii) in the Corollary says that the value $\mathrm{val}(\mathcal{A})$ is real if $\mathcal{A}^2 = 1$.

Remark. Numerical computations reveal that not all $\mathrm{val}(w)$ are real.

We give three examples.

Example 1. The minimal discriminant for which there appears a non-real value is $D = 136$. The wide class number h is 2, and the narrow one h^+ is 4. A representative of the Γ-equivalence class of numbers of discriminant 136 is given by

$$\sqrt{34},\quad \frac{-4+\sqrt{34}}{18},\quad \frac{-1+\sqrt{34}}{11},\quad \frac{1+\sqrt{34}}{11},$$

and these are grouped into two wide ($\mathrm{PGL}_2(\mathbf{Z})$-equivalence) classes:

$$\left\{\sqrt{34},\ \frac{-4+\sqrt{34}}{18}\right\},\quad \left\{\frac{-1+\sqrt{34}}{11},\ \frac{1+\sqrt{34}}{11}\right\}.$$

The narrow class group $Cl^+(136)$ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$ and is generated by the class corresponding to $(-1+\sqrt{34})/11$. The values of val at this generator and its inverse $(1+\sqrt{34})/11$ (this is also a generator of $Cl^+(136)$) are computed as

$$\mathrm{val}\left(\frac{-1+\sqrt{34}}{11}\right) = 710.600\,451\,944\,002\,489\ldots - 0.519\,793\,828\,196\,1062\ldots i,$$

$$\mathrm{val}\left(\frac{1+\sqrt{34}}{11}\right) = 710.600\,451\,944\,002\,489\ldots + 0.519\,793\,828\,196\,1062\ldots i,$$

the two being conjugate with each other as follows from part (iii) of the Proposition.

The values at other two points are

$$\mathrm{val}(\sqrt{34}) = \mathrm{val}\left(\frac{-4+\sqrt{34}}{18}\right) = 720.290\,035\,004\,450\,662\,39\ldots,$$

values being identical because $(-4+\sqrt{34})/18$ and $-\sqrt{34} = (\sqrt{34})'$ are $\mathrm{PSL}_2(\mathbf{Z})$-equivalent.

Example 2. Consider the discriminant $D = 145$. In this case, we have $h = h^+ = 4$. As representative numbers, we may choose

$$\frac{1+\sqrt{145}}{2}, \quad \frac{1+\sqrt{145}}{6}, \quad \frac{-5+\sqrt{145}}{12}, \quad \frac{7+\sqrt{145}}{16}.$$

By the Corollary part (i) we know that all values of val at these points are real. Numerically, they are given as

$$\mathrm{val}\left(\frac{1+\sqrt{145}}{2}\right) = 720.484\,777\,347\,009\,813\ldots,$$
$$\mathrm{val}\left(\frac{1+\sqrt{145}}{6}\right) = 715.729\,503\,630\,174\,741\ldots,$$
$$\mathrm{val}\left(\frac{-5+\sqrt{145}}{12}\right) = 708.568\,357\,453\,922\,648\ldots,$$
$$\mathrm{val}\left(\frac{7+\sqrt{145}}{16}\right) = 715.729\,503\,630\,174\,741\ldots.$$

The class group is isomorphic to $\mathbf{Z}/4\mathbf{Z}$. This is seen from the fact that, for $w_1 = (1+\sqrt{145})/6$, $-w_1'$ is equivalent not to w_1 but to $w_2 = (7+\sqrt{145})/16$. Hence $\mathrm{val}(w_1) = \mathrm{val}(w_2)$.

Example 3. Consider $D = 520$. In this case, again, we have $h = h^+ = 4$. As representative numbers, we may choose

$$\sqrt{130}, \quad \frac{-1+\sqrt{130}}{3}, \quad \frac{-3+\sqrt{130}}{11}, \quad \frac{-5+\sqrt{130}}{15},$$

and whose 'values' are given numerically by

$$\mathrm{val}(\sqrt{130}) = 721.700\,344\,576\,590\,835\ldots,$$
$$\mathrm{val}\left(\frac{-1+\sqrt{130}}{3}\right) = 719.032\,996\,230\,455\,907\ldots,$$
$$\mathrm{val}\left(\frac{-3+\sqrt{130}}{11}\right) = 713.022\,954\,982\,182\,920\ldots,$$
$$\mathrm{val}\left(\frac{-5+\sqrt{130}}{15}\right) = 716.888\,481\,219\,718\,920\ldots.$$

In this case, the class group is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, and all values appear to be distinct.

3. Experiments related to Markoff numbers

First let us recall Markoff's theory. The classical theorem of Hurwitz asserts that, for any real irrational number α, there exist infinitely many rational numbers p/q that satisfy

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

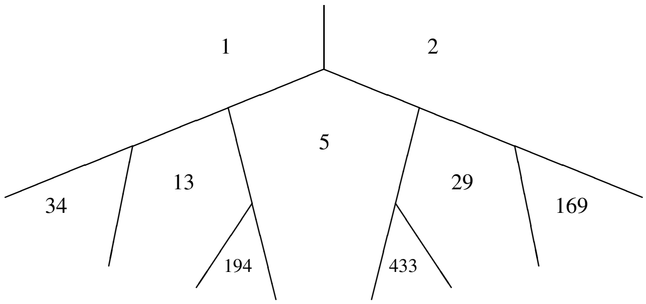


FIGURE 1. The tree of Markoff numbers.

The constant $1/\sqrt{5}$ is best possible. But if we exclude as α the numbers that are $\mathrm{PGL}_2(\mathbf{Z})$-equivalent to the golden ratio $(1+\sqrt{5})/2$, the constant $1/\sqrt{5}$ improves to $1/\sqrt{8}$. If we also exclude the numbers that are $\mathrm{PGL}_2(\mathbf{Z})$-equivalent to $\sqrt{2}$, then we can take $5/\sqrt{221}$ as the constant. In general, this continues as follows. There is an infinite sequence of integers called Markoff numbers,

$$\{m_i\}_{i=1}^{\infty} = \{1, 2, 5, 13, 29, 34, 89, 169, 194, 233, \ldots\},$$

and associated quadratic irrationalities θ_i and monotonically increasing L_i whose limit is 3, with the following property: For any i, if the number α is not $\mathrm{PGL}_2(\mathbf{Z})$-equivalent to any of $\theta_1, \theta_2, \ldots, \theta_{i-1}$, then there exist infinitely many rational numbers p/q that satisfy

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{L_i q^2}.$$

Explicitly, the Markoff numbers m_i appear as solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz, \tag{4}$$

and

$$L_i = \sqrt{9 - 4/m_i^2}, \quad \theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i^2 - 4}}{2m_i}, \tag{5}$$

where k_i is an integer that satisfies $a_i k_i \equiv b_i (\mathrm{mod}\, m_i)$ and here (a_i, b_i, m_i) is a solution of equation (4) with m_i maximal. If (p, q, r) is a solution of (4), then $(p, q, 3pq - r)$ and $(p, r, 3pr - q)$ are also solutions. This gives to all solutions the structure of a tree, and we can arrange Markoff numbers like in Figure 1.

We computed several values of $\mathrm{val}(\theta_i)$, and observed the following.

Observation

(iv) Only real values are

$$\mathrm{val}(\theta_1) = \mathrm{val}\left(\frac{-1+\sqrt{5}}{2}\right) = 706.324\,813\,54\ldots$$

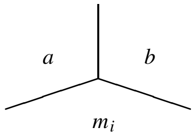
and

$$\mathrm{val}(\theta_2) = \mathrm{val}(-1+\sqrt{2}) = 709.892\,890\,91\ldots.$$

No other values $\mathrm{val}(\theta_i)$ $(i \geq 3)$ seem to be real.

Note that in Markoff's theory only $\mathrm{PGL}_2(\mathbf{Z})$-equivalence class is relevant, but we need $\mathrm{PSL}_2(\mathbf{Z})$-equivalence to distinguish non-real $\mathrm{val}(\theta_i)$ and its conjugate. Here, the order of (a_i, b_i) in the definition of θ_i in (5) becomes relevant. We introduce the following refinement.

Let (a, b, m_i) be the Markoff triple associated to the ith Markoff number m_i and assume that the order of a and b is chosen so that their positions in the tree is like



(b is on the right of a, so, (13, 5, 194) for 194, (5, 29, 433) for 433, etc.) Define two numbers $\theta_{i,1}$ and $\theta_{i,2}$ by

$$\theta_{i,1} = \frac{-3m_i + 2k_{i,1} + \sqrt{9m_i^2 - 4}}{2m_i} \quad \text{and} \quad \theta_{i,2} = \frac{-3m_i + 2k_{i,2} + \sqrt{9m_i^2 - 4}}{2m_i}$$

with $k_{i,1}$ and $k_{i,2}$ being integers that satisfy

$$ak_{i,1} \equiv b \pmod{m_i} \quad \text{and} \quad bk_{i,2} \equiv a \pmod{m_i}$$

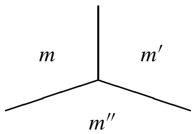
respectively.

Observation

(v) The imaginary part of $\mathrm{val}(\theta_{i,1})$ (respectively $\mathrm{val}(\theta_{i,2})$) is always positive (respectively negative).

Observation

(vi) Suppose three Markoff numbers $m,\ m',\ m''$ are in the position like



in the Markoff tree, and let $\theta_1,\ \theta_2,\ \theta_1',\ \theta_2',\ \theta_1'',\ \theta_2''$ be the associated (refined) quadratic numbers. Then, for $j = 1, 2$, both the real and the imaginary parts of θ_j'' lie between those of θ_j and θ_j' (the case of $m = 1,\ m' = 2,\ m'' = 5$ is exceptional, where the imaginary parts of $\mathrm{val}(\theta_j)$ and $\mathrm{val}(\theta_j')$ are both 0, while the real part of $\mathrm{val}(\theta_j'')$ is indeed in between those of $\mathrm{val}(\theta_j)$ and $\mathrm{val}(\theta_j')$).

Hence, all real parts of $\mathrm{val}(\theta_{i,j})$ $(j = 1, 2)$, conjecturally, lie in the interval

$$[706.324\,8135\ldots, 709.892\,8909\ldots]$$

and imaginary parts in

$$[-0.267\,0397\ldots, 0.267\,0397\ldots],$$

where 0.267 0397 . . . is the imaginary part of

$$\mathrm{val}(\theta_{3,1}) = \mathrm{val}((-11+\sqrt{221})/10) = 708.909\,919\,72\ldots + 0.267\,039\,735\ldots i.$$

Choose any Markoff number m. This determines a connected unbounded region R in the tree. If we trace the edges of R downward, we obtain the sequence of Markoff numbers associated to the neighboring region with respect to those edges. Let

$$n_1^L, n_2^L, n_3^L, \ldots \quad \text{and} \quad n_1^R, n_2^R, n_3^R, \ldots$$

be those sequences corresponding to the left and the right edges respectively. (When $m = 1$ (respectively $m = 2$), only the sequence $\{n_k^R\}$ (respectively $\{n_k^L\}$) occur.)

Observation

(vii) Let $\theta_1^{(m)}$ and $\theta_2^{(m)}$ be the Markoff irrationalities associated to m as explained above (by fixing the order of a and b in the triple (a, b, m)), and similarly $\theta_{k,j}^L$ $(j = 1, 2)$ (respectively $\theta_{k,j}^R$ $(j = 1, 2)$) the irrationalities associated to n_k^L (respectively n_k^R). Then, we surmise

$$\lim_{k\to\infty} \mathrm{val}(\theta_{k,1}^R) = \mathrm{val}(\theta_1^{(m)}) \quad \text{and} \quad \lim_{k\to\infty} \mathrm{val}(\theta_{k,2}^L) = \mathrm{val}(\theta_2^{(m)}).$$

Below, we repeat the observations made at the beginning of the paper, in the form of several questions:

(i) Is $\mathrm{val}((1+\sqrt{5})/2) = 706.324\,813\,540\,81\ldots$ minimal (in absolute value) among all the values of $j(\tau)$ at real quadratics? Do all real values of $\mathrm{val}(w)$, or all absolute values or real parts of $\mathrm{val}(w)$, lie in the interval $[706.324\,813\,540\,81\ldots, 744]$? If this is the case, is 744 the best possible upper bound?

(ii) Does $\mathrm{val}(w)$ possess some information concerning the Diophantine approximation of w? For instance, does $\mathrm{val}(w)$ increase as the rational approximation of w improves?

(iii) Does the imaginary part of $\mathrm{val}(w)$ always lie in the interval $(-1, 1)$? What is the distribution of the imaginary parts?

PROBLEM *Formulate rigorous statements and find proofs of them that answer all of these questions and, above all, find an arithmetic meaning of* $\mathrm{val}(w)$.

Remark. (1) Concerning the nature of the value $\mathrm{val}(w)$, numerical experiments suggest that it is very unlikely that $\mathrm{val}(w)$ is itself an algebraic number. The author has spent a fair amount of time, using ‘lindep’ or ‘algdep’ facilities of Pari-GP, or ‘Plouffe’s inverter’ website, to see if any multiplicative combination of $\mathrm{val}(w)$, $\log\varepsilon$, π, etc. becomes algebraic, but all in vain so far.

(2) Recent work of Duke *et al* [**2**] reveals that the ‘trace’ of $\mathrm{val}(w)$ appears as the Fourier coefficient of a weakly harmonic modular forms of weight 1/2. It would be an important problem to understand our observations in light of their results.

In Tables 1–7, we present some values of $\mathrm{val}(w)$. The computations were carried out using Mathematica. Figure 2 shows some values at the Markoff irrationalities.

We denote by $[b_1, b_2, \ldots, b_n]$ a purely periodic (ordinary) continued fraction of period length n. For example, we have $[1] = (1+\sqrt{5})/2$, $[2, 1] = 1+\sqrt{3}$, etc. The fundamental unit of norm 1 (a generator of U_D in Section 2) of the order O_D of discriminant D is denoted by ε.

TABLE 1. Values of $\mathrm{val}(w)$ at $w = [n]$.

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[1]	5	706.324 813 540 812 582 055 9603 . . .	0.962 423 650 1192 . . .
[2]	8	709.892 890 919 912 336 805 9253 . . .	1.762 747 174 0390 . . .
[3]	13	713.222 719 212 910 637 526 0272 . . .	2.389 526 434 5742 . . .
[4]	20	715.865 831 050 964 456 788 2877 . . .	2.887 270 950 3576 . . .
[5]	29	717.916 551 088 562 709 794 6754 . . .	3.294 462 292 7421 . . .
[6]	40	719.529 219 514 924 156 581 2037 . . .	3.636 892 918 4641 . . .
[7]	53	720.824 755 382 901 692 908 9184 . . .	3.931 440 943 2993 . . .
[8]	68	721.887 832 620 286 958 890 5005 . . .	4.189 425 094 5222 . . .
[9]	85	722.776 891 456 521 926 283 0724 . . .	4.418 695 417 2306 . . .
[10]	104	723.532 770 090 746 496 037 8584 . . .	4.624 876 682 5455 . . .
[20]	404	727.629 600 004 732 546 482 4629 . . .	5.996 445 900 5959 . . .
[30]	904	729.431 443 862 573 248 095 1697 . . .	6.804 613 290 9611 . . .
[50]	2504	731.242 602 752 474 100 559 3885 . . .	7.824 845 531 2825 . . .
[100]	10 004	733.111 306 559 737 273 613 0899 . . .	9.210 540 341 9828 . . .

TABLE 2. Values of $\mathrm{val}(w)$ at $w = [n, 1]$.

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[2, 1]	12	709.792 359 008 032 010 270 2826 . . .	1.316 957 896 9248 . . .
[3, 1]	21	713.246 137 271 926 341 337 2589 . . .	1.566 799 236 9724 . . .
[4, 1]	32	715.876 486 180 014 188 035 1424 . . .	1.762 747 174 0390 . . .
[5, 1]	45	717.883 409 637 447 348 654 6884 . . .	1.924 847 300 2384 . . .
[6, 1]	60	719.455 961 655 235 800 385 4302 . . .	2.063 437 068 8955 . . .
[7, 1]	77	720.721 568 296 248 954 455 0810 . . .	2.184 643 791 6051 . . .
[8, 1]	96	721.764 036 803 803 548 916 9855 . . .	2.292 431 669 5611 . . .
[9, 1]	117	722.639 624 217 652 446 518 1309 . . .	2.389 526 434 5742 . . .
[10, 1]	140	723.387 187 954 432 922 287 5427 . . .	2.477 888 730 2884 . . .
[20, 1]	480	727.493 557 432 673 052 183 8984 . . .	3.088 969 904 8446 . . .
[30, 1]	1020	729.324 063 137 304 363 666 7693 . . .	3.464 757 906 6758 . . .
[50, 1]	2700	731.170 341 715 375 608 810 5933 . . .	3.950 873 690 7744 . . .
[100, 1]	10 400	733.072 896 468 766 515 552 2285 . . .	4.624 876 682 5455 . . .

TABLE 3. Values of $\mathrm{val}(w)$ at $w = [n, 2]$.

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[3, 2]	60	711.927 516 399 581 905 655 3017 . . .	2.063 437 068 8955 . . .
[4, 2]	24	713.825 864 287 342 036 491 8902 . . .	2.292 431 669 5611 . . .
[5, 2]	140	715.400 787 446 589 501 269 6492 . . .	2.477 888 730 2884 . . .
[6, 2]	48	716.695 284 423 882 570 542 4260 . . .	2.633 915 793 8496 . . .
[7, 2]	252	717.771 120 164 298 940 237 6217 . . .	2.768 659 383 3135 . . .
[8, 2]	80	718.678 601 577 902 203 841 7819 . . .	2.887 270 950 3576 . . .
[9, 2]	396	719.455 234 695 205 003 389 4397 . . .	2.993 222 846 1263 . . .
[10, 2]	120	720.128 621 394 196 009 353 6607 . . .	3.088 969 904 8446 . . .

TABLE 4. Values of $\mathrm{val}(w)$ at $w = [2, 1, \ldots, 1]$.

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[2]	8	709.892 890 919 912 336 805 9253 ...	1.762 747 174 0390 ...
[2, 1]	12	709.792 359 008 032 010 270 2826 ...	1.316 957 896 9248 ...
[2, 1, 1]	40	708.513 448 134 892 190 619 8907 ...	3.636 892 918 4641 ...
[2, 1, 1, 1]	96	708.156 050 841 666 154 768 9422 ...	2.292 431 669 5611 ...
[2, 1, 1, 1, 1]	260	707.806 465 621 023 832 295 3785 ...	5.552 944 561 4474 ...
[2, 1, 1, 1, 1, 1]	672	707.597 854 238 026 263 880 5993 ...	3.256 613 954 8000 ...
[2, 1, 1, 1, 1, 1, 1]	1768	707.430 561 224 434 932 261 1838 ...	7.476 472 060 5230 ...

TABLE 5. Values of $\mathrm{val}(w)$ at $w = [3, 1, \ldots, 1]$.

w	D	$\mathrm{val}(w)$	$\log \varepsilon$
[3]	13	713.222 719 212 910 637 526 0272 ...	2.389 526 434 5742 ...
[3, 1]	21	713.246 137 271 926 341 337 2589 ...	1.566 799 236 9724 ...
[3, 1, 1]	17	711.046 084 409 655 031 887 9502 ...	4.189 425 094 5222 ...
[3, 1, 1, 1]	165	710.336 609 396 122 525 208 7583 ...	2.558 978 977 0286 ...
[3, 1, 1, 1, 1]	445	709.647 535 497 919 284 996 8007 ...	6.093 564 674 4388 ...
[3, 1, 1, 1, 1, 1]	288	709.211 854 158 535 704 218 8756 ...	3.525 494 348 0781 ...
[3, 1, 1, 1, 1, 1, 1]	3029	708.859 309 167 215 572 179 0085 ...	8.015 327 199 8839 ...

TABLE 6. First several non-real values.

w	D	$\mathrm{val}(w)$
$(12+\sqrt{34})/11$	136	710.600 451 944 002 489 45 ... + 0.519 793 828 196 106 20 ... i
$(10+\sqrt{34})/11$	136	710.600 451 944 002 489 45 ... − 0.519 793 828 196 106 20 ... i
$(33+\sqrt{205})/34$	205	714.160 340 182 257 155 92 ... + 0.753 639 139 590 380 68 ... i
$(25+\sqrt{205})/30$	205	714.160 340 182 257 155 92 ... − 0.753 639 139 590 380 68 ... i
$(21+\sqrt{221})/22$	221	708.909 919 720 708 747 30 ... + 0.267 039 735 460 289 96 ... i
$(23+\sqrt{221})/22$	221	708.909 919 720 708 747 30 ... − 0.267 039 735 460 289 96 ... i
$(47+\sqrt{305})/56$	305	716.138 986 938 485 793 03 ... + 0.821 841 933 596 968 10 ... i
$(35+\sqrt{305})/46$	305	716.138 986 938 485 793 03 ... − 0.821 841 933 596 968 10 ... i
$(23+\sqrt{79})/25$	316	712.659 485 826 877 025 03 ... + 0.325 455 537 687 324 63 ... i
$(13+\sqrt{79})/15$	316	712.659 485 826 877 025 03 ... − 0.325 455 537 687 324 63 ... i
$(17+\sqrt{79})/15$	316	712.659 485 826 877 025 03 ... + 0.325 455 537 687 324 63 ... i
$(17+\sqrt{79})/21$	316	712.659 485 826 877 025 03 ... − 0.325 455 537 687 324 63 ... i

TABLE 7. First several values at Markoff irrationalities.

i	m_i	$\theta_{i,1}$	$\mathrm{val}(\theta_{i,1})$
1	1	$(-3+\sqrt{5})/2$	706.324 813 540 812 582 05 . . .
2	2	$-1+\sqrt{2}$	709.892 890 919 912 336 80 . . .
3	5	$(-11+\sqrt{221})/10$	708.909 919 720 708 747 . . . + 0.267 039 735 460 289 . . . i
4	13	$(-29+\sqrt{1517})/26$	708.257 588 242 846 779 . . . + 0.228 635 826 664 936 . . . i
5	29	$(-63+\sqrt{7565})/58$	709.302 611 667 387 656 . . . + 0.165 196 473 942 199 . . . i
6	34	$(-19+5\sqrt{26})/17$	707.858 372 382 696 744 . . . + 0.184 765 335 383 899 . . . i
7	89	$(-199+\sqrt{71\,285})/178$	707.594 565 998 876 317 . . . + 0.153 386 774 906 169 . . . i
8	169	$(-367+\sqrt{257\,045})/338$	709.469 768 024 657 232 . . . + 0.118 518 079 083 046 . . . i
9	194	$(-108+\sqrt{21\,170})/97$	708.534 665 666 479 421 . . . + 0.245 013 213 468 323 . . . i
10	233	$(-521+\sqrt{488\,597})/466$	707.408 028 846 873 175 . . . + 0.130 903 420 887 032 . . . i

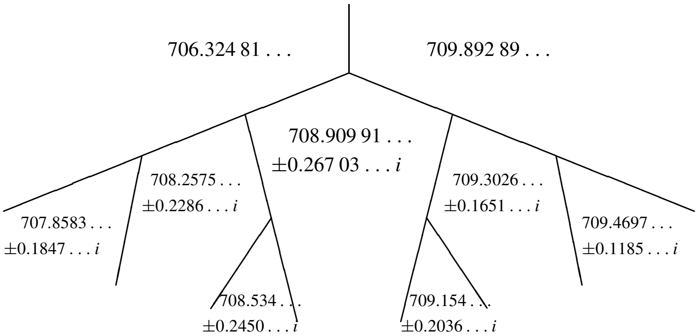


FIGURE 2. Values in the Markoff tree.

REFERENCES

[1] R. Dedekind. Schreiben an Borchardt über die Theorie der elliptischen Modulfunktionen. J. Reine Angew. Math. **83** (1877), 265–292.

[2] W. Duke, Ö. Imamoḡlu and Á. Tóth. Cycle integrals of the j-function and weakly harmonic modular forms. Preprint, 2008.

[3] E. Hecke. Darstellung von Klassenzahlen als Perioden von Integralen 3. Gattung aus dem Gebiet der elliptischen Modulfunktionen. Abh. Math. Sem. Hamburg **4** (1925), 211–223.

Masanobu Kaneko
Faculty of Mathematics
Kyushu University 33
Fukuoka 812-8581
Japan
(E-mail: mkaneko@math.kyushu-u.ac.jp)