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A VARIATION OF EULER'S APPROACH TO VALUES OF THE RIEMANN ZETA FUNCTION

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Dedicated to Leonhard Euler on the 296th anniversary of his birthday

1. Introduction

Some time around 1740, Euler [**3–5**] took a decisive step toward laying the groundwork for the functional equation of the Riemann zeta function when he discovered a marvellous method of calculating the values of the (absolutely!) divergent series

$$\begin{aligned} \text{‘}1+1+1+1+1+\cdots\text{’} &= -\tfrac{1}{2}, \\ \text{‘}1+2+3+4+5+\cdots\text{’} &= -\tfrac{1}{12}, \\ \text{‘}1+4+9+16+25+\cdots\text{’} &= 0, \\ \text{‘}1+8+27+64+125+\cdots\text{’} &= \tfrac{1}{120}, \quad \text{etc.} \end{aligned}$$

In modern terms, these are the values at non-positive integer arguments of the Riemann zeta function $\zeta(s)$, defined by the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots,$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. With no concept of functions of complex variables, to say nothing of analytic continuation, Euler proceeded as described below to give meaning to divergent series of the above type and to evaluate the values of them.

First, he directed his attention to the 'less divergent' alternating series

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \text{etc.}, \tag{$\odot$}$$

because its convergent counterpart

$$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \text{etc.}, \tag{☽}$$

does indeed have faster convergence and is related to the original series by the simple equation

$$\zeta^*(s) = (1 - 2^{1-s})\zeta(s), \tag{1}$$

where

$$\zeta^*(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \cdots .$$

Then, he observed that the value in ⊙ is obtained as a 'limit' of the power series

$$1^m - 2^m x + 3^m x^2 - 4^m x^3 + 5^m x^4 - 6^m x^5 + \text{etc.} \tag{2}$$

as $x \to 1$, because, although the series itself converges only for $|x| < 1$, it has an expression as a rational function (or analytic continuation as we now term it), finite at $x = 1$, which is obtained for a given value of m by applying the operator $((d/dx)x)^m$ (or in terms of the *Euler operator* $x(d/dx)$, the operator $x^{-1}(x(d/dx))^m x$) to the geometric series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \quad (|x| < 1). \tag{3}$$

For instance, if we substitute $x = 1$ into (3), we formally find

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + 1 - \cdots = \zeta^*(0),$$

and hence, in view of (1), we have $\zeta(0) = -1/2$. The next few cases are

$$\begin{aligned} \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots , \\ \frac{1-x}{(1+x)^3} &= 1 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - \cdots , \\ \frac{1-4x+x^2}{(1+x)^4} &= 1 - 2^3x + 3^3x^2 - 4^3x^3 + 5^3x^4 - \cdots , \\ &\vdots \end{aligned}$$

which imply

$$\zeta^*(-1) = \tfrac{1}{4}, \quad \zeta^*(-2) = 0, \quad \zeta^*(-3) = -\tfrac{1}{8} \cdots ,$$

and hence

$$\zeta(-1) = -\tfrac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \tfrac{1}{120} \cdots .$$

Later, with the power of complex integral and the theory of analytic continuation, Euler's method found a justification from a modern point of view. In fact, in

[7, Theorem 2.5], it is shown that the alternating series $\zeta^*(s)$ is summable (in the sense of Abel) to $(1-2^{1-s})\zeta(s)$ for all values of s.

In this article, aiming to evaluate $\zeta(-m)$ as a value of the analytically continued function $\zeta(s)$ in the most elementary, yet rigorous, way possible, we present an alternative approach. We introduce and investigate a q-analogue of the Riemann zeta function. As becomes clear in the course of our study, this function serves very well for the purpose of not only computing $\zeta(-m)$, but also providing a q-analogue which is valid for *all* $s \in \mathbb{C}$; in other words, in the classical limit $q \to 1$ the value approaches to $\zeta(s)$ for all s.

To be more specific, we consider a series that is similar to that obtained from (2) by substituting q^t for x (in which case the operator (d/dt) essentially plays the role of the Euler operator $x(d/dx)$), and we replace n^m by the q-integer $[n]_q := (1-q^n)/(1-q)$ raised by the power $-s$. (Recall that Euler is the 'grand master' of q.) Thus, we consider the series

$$f_q(s,t) := \sum_{n=1}^{\infty} \frac{q^{nt}}{[n]_q^s} = \frac{q^t}{[1]_q^s} + \frac{q^{2t}}{[2]_q^s} + \frac{q^{3t}}{[3]_q^s} + \frac{q^{4t}}{[4]_q^s} + \cdots . \tag{4}$$

Throughout the paper, we assume $0 < q < 1$, so that the series (4) converges absolutely for any $s \in \mathbb{C}$ and $\mathrm{Re}(t) > 0$. If $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(t) > 0$, the series obviously converges to $\zeta(s)$ as $q \uparrow 1$. This suggests that we should regard the function $f_q(s,t)$ as a q-analogue of the Riemann zeta function $\zeta(s)$, but we put off elucidating the precise analogy until we restrict our consideration to the special case $t = s-1$. Before considering this special case, we establish below the meromorphic continuation of $f_q(s,t)$ as a function of the two variables s and t. This is carried out quite easily by use of the binomial theorem.

PROPOSITION 1. *Let $0 < q < 1$. As a function of $(s,t) \in \mathbb{C}^2$, $f_q(s,t)$ is continued meromorphically via the series expansion*

$$\begin{aligned} f_q(s,t) &= (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{t+r}}{1-q^{t+r}} \\ &= (1-q)^s \left(\frac{q^t}{1-q^t} + s\frac{q^{t+1}}{1-q^{t+1}} + \frac{s(s+1)}{2}\frac{q^{t+2}}{1-q^{t+2}} + \cdots \right), \end{aligned}$$

which has poles of order 1 *at all $t \in \mathbb{Z}_{\le 0} + 2\pi i\mathbb{Z}/\log q := \{a + 2\pi i b/\log q \mid a, b \in \mathbb{Z}, a \le 0\}$.*

Proof. We simply apply the binomial expansion

$$(1-q^n)^{-s} = \sum_{r=0}^{\infty} \binom{s+r-1}{r} q^{nr}$$

to (4) and change the order of the summations to obtain

$$\begin{aligned} f_q(s,t) &= (1-q)^s \sum_{n=1}^{\infty} \frac{q^{nt}}{(1-q^n)^s} \\ &= (1-q)^s \sum_{n=1}^{\infty} q^{nt} \sum_{r=0}^{\infty} \binom{s+r-1}{r} q^{nr} \\ &= (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \sum_{n=1}^{\infty} q^{n(t+r)} \\ &= (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{t+r}}{1-q^{t+r}}. \end{aligned}$$

The other assertion follows readily from this.

Remark. It is worth noting that the function $f_q(s,t)$ can be expressed as a (beta-like) Jackson integral. In fact, we have the identity

$$q^{-t}(1-q)^{1-s} f_q(s,t) = (1-q) \sum_{j=0}^{\infty} \frac{q^{jt}}{(1-q^{j+1})^s} = \int_0^1 x^{t-1}(1-qx)^{-s}\, d_q x.$$

In the next section, we specialize to the case $t = s-1$ and establish a formula for the value at $s = -m \in \mathbb{Z}_{\le 0}$ in Proposition 2 and its limit as $q \uparrow 1$ in Theorem 1. Then we prove in Theorem 2 that the limit as $q \uparrow 1$ is equal to $\zeta(s)$ for any $s \in \mathbb{C}$ other than 1.

2. Main results

We now consider the case $t = s-1$. For $s = -m \in \mathbb{Z}_{\le 0}$, the point $(s,t) = (-m, -m-1)$ lies on the pole divisor $t = -m-1$ of $f_q(s,t)$. Nevertheless, a sort of 'miracle' occurs by which this point turns out to be what is called a 'point of indeterminacy'; in other words, the function $f_q(s, s-1)$ has a finite limit as $s \to -m$

and, moreover, this limit approaches the 'correct' value, $\zeta(-m)$, as $q \uparrow 1$. What is more, the function $f_q(s, s-1)$ converges as $q \uparrow 1$ to $\zeta(s)$ for *any* s! These results, which we prove in quite elementary ways (using only methods available to Euler himself), reveal that it is quite natural to regard the function $f_q(s, s-1)$ as the 'true' q-analogue of the Riemann zeta function and, for this reason, hereafter we write $f_q(s, s-1)$ as $\zeta_q(s)$:

$$\zeta_q(s) := f_q(s, s-1) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{[n]_q^s} = \frac{q^{s-1}}{[1]_q^s} + \frac{q^{2(s-1)}}{[2]_q^s} + \frac{q^{3(s-1)}}{[3]_q^s} + \frac{q^{4(s-1)}}{[4]_q^s} + \cdots .$$

Remark. (1) The proper choice of t seems to be essential for obtaining a well-behaved q-analogue of $\zeta(s)$. For example, the choice $t = s$ adopted in [**9**] requires an extra term to adjust the convergence when $q \uparrow 1$ and the point $(s, t) = (-m, -m)$ is not a point of indeterminacy for any $m \in \mathbb{Z}_{\geq 0}$. The choices $t = s-2, s-3, s-4, \ldots$ seem as good as the choice $t = s-1$ in defining $\zeta_q(s)$, as long as $s = -m$; but for such choices, extra poles appear at $s = 2, 3, 4, \ldots$. However, these poles disappear in the limit $q \uparrow 1$. For example, with $t = s-2$ the residue at the simple pole $s = 2$ is $-(1-q)^2/\log q$, which goes to 0 as $q \uparrow 1$. It is still not known how the behavior of $\zeta_q(s)$ depends on the value of t chosen in defining it.

(2) If we define the q-analogue $\zeta_q^*(s)$ of the alternating series $\zeta^*(s)$ given in Section 1 by

$$\zeta_q^*(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{q^{n(s-1)}}{[n]_q^s},$$

the identity corresponding to (1) takes the form

$$\zeta_q^*(s) = \zeta_q(s) - 2(1+q)^{-s} \zeta_{q^2}(s).$$

In contrast to the situation considered by Euler, however, introducing $\zeta_q^*(s)$ is not helpful because the relation between $\zeta_q^*(s)$ and $\zeta_q(s)$ is complicated by the presence of ζ_{q^2}.

When specialized to the case $t = s-1$, the formula in Proposition 1 becomes

$$\begin{aligned} \zeta_q(s) &= (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{s+r-1}}{1-q^{s+r-1}} \\ &= (1-q)^s \left(\frac{q^{s-1}}{1-q^{s-1}} + s \frac{q^s}{1-q^s} + \frac{s(s+1)}{2} \frac{q^{s+1}}{1-q^{s+1}} + \cdots \right). \end{aligned} \tag{5}$$

PROPOSITION 2. (1) *The function $\zeta_q(s)$ has a simple pole at points in $1+2\pi i\mathbb{Z}/\log q$ and in the set $\{a+2\pi i b/\log q | a, b \in \mathbb{Z}, a \le 0, b \ne 0\}$. In particular, $s = 1$ is a simple pole of $\zeta_q(s)$ with residue $(q-1)/\log q$.*

(2) *For $m \in \mathbb{Z},\ m \ge 0$, the limiting value $\lim_{s\to -m} \zeta_q(s) =: \zeta_q(-m)$ exists and is given explicitly by*

$$\zeta_q(-m) = (1-q)^{-m}\left\{\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{1}{q^{m+1-r}-1}+\frac{(-1)^{m+1}}{(m+1)\log q}\right\}. \tag{6}$$

Proof. Assertion (1) is straightforward from (5), with the formula $\lim_{y\to 0} y/(1-q^y) = -1/\log q$ used for the residue at $s = 1$. For (2), note that the terms for which $r \ge m+2$ in the sum in (5) vanish as $s \to -m$, because $\binom{-m+r-1}{r} = 0$ and $1-q^{-m+r-1} \ne 0$. On the other hand, for $r = m+1$, we have $\lim_{s\to -m}(s+m)/(1-q^{s+m}) = -1/\log q$, and hence

$$\lim_{s\to -m}\binom{s+m}{m+1}\frac{q^{s+m}}{1-q^{s+m}} = \frac{(-1)^m m!}{(m+1)!}\left(-\frac{1}{\log q}\right) = \frac{(-1)^{m+1}}{(m+1)\log q}.$$

The rest of the derivation of (6) is clear. □

Before presenting the general formula for $\lim_{q\uparrow 1}\zeta_q(-m)$, let us consider the first three cases.

Example 1. As stated in Proposition 2, $\zeta_q(s)$ has a simple pole at $s = 1$ with residue $(q-1)/\log q$, which converges to 1 as $q \to 1$. This agrees with the well-known fact that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

Example 2. By (6), we have

$$\zeta_q(0) = \frac{1}{q-1} - \frac{1}{\log q}.$$

Then, because

$$\frac{1}{\log q} = \frac{1}{\log(1+(q-1))} = \frac{1}{(q-1)-(q-1)^2/2+\cdots} = \frac{1}{q-1}+\frac{1}{2}+O(q-1),$$

we find

$$\lim_{q\to 1}\zeta_q(0) = -\tfrac{1}{2}.$$

This agrees with Euler's result, $\zeta(0) = -1/2$.

Example 3. Again by (6), we have

$$\begin{aligned}
\zeta_q(-1) &= (1-q)^{-1}\left(\frac{1}{q^2-1} - \frac{1}{q-1} + \frac{1}{2\log q}\right)\\
&= \frac{1}{1-q}\left(\frac{1}{q-1}\frac{1}{2+q-1} - \frac{1}{q-1} + \frac{1}{2\log q}\right)\\
&= \frac{1}{1-q}\bigg(\frac{1}{2(q-1)} - \frac{1}{4} + \frac{q-1}{8} + \cdots - \frac{1}{q-1}\\
&\quad + \frac{1}{2(q-1)} + \frac{1}{4} - \frac{q-1}{24} + \cdots\bigg)\\
&\longrightarrow -\frac{1}{12} \quad \text{as } q \to 1,
\end{aligned}$$

in accordance with $\zeta(-1) = -1/12$.

Let the Bernoulli numbers B_k be defined by the generating series

$$\frac{te^t}{e^t-1}\left(=\frac{t}{1-e^{-t}}\right) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The first values are

$$B_0 = 1, \quad B_1 = \tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0, \quad B_4 = -\tfrac{1}{30}, \quad B_5 = 0,$$
$$B_6 = \tfrac{1}{42}, \quad B_7 = 0, \ \ldots.$$

We are now in a position to state the general formula for $\lim_{q\uparrow 1} \zeta_q(-m)$.

THEOREM 1. *For each non-negative integer m, we have*

$$\lim_{q\uparrow 1} \zeta_q(-m) = -\frac{B_{m+1}}{m+1}.$$

Proof. With (6), the assertion of the theorem becomes

$$\lim_{q\to 1}(1-q)^{-m}\left\{\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{1}{q^{m+1-r}-1} + \frac{(-1)^{m+1}}{(m+1)\log q}\right\} = -\frac{B_{m+1}}{m+1}.$$

(Note here that because the sum is finite, we can replace the limit $q \uparrow 1$ by $q \to 1$.) Multiplying both sides by $(-1)^{m+1}(m+1)$ and making the replacement $r \to m+1-r$, we see that this is equivalent to

$$\lim_{q\to 1}(1-q)^{-m}\left\{(m+1)\sum_{r=1}^{m+1}(-1)^r\binom{m}{r-1}\frac{1}{q^r-1} + \frac{1}{\log q}\right\} = (-1)^m B_{m+1}.$$

Then, writing

$$\frac{1}{q^r - 1} = \frac{1}{r} \frac{r \log q}{e^{r \log q} - 1} \frac{1}{\log q}$$

and using

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} (-1)^k B_k \frac{t^k}{k!},$$

we have

$$\begin{aligned}
&(m+1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{q^r - 1} \\
&\quad = (m+1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{r} \sum_{k=0}^{\infty} (-1)^k B_k \frac{(r \log q)^k}{k!} \frac{1}{\log q} \\
&\quad = \sum_{k=0}^{\infty} \left(\sum_{r=1}^{m+1} (-1)^r \binom{m+1}{r} r^k \right) (-1)^k B_k \frac{(\log q)^{k-1}}{k!}.
\end{aligned}$$

Because the inner sum on the right-hand side of this expression can be calculated as

$$\begin{aligned}
\sum_{r=1}^{m+1} (-1)^r \binom{m+1}{r} r^k &= \left(\left(x \frac{d}{dx} \right)^k ((1-x)^{m+1} - 1) \right) \Big|_{x=1} \\
&= \begin{cases} -1 & \text{if } k = 0, \\ 0 & \text{if } 0 < k < m+1, \\ (-1)^{m+1}(m+1)! & \text{if } k = m+1, \end{cases}
\end{aligned}$$

we find

$$\begin{aligned}
&(m+1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{q^r - 1} \\
&\quad = -\frac{1}{\log q} + B_{m+1} (\log q)^m + O((\log q)^{m+1}) \quad (\text{as } q \to 1).
\end{aligned}$$

From this and the expansion $\log q = q - 1 + O((q-1)^2)$ $(q \to 1)$, we obtain the desired result. $\square$

Remark. In view of Theorem 1, it is natural to define the *q-Bernoulli numbers $B_m(q)$* by

$$B_m(q) := -m \zeta_q(1-m) \quad (m \geq 1).$$

By (6) (making the replacements $m \to m-1$ and $r \to m-r$) we obtain the closed formula

$$B_m(q) = (q-1)^{-m+1}\left\{\sum_{r=1}^{m}(-1)^r\binom{m}{r}\frac{r}{q^r-1}+\frac{1}{\log q}\right\}$$
$$= (q-1)^{-m+1}\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{r}{q^r-1} \quad (m \geq 1).$$

Here, the term with $r = 0$ is understood to be $1/\log q$ (the limiting value of the summand in the limit $r \to 0$). This suggests that we define

$$B_0(q) := \frac{q-1}{\log q}.$$

With this, the q-Bernoulli numbers $\{B_m(q)\}_{m\geq 0}$ satisfy the recursion relation

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}q^m B_m(q) = (-1)^n B_n(q) + \delta_{1n} \quad (n \geq 0),$$

where δ_{1n} is 1 if $n = 1$ and 0 otherwise and the generating function

$$F_q(t) := \sum_{m=0}^{\infty} B_m(q)\frac{t^m}{m!}$$

satisfies the relation

$$F_q(qt) = e^t F_q(t) - te^t.$$

The q-Bernoulli numbers defined here are essentially (i.e. up to the factor $(-1)^m$) the same as those introduced by Tsumura [**8**].

The following fundamental relation, in addition to being important in its own right, guarantees that our computation at negative integers above does give us the correct values that we sought to obtain on a rigorous basis.

THEOREM 2. *For any $s \in \mathbb{C}$, excluding $s = 1$, we have*

$$\lim_{q\uparrow 1}\zeta_q(s) = \zeta(s).$$

Example. We now give some numerically computed examples directly illustrating this relation. Setting $s = 1/2$ and $q = 0.999$ in (5), the sum of the first 10^5 terms gives $-1.460\,145\,273\,95\cdots$. Then, setting $q = 0.999\,99$ and taking the first 10^7 terms, we obtain $-1.460\,352\,417\cdots$, which agrees with the actual value $\zeta(1/2) = -1.460\,354\,5088\cdots$ to the fifth decimal place.

Next, consider the point $s = 1/2 + 14.1347i$ near the first non-trivial zero $(= 1/2 + 14.134\,725\,141\,734\,693\,790\,457\,251\,983\,562\cdots i)$ of $\zeta(s)$. For $q = 0.9999$, the first 10^5 terms give the absurdly large value $108\,35.552\cdots + 102\,70.785\cdots i$, while the first 10^6 terms give $-0.000\,306\,477\cdots + 0.000\,794\,677\cdots i$. (The actual value is $\zeta(1/2 + 14.1347i) = 0.000\,003\,135\,364\cdots - 0.000\,019\,693\,36\cdots i$.) Then if we set $s = 1/2 + 14.134\,725i$ and $q = 0.999\,99$, the first 2×10^6 terms give $-0.469\,0527\cdots - 0.466\,9811\cdots i$, and the first 5×10^6 terms give $-0.000\,031\,064\cdots + 0.000\,081\,2513\cdots i$. (The actual value is $\zeta(1/2 + 14.134725i) = 0.000\,000\,017\,674\cdots - 0.000\,000\,111\,02\cdots i$.)

Combining Theorems 1 and 2, we readily obtain the following.

COROLLARY. *For each non-negative integer m, we have*

$$\text{‘}1^m + 2^m + 3^m + 4^m + 5^m + \cdots\text{’} = \zeta(-m) = -\frac{B_{m+1}}{m+1}.$$

Remarks. (1) We can also define a q-analogue of the Hurwitz zeta function

$$\zeta(s; a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

by

$$\zeta_q(s; a) = \sum_{n=0}^{\infty} \frac{q^{(n+a)(s-1)}}{[n+a]_q^s}$$

and prove the identity

$$\lim_{q\uparrow 1} \zeta_q(s; a) = \zeta(s; a)$$

for any $s \neq 1$, as well as the formula

$$\lim_{q\uparrow 1} \zeta_q(-m; a) = -\frac{B_{m+1}(a)}{m+1}$$

for integers $m \leq 0$. Here, the Bernoulli polynomial $B_k(x)$ is defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}. \tag{7}$$

We can now define q-Bernoulli polynomials (and derive elementary formulas), in analogy to the q-Bernoulli numbers defined in the remark given after Theorem 1.

However, to make our presentation as concise as possible, we restrict ourselves to the case of the Riemann zeta function.

(2) It is interesting to note that the limit

$$\lim_{q\uparrow 1}(1-q)^k \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n} = (k-1)!\zeta(k) \quad (\forall k \geq 2, k \in \mathbb{Z}) \tag{8}$$

is derived easily from the relation

$$\lim_{q\uparrow 1} \zeta_q(k) = \zeta(k). \tag{9}$$

(Equation (9) directly follows from the definition without appealing to Theorem 2, because we are in the region of absolute convergence, due to the condition $k \geq 2$.) In fact, if we set $s = 2$ in (5) and make the replacement $r + 1 \to n$, we have

$$\zeta_q(2) = (1-q)^2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

which gives the desired limit for the case $k = 2$. For general values of k, we similarly set $s = k$ in (5) and make the replacement $k + r - 1 \to n$ to find

$$\zeta_q(k) = (1-q)^k \sum_{n=1}^{\infty} \binom{n}{k-1} \frac{q^n}{1-q^n}.$$

(Observe that $\binom{n}{k-1} = 0$ for $n = 1, 2, \ldots, k-2$.) We then note that

$$\binom{n}{k-1} = \frac{n^{k-1}}{(k-1)!} + [\text{lower degree terms}],$$

and, on taking the limit $q \uparrow 1$, the sums coming from the lower degree terms vanish, as can be shown inductively, hence we obtain the conclusion.

When k is even and $k \geq 4$, the series

$$\sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{(1-q^n)} = \sum_{n=1}^{\infty} \Bigl(\sum_{d|n} d^{k-1}\Bigr) q^n$$

constitutes the Fourier series for the Eisenstein series $G_k(\tau)$ of weight k on the modular group, with constant term $-B_k/2k$ $(= \zeta(1-k)/2)$. Here τ is a variable in the upper-half plane and is related to q as $q = e^{2\pi i\tau}$. The modularity condition amounts to the transformation formula $G_k(-1/\tau) = \tau^k G_k(\tau)$, which can be derived from (as shown by Hecke [**6**]) the functional equation for the corresponding Dirichlet series $\varphi(s) := \zeta(s)\zeta(s+1-k)$,

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\varphi(k-s).$$

(When k is odd, the functional equation for $\varphi(s)$ does not take this form and, therefore, in this case the series $\sum_{n=1}^{\infty} n^{k-1}q^n/(1-q^n)$ cannot be the Fourier series of a modular form.) Hecke also showed that the residue of $\varphi(s)$ at the simple pole $s = k$ is equal to $(2\pi i)^k c_0/(k-1)!$, where c_0 is the constant term of the corresponding modular form. In our case, the residue is $\zeta(k)$ and thus the constant term of $G_k(\tau)$ is $(k-1)!\zeta(k)/(2\pi i)^k = -B_k/2k$, as expected. As an alternative method to determine the constant term, we can use (8) as follows. First, set $\tau = it$ with $t > 0$. Then, $e^{2\pi i(-1/it)} \to 0$ as $t \to 0$ and hence

$$\begin{aligned} c_0 &= \lim_{t\to 0} G_k\left(-\frac{1}{it}\right) = \lim_{t\to 0}\,(it)^k G_k(it) = \lim_{q\uparrow 1} \frac{(it)^k}{(1-q)^k}(1-q)^k G_k(it) \\ &= \frac{1}{(2\pi i)^k}(k-1)!\zeta(k). \end{aligned}$$

Proof of Theorem 2. Recall the celebrated *summation formula of Euler* [**1, 2**] (obtained later by Maclaurin (cf. [**10**, Section 7.21]), which is proved by simply repeating integration by parts). For a C^∞-function $f(x)$ on $[1,\infty)$ and arbitrary integers $M \geq 0$, $N \geq 1$, we have

$$\begin{aligned} \sum_{n=1}^{N} f(n) = \int_1^N f(x)\,dx + \frac{1}{2}(f(1)+f(N)) + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!}(f^{(k)}(N) - f^{(k)}(1)) \\ - \frac{(-1)^{M+1}}{(M+1)!}\int_1^N \widetilde{B}_{M+1}(x) f^{(M+1)}(x)\,dx, \end{aligned} \tag{10}$$

where $\widetilde{B}_{M+1}(x)$ is the 'periodic Bernoulli polynomial' defined by

$$\widetilde{B}_k(x) = B_k(x-[x]) \quad ([x] \text{ being the largest integer not exceeding } x).$$

Recall that the Bernoulli polynomials $B_k(x)$ are defined by the generating series (7):

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x, \ldots .$$

As is well known, by choosing $f(x) = x^{-s}$ and taking the limit $N \to \infty$, we obtain the analytic continuation of $\zeta(s)$ to the region satisfying $\mathrm{Re}(s) > -M$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!}(s)_k - \frac{(s)_{M+1}}{(M+1)!}\int_1^\infty \widetilde{B}_{M+1}(x) x^{-s-M-1}\,dx, \tag{11}$$

where $(s)_k := s(s+1)\cdots(s+k-1)$. Since we can choose M arbitrarily large, this gives the analytic continuation of $\zeta(s)$ to the whole s-plane, showing that there is a (unique) simple pole at $s=1$ with residue 1.

Now we set $f(x) = q^{x(s-1)}/(1-q^x)^s$ and $M = 1$ in (10). Then, assuming $\mathrm{Re}(s) > 1$ and noting

$$f'(x) = \log q \cdot q^{x(s-1)} \frac{s-1+q^x}{(1-q^x)^{s+1}},$$
$$f''(x) = (\log q)^2 q^{x(s-1)} \frac{s(s+1) - 3s(1-q^x) + (1-q^x)^2}{(1-q^x)^{s+2}},$$

and in general $f^{(k)}(x) = (\log q)^k q^{x(s-1)}(1-q^x)^{-s-k} \times$ (a polynomial in s and q^x), we see that we can take the limit $N \to \infty$; doing so yields

$$\sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{(1-q^n)^s} = \int_1^{\infty} \frac{q^{x(s-1)}}{(1-q^x)^s}\,dx + \frac{1}{2}\frac{q^{s-1}}{(1-q)^s} - \frac{1}{12}(\log q) q^{s-1}\frac{s-1+q}{(1-q)^{s+1}} - \frac{(\log q)^2}{2}\int_1^{\infty} \widetilde{B}_2(x) q^{x(s-1)} \frac{s(s+1)-3s(1-q^x)+(1-q^x)^2}{(1-q^x)^{s+2}}\,dx$$

for $\mathrm{Re}(s) > 1$. The first integral on the right-hand side is evaluated as

$$\int_1^{\infty} \frac{q^{x(s-1)}}{(1-q^x)^s}\,dx = \int_1^{\infty} \frac{q^{-x}}{(q^{-x}-1)^s}\,dx = \left[\frac{(q^{-x}-1)^{1-s}}{(s-1)\log q}\right]_1^{\infty} = -\frac{q^{s-1}(1-q)^{1-s}}{(s-1)\log q}.$$

We therefore obtain

$$\begin{aligned}\zeta_q(s) &= (1-q)^s \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{(1-q^n)^s} \\ &= \frac{q^{s-1}}{s-1}\frac{q-1}{\log q} + \frac{q^{s-1}}{2} + \frac{q^{s-1}}{12}\frac{\log q}{q-1}(s-1+q) \\ &\quad - (1-q)^s \frac{(\log q)^2}{2}\int_1^{\infty} \widetilde{B}_2(x) q^{x(s-1)} \frac{s(s+1)-3s(1-q^x)+(1-q^x)^2}{(1-q^x)^{s+2}}\,dx.\end{aligned} \tag{12}$$

Unlike in the classical case represented by (11), the integral in (12) cannot be made to converge by simply choosing M sufficiently large instead of $M = 1$, because the presence of the factor $q^{x(s-1)}$ in $f^{(M+1)}(x)$ implies that necessarily $\mathrm{Re}(s) > 1$.

Therefore, in this case we use the Fourier expansion of the periodic Bernoulli polynomials† (cf. [**10**, Ch. IX, Miscellaneous Exercise 12]),

$$\widetilde{B}_k(x) = -k! \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e^{2\pi i n x}}{(2\pi i n)^k}. \tag{13}$$

This equality is valid for all real numbers x when $k \geq 2$, in which case the sum is absolutely and uniformly convergent. Substituting this (with $k = 2$) into (12) and interchanging the summation and the integration, we find

$$\begin{aligned}\zeta_q(s) = {} & \frac{q^{s-1}}{s-1}\frac{q-1}{\log q} + \frac{q^{s-1}}{2} + \frac{q^{s-1}}{12}\frac{\log q}{q-1}(s-1+q) + (1-q)^s(\log q)^2 \\ & \times \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{(2\pi i n)^2}\int_1^\infty e^{2\pi i n x} q^{x(s-1)} \frac{s(s+1) - 3s(1-q^x) + (1-q^x)^2}{(1-q^x)^{s+2}}\, dx.\end{aligned}$$

Further, we make the change of variable $q^x = u$ in the integral to obtain

$$\begin{aligned}\zeta_q(s) = {} & \frac{q^{s-1}}{s-1}\frac{q-1}{\log q} + \frac{q^{s-1}}{2} + \frac{q^{s-1}}{12}\frac{\log q}{q-1}(s-1+q) \\ & - (1-q)^s \log q \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{(2\pi i n)^2}\{s(s+1)b_q(s-1+\delta n, -s-1) \\ & - 3sb_q(s-1+\delta n, -s) + b_q(s-1+\delta n, -s+1)\},\end{aligned} \tag{14}$$

where $\delta = 2\pi i/\log q$ and

$$b_t(\alpha, \beta) = \int_0^t u^{\alpha-1}(1-u)^{\beta-1}\, du,$$

which is referred to as the incomplete beta function‡. Note that each of the incomplete beta integrals in (14) converges absolutely for $\mathrm{Re}(s) > 1$ and is uniformly bounded with respect to n:

$$|b_q(s-1+\delta n, -s+\nu)| \leq \int_0^q u^{\sigma-2}(1-u)^{-\sigma+\nu-1}\, du \quad (\forall n, \sigma = \mathrm{Re}(s), \nu = -1, 0, 1).$$

Hence the sum converges absolutely.

†The idea of replacing $\widetilde{B}_2(x)$ in the integral by its Fourier expansion is due to Ueno and Nishizawa [**9**]. However, the argument used here that follows this replacement, which only uses integration by parts (and no confluent hypergeometric functions or the like), seems to be quite different from that in [**9**].

‡We remind the reader that the beta integral is often called the *Euler integral*.

Now, repeated use of integration by parts yields the formula

$$\begin{aligned}
b_t(\alpha,\beta) &= \int_0^t \left(\frac{u^\alpha}{\alpha}\right)'(1-u)^{\beta-1}\,du \\
&= \frac{1}{\alpha}t^\alpha(1-t)^{\beta-1} - \frac{1-\beta}{\alpha}\int_0^t u^\alpha(1-u)^{\beta-2}\,du \\
&= \frac{1}{\alpha}t^\alpha(1-t)^{\beta-1} - \frac{1-\beta}{\alpha}\int_0^t \left(\frac{u^{\alpha+1}}{\alpha+1}\right)'(1-u)^{\beta-2}\,du \\
&\vdots \\
&= \sum_{k=1}^{M-1}(-1)^{k-1}\frac{(1-\beta)_{k-1}}{(\alpha)_k}t^{\alpha+k-1}(1-t)^{\beta-k} \\
&\quad + (-1)^{M-1}\frac{(1-\beta)_{M-1}}{(\alpha)_{M-1}}\beta_t(\alpha+M-1,\beta-M+1)
\end{aligned}$$

for any $M \geq 2$. Applying this to $b_q(s-1+\delta n, -s-1)$, we have (note that $q^{\delta n} = 1$)

$$\begin{aligned}
b_q(s-1+\delta n, -s-1) = \sum_{k=1}^{M-1}(-1)^{k-1}\frac{(s+2)_{k-1}}{(s-1+\delta n)_k}q^{s+k-2}(1-q)^{-s-1-k} \\
+ (-1)^{M-1}\frac{(s+2)_{M-1}}{(s-1+\delta n)_{M-1}}b_q(s-2+M+\delta n, -s-M).
\end{aligned}$$

This allows us to carry out the analytic continuation of $b_q(s-1+\delta n, -s-1)$ as a function of s into the region $\mathrm{Re}(s) > 2-M$. From this we have

$$\begin{aligned}
&\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{s(s+1)}{(2\pi in)^2}b_q(s-1+\delta n, -s-1) \\
&\quad = \sum_{k=1}^{M-1}(-1)^{k-1}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{(2\pi in)^2}\frac{(s)_{k+1}}{(s-1+\delta n)_k}q^{s+k-2}(1-q)^{-s-1-k} \\
&\quad + (-1)^{M-1}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{(2\pi in)^2}\frac{(s)_{M+1}}{(s-1+\delta n)_{M-1}} \\
&\quad \times \int_0^q u^{s-3+M+\delta n}(1-u)^{-s-M-1}\,du
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{M-1} (-1)^{k-1} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{(2\pi i n)^2} \frac{(s)_{k+1}}{(s-1+\delta n)_k} q^{s+k-2}(1-q)^{-s-1-k} \\
&\quad - (-1)^{M-1} \log q \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{(2\pi i n)^2} \frac{(s)_{M+1}}{(s-1+\delta n)_{M-1}} \\
&\quad \times \int_1^\infty e^{2\pi i n x} q^{x(s-2+M)}(1-q^x)^{-s-M-1}\, dx.
\end{aligned}
$$

Then, using

$$
\lim_{q\to 1} \frac{\log q}{1-q} = -1, \quad \lim_{q\to 1} (1-q)^k (s-1+\delta n)_k = (-2\pi i)^k, \quad \lim_{q\to 1} \frac{1-q^x}{1-q} = x,
$$

we obtain, for $\mathrm{Re}(s) > 2 - M$,

$$
\begin{aligned}
&\lim_{q\uparrow 1} (1-q)^s \log q \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{s(s+1)}{(2\pi i n)^2} b_q(s-1+\delta n, -s-1) \\
&= \sum_{k=1}^{M-1} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(s)_{k+1}}{(2\pi i n)^{k+2}} - \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(s)_{M+1}}{(2\pi i n)^{M+1}} \int_1^\infty e^{2\pi i n x} x^{-s-M-1}\, dx \\
&= -\sum_{k=1}^{M-1} \frac{B_{k+2}}{(k+2)!} (s)_{k+1} + \frac{(s)_{M+1}}{(M+1)!} \int_1^\infty \widetilde{B}_{M+1}(x) x^{-s-M-1}\, dx.
\end{aligned}
$$

In the last equality, we have used (13) and its form in the case $x = 1$,

$$
\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{(2\pi i n)^k} = -\frac{B_k}{k!},
$$

valid for all $k \geq 2$. We carry out the same procedure for the terms containing $b_q(s-1+\delta n, -s)$ and $b_q(s-1+\delta n, -s+1)$. As it turns out, however, the contributions from these two vanish when we take $q \uparrow 1$, because the powers of $1-q$ that they contain are lower than those from $b_q(s-1+\delta n, -s-1)$. We therefore obtain, for

$\mathrm{Re}(s) > 2 - M$,

$$\begin{aligned}\lim_{q\uparrow 1}\zeta_q(s) &= \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} + \sum_{k=2}^{M}\frac{B_{k+1}}{(k+1)!}(s)_k - \frac{(s)_{M+1}}{(M+1)!}\\ &\quad\times\int_1^\infty \widetilde{B}_{M+1}(x)x^{-s-M-1}\,dx\\ &= \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{M}\frac{B_{k+1}}{(k+1)!}(s)_k - \frac{(s)_{M+1}}{(M+1)!}\int_1^\infty \widetilde{B}_{M+1}(x)x^{-s-M-1}\,dx.\end{aligned}$$

This coincides with formula (11) for $\zeta(s)$, which is valid for $\mathrm{Re}(s) > -M$ and thus the theorem is established, as the integer M can be chosen arbitrarily large. □

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