On the local factor of the zeta function of quadratic orders

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Abstract

We prove by an elementary method the Riemann hypothesis for the local Euler factor of the zeta function of quadratic orders.

Let K be a quadratic number field, $\mathcal{O}_K$ its ring of integers, and $\mathcal{O}_f$ for each integer $f \geq 1$ the order of conductor f in $\mathcal{O}_K = \mathcal{O}_1$. As shown in [1] and [3], the zeta function $\zeta_{\mathcal{O}_f}(s)$ of $\mathcal{O}_f$, which is defined by

$$\zeta_{\mathcal{O}_f}(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where the sum extends over all *proper* ideals $\mathfrak{a}$ of $\mathcal{O}_f$ with norm $N(\mathfrak{a})$, has the following form:

$$\zeta_{\mathcal{O}_f}(s) = \zeta_K(s) \cdot \prod_{p|f} \frac{(1-p^{-s})(1-\chi(p)p^{-s}) - p^{n_p-1-2n_ps}(1-p^{1-s})(\chi(p)-p^{1-s})}{(1-p^{1-2s})}.$$

Here, $\zeta_K(s)$ is the Dedekind zeta function of K, the product is over the prime factors p of the conductor f, with n_p being the exact power of p in f, and χ is the Dirichlet character corresponding to the extension $K/\mathbf{Q}$.

The main purpose of this short note is to provide proofs of the following properties, including the "Riemann hypothesis," for the local factor

$$\varepsilon_{f,p}(s) := \frac{(1-p^{-s})(1-\chi(p)p^{-s}) - p^{n_p-1-2n_ps}(1-p^{1-s})(\chi(p)-p^{1-s})}{(1-p^{1-2s})}$$

of $\zeta_{\mathcal{O}_f}(s)/\zeta_K(s)$:

Theorem. 1) *The function $\varepsilon_{f,p}(s)$ is a polynomial of degree $2n_p$ in p^{-s} and satisfies the functional equation*

$$\varepsilon_{f,p}(1-s) = p^{-n_p(1-2s)}\varepsilon_{f,p}(s).$$

2) *All the zeros of $\varepsilon_{f,p}(s)$ lie on the line $Re(s) = 1/2$.*

Proof. Setting $u = p^{-s}$ and $n = n_p$, we rewrite $\varepsilon_{f,p}(s)$ as the function $P_n(u)$, given as follows:

$$\begin{aligned} P_n(u) &= \frac{(1-u)(1-\chi(p)u) - p^{n-1}u^{2n}(1-pu)(\chi(p)-pu)}{1-pu^2} \\ &= \frac{1-p^{n+1}u^{2(n+1)} - (1+\chi(p))u(1-p^nu^{2n}) + \chi(p)u^2(1-p^{n-1}u^{2(n-1)})}{1-pu^2}. \end{aligned}$$

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The numerator of this expression vanishes if we set $u = \pm 1/\sqrt{p}$ and hence is divisible by the denominator $1 - pu^2$. Thus $P_n(u)$ is indeed a polynomial of degree $2n$. By direct division, we find

$$P_n(u) = 1 - (1 + \chi(p))u + \cdots - p^{n-1}(1 + \chi(p))u^{2n-1} + p^n u^{2n}.$$

The functional equation can be verified straightforwardly. This ends the proof of assertion 1).

To prove the Riemann hypothesis 2), put $s = 1/2 + it/\log p$. Then we have $u = p^{-1/2}e^{-it}$ and

$$P_n(u) = \frac{1 - e^{-2(n+1)it} - (1 + \chi(p))p^{-1/2}e^{-it}(1 - e^{-2nit}) + \chi(p)p^{-1}e^{-2it}(1 - e^{-2(n-1)it})}{1 - e^{-2it}}.$$

Then, using the relation

$$\frac{1 - e^{-2mit}}{1 - e^{-2it}} = \frac{e^{-mit}}{e^{-it}} \frac{\sin mt}{\sin t},$$

we obtain

$$P_n(u) = \frac{e^{-nit}}{p} \left(p \, \frac{\sin(n+1)t}{\sin t} - \sqrt{p}\,(1 + \chi(p)) \, \frac{\sin nt}{\sin t} - \chi(p) \, \frac{\sin(n-1)t}{\sin t} \right).$$

We have to show that if the right-hand side of this is zero then t is real. Recall that, for any integer $m \geq 0$, the quotient $\sin(m+1)t/\sin t$ is a polynomial of degree m in $x = \cos t$, which is referred to as the Chebyshev polynomial of the second kind, denoted by $U_m(x)$. The first several of these are as follows:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x.$$

Note that the function $U_m(x)$ can also be defined for $m < 0$; in particular, we have $U_{-1}(x) = 0$ and $U_{-2}(x) = -1$. Using the $U_m(x)$, the proof is reduced to showing that all the roots of the polynomial (of degree n)

$$Q_n(x) := p\,U_n(x) - \sqrt{p}\,(1 + \chi(p))\,U_{n-1}(x) + \chi(p)\,U_{n-2}(x) \quad (n \geq 1)$$

are in the real interval $[-1, 1]$.

Because of the recurrence of the Chebyshev polynomials

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

the polynomials $Q_n(x)$ satisfy the same recurrence:

$$Q_n(x) = 2xQ_{n-1}(x) - Q_{n-2}(x) \quad (n \geq 2),$$

with $Q_0(x) = p - \chi(p)$. We show that the n roots of $Q_n(x)$ are all in the interval $(-1, 1)$ by making use of the theorem of Sturm (cf. [2, §92]), utilizing the fact that $Q_n(x), Q_{n-1}(x), \ldots, Q_0(x)$ forms a "Strum sequence". Because $U_n(1) = n + 1$ and $U_n(-1) = (-1)^n(n + 1)$, we have

$$\begin{aligned} Q_n(1) &= p(n+1) - \sqrt{p}(1 + \chi(p))n + \chi(p)(n-1) \\ &= (\sqrt{p} - 1)(\sqrt{p} - \chi(p))n + p - \chi(p) > 0 \end{aligned}$$

and

$$\begin{aligned} Q_n(-1) &= p(-1)^n(n+1) - \sqrt{p}(1 + \chi(p))(-1)^{n-1}n + \chi(p)(-1)^{n-2}(n-1) \\ &= (-1)^n \left\{ p(n+1) + \sqrt{p}(1 + \chi(p))n + \chi(p)(n-1) \right\} \\ &= (-1)^n \left\{ (\sqrt{p} + 1)(\sqrt{p} + \chi(p))n + p - \chi(p) \right\}. \end{aligned}$$

The sign of the last expression is $(-1)^n$, and hence the number of "variations," as defined in [2, §92], is n. Then, noting the theorem of Sturm, we conclude that $Q_n(x)$ has n roots in the interval $(-1, 1)$. (Note that, since the degree of $Q_n(x)$ is n, condition 4 of [2, §92] need not be verified.) ∎

Remarks and questions. 1) It is amusing that the properties stated in the above theorem are precisely those enjoyed by the congruence zeta function (or, rather, its essential part) of a curve of genus $n = n_p$ over the prime field $\mathbf{F}_p$. This naturally leads us to wonder if $\varepsilon_{f,p}(s)$ admits a cohomological (or any other "nice") interpretation and if the above theorem can be proved "conceptually" using such an interpretation.

2) The theorem proved here shows in particular that the quotient $\zeta_{\mathcal{O}_f}(s)/\zeta_K(s)$ is entire and that the Riemann hypothesis holds for $\zeta_{\mathcal{O}_f}(s)$ only if it holds for $\zeta_K(s)$. It is known that for a Galois extension k'/k of number fields, the quotient of the Dedekind zeta functions $\zeta_{k'}(s)/\zeta_k(s)$ is entire. Thus the zeta function of the *over* field is divisible by that of the base field. On the contrary, in the case considered above, the zeta function of the *subring* $\mathcal{O}_f$ is divisible by that of the over ring. What is the reason for this?

3) The generating function of the polynomials $P_n(x)$ takes the simple form

$$F(u, X) := 1 + \sum_{n=1}^{\infty} P_n(u) X^n = \frac{(1 - uX)(1 - \chi(p)uX)}{(1 - X)(1 - pu^2X)},$$

and the functional equation for $P_n(u)$, which is written as

$$p^n u^{2n} P_n(\frac{1}{pu}) = P_n(u),$$

is encoded as

$$F(\frac{1}{pu}, pu^2X) = F(u, X).$$

4) For another zeta function

$$\zeta^*_{\mathcal{O}_f}(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs over *all* (not necessarily proper) ideals of $\mathcal{O}_f$, the Euler product is (*cf.* [3])

$$\zeta_K(s) \cdot \prod_{p|f} \frac{1 - p^{(n_p+1)(1-2s)} - \chi(p)p^{-s}(1 - p^{n_p(1-2s)})}{1 - p^{1-2s}}.$$

It can be shown similarly that the local factor

$$\frac{1 - p^{(n_p+1)(1-2s)} - \chi(p)p^{-s}(1 - p^{n_p(1-2s)})}{1 - p^{1-2s}}$$

possesses the same properties as in the theorem.

5) It would be nice to have a generalization of our theorem to the zeta functions of orders of number fields of higher degree.

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