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On Poly-Bernoulli Numbers

by

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1. Main theorems

In our previous paper [1], we defined and studied "poly-Bernoulli numbers" which generalize the classical Bernoulli numbers. As a continuation, we present here two results, one of which is a further investigation of Clausen-von Staudt type theorem that was treated only in "di-Bernoulli" case in [1], the other being a combinatorial closed formula for negative index poly-Bernoulli numbers.

Poly-Bernoulli numbers $B_n^{(k)}$ $(n=0, 1, 2, \cdots)$ are defined for each integer k by the generating series

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where $Li_k(z)=\sum_{m=1}^{\infty} \frac{z^m}{m^k}$. Table of values of $B_n^{(k)}$ for small k and n will be given at the end of the paper. In [1], we obtained an explicit formula for $B_n^{(k)}$:

$$B_n^{(k)}=(-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}, \tag{1}$$

where $\left\{ {n \atop m} \right\}$ is an integer referred to as the Stirling number of the second kind ("Stirling subset number" in Knuth's terminology, we adopt his notation [2]).

Let p be a prime number. First of all, it is clear from the above formula (1) that the $B_n^{(k)}$ is p-integral when p is larger than $n+1$. Our first theorem gives an information on the p-part of $B_n^{(k)}$ for $p \le n+1$.

THEOREM 1 (Clausen-von Staudt type theorem). *Assume $k \ge 2$. Let p be a prime number satisfying $k+2 \le p \le n+1$.*

(i) *If $n \equiv 0 \bmod (p-1)$, then $p^k B_n^{(k)}$ is a p-adic integer and satisfies*

$$p^k B_n^{(k)} \equiv -1 \mod p\mathbb{Z}_p .$$

(ii) *If $n \not\equiv 0 \bmod (p-1)$, then $p^{k-1} B_n^{(k)}$ is a p-adic integer. It satisfies*

$$p^{k-1}B_n^{(k)} \equiv \begin{cases} \dfrac{1}{p}\left\{ {n \atop p-1} \right\} - \dfrac{n}{2^k} \mod p\mathbb{Z}_p\,, & \textit{if } n \equiv 1 \mod (p-1) \\ \dfrac{(-1)^{n-1}}{p}\left\{ {n \atop p-1} \right\} \mod p\mathbb{Z}_p & \textit{otherwise}\,. \end{cases}$$

REMARK 1. That the $p^{k-1}B_n^{(k)}(p-1\nmid n)$ and $p^kB_n^{(k)}(p-1\mid n)$ are p-integral $(k+2\le p)$ has also been obtained independently by Roberto Sánchez-Peregrino in [3].

2. If $n \not\equiv 0,\ 1 \mod (p-1)$, the congruence in (ii) may be written as

$$p^{k-1}B_n^{(k)} \equiv (n-n')\frac{B_n^{(1)}}{n} \mod p\mathbb{Z}_p\,,$$

where n' is a unique integer with $n' \equiv n \mod p-1$ and $1<n'<p$. Actually, it was shown in [1] the congruence

$$\frac{(-1)^{n-1}}{p}\left\{ {n \atop p-1} \right\} \equiv (n-n')\frac{B_n^{(1)}}{n} \mod p\mathbb{Z}_p$$

if $n \not\equiv 0,\ 1 \mod (p-1)$ (the assumption made there that n being even can be loosened to the present one).

3. When $p>n+1$, the formula (1) shows that the congruence $B_n^{(k)} \equiv B_n^{(k')} \mod p$ holds for any integers k and k' satisfying $k \equiv k' \mod (p-1)$.

The number $B_n^{(k)}$ is a positive integer when k is non-positive. Our second theorem is a closed formula (which is completely different from (1)) for this integer.

THEOREM 2 (Closed formula). *For any* $n, k \ge 0$, *we have*

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}.$$

REMARK. This formula gives another proof of the symmetry $B_n^{(-k)} = B_k^{(-n)}$ mentioned in [1].

The proofs of Theorems 1 and 2 will be given in §2 and §3 respectively.

2. Proof of Clausen-von Staudt type theorem

Let $k\ge 2$ and p be a prime number satisfying $k+2\le p\le n+1$. To prove Theorem 1, we estimate the p-order of each summand $\dfrac{(-1)^m m!\left\{ {n \atop m} \right\}}{(m+1)^k}$ in (1), which we denote hereafter by $b_n^{(k)}(m)$. We prove (i) and (ii) simultaneously. The p-order of an integer a is denoted by $ord_p(a)$ with the convention $ord_p(p^t)=t$. Write $m+1=ap^e$, $(a,p)=1$, $e\ge 0$. If $e=0$, then $b_n^{(k)}(m)$ is p-integral. We can ignore this term, because by the assumption $k\ge 2$ we have $p^{k-1}b_n^{(k)}(m)\equiv 0 \mod p\mathbb{Z}_p$. Since $\left\{ {n \atop m} \right\}$ is an integer, we have

$$ord_p(b_n^{(k)}(m)) \geq ord_p\left(\frac{m!}{(m+1)^k}\right).$$

First, assume $e \geq 2$. We show that $p^k b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$ and moreover $p^{k-1} b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$ if $n \not\equiv 0 \bmod (p-1)$. Using $ord_p(m!) = \sum_{j=1}^{\infty} \left[\frac{m}{p^j}\right]$, we have

$$\begin{aligned}
ord_p\left(\frac{m!}{(m+1)^k}\right) &= \sum_{j=1}^{\infty} \left[\frac{m}{p^j}\right] - ek \\
&\geq \left[\frac{m}{p}\right] - ek = \left[\frac{ap^e - 1}{p}\right] - ek = ap^{e-1} - 1 - ek \\
&\geq p^e - 1 - ek = (1 + p - 1)^{e-1} - 1 - ek \\
&\geq 1 + (e-1)(p-1) - 1 - ek = (e-1)(p-1) - ek \\
&\geq (e-1)(k+1) - ek = -k + e - 1 \\
&\geq -k + 1 .
\end{aligned}$$

Thus we get $ord_p\left(\frac{m!}{(m+1)^k}\right) \geq -k+1$ and so $p^{k-1} b_n^{(k)}(m)$ is p-integral. Hence $p^k b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$. If any one of the above inequalities is strict (i.e. '$>$'), then we get $p^{k-1} b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$. Then only case when the equalities hold everywhere is when $e=2$, $m+1=p^2$, and $p=k+2$. In this case, the following lemma ($a=p$) implies $p^{k-1} b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$ if $n \not\equiv 0 \bmod (p-1)$.

LEMMA. *Let n and a be natural numbers. We have the congrunce*

$$\left\{ {n \atop ap-1} \right\} \equiv \begin{cases} \dbinom{c-1}{a-1} \bmod p & \text{if } n = a-1+c(p-1) \text{ for some } c \geq a \\ 0 \bmod p & \text{otherwise} . \end{cases}$$

Proof. Use the following formula for a generating function of $\left\{ {n \atop m} \right\}$ ([4, (7.47)]):

$$\sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} x^n = \frac{x^m}{(1-x)(1-2x)\cdots(1-mx)} . \tag{2}$$

If $m = ap-1$, the right-hand side of this formula is congruent modulo p to

$$\frac{x^{ap-1}}{(1-x^{p-1})^a} = x^{ap-1} \sum_{i=0}^{\infty} \binom{a+i-1}{i} x^{i(p-1)} = \sum_{i=0}^{\infty} \binom{a+i-1}{a-1} x^{a-1+(a+i)(p-1)}$$

(we have used $(1-x)(1-2x)\cdots(1-(p-1)x) \equiv 1 - x^{p-1} \bmod p$). Putting $a+i=c$, we obtain the lemma.

Now suppose $e=1$ ($m = ap-1$). If $a \geq 3$, then $p^2 \mid (ap-1)!$. Hence $ord_p\left(\frac{m!}{(m+1)^k}\right) >$

$-k+1$, from which follows $p^{k-1}b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$. If $a=2$, then $ord_p(m!)=1$ and $ord_p\left(\frac{m!}{(m+1)^k}\right)=1-k$. Hence $ord_p(b_n^{(k)}(m))=1-k+\mathrm{ord}_p\left(\left\{ {n \atop m} \right\}\right)$ and so $p^k b_n^{(k)}(m) \equiv 0$ $\bmod p\mathbb{Z}_p$. If $n \not\equiv 1 \bmod (p-1)$, we see from the above lemma ($a=2$) that $\left\{ {n \atop 2p-1} \right\} \equiv 0$ $\bmod p$. From this we have $p^{k-1}b_n^{(k)}(m) \equiv 0 \bmod p\mathbb{Z}_p$ if $n \not\equiv 1 \bmod (p-1)$. If $n \equiv 1$ $\bmod (p-1)$ and $n=1+c(p-1)$ with $c \geq 2$ ($c=1$ cannot occur because $n \geq m$), then we see by the lemma that $\left\{ {n \atop 2p-1} \right\} \equiv c-1 \equiv -n \bmod p$. From this, we obtain

$$p^{k-1}b_n^{(k)}(m)=p^{k-1}\frac{(-1)^{2p-1}(2p-1)!\left\{ {n \atop 2p-1} \right\}}{(2p)^k}$$

$$\equiv \frac{n}{2^k} \quad \bmod p\mathbb{Z}_p .$$

Finally, the case $a=1$ ($m=p-1$) gives us $b_n^{(k)}(m)=\frac{(p-1)!\left\{ {n \atop p-1} \right\}}{p^k}$. From the lemma, we have $\left\{ {n \atop p-1} \right\} \equiv 0 \bmod p$ if $n \not\equiv 0 \bmod (p-1)$ and thus $p^{k-1}b_n^{(k)}(m) \equiv -\frac{1}{p}\left\{ {n \atop p-1} \right\} \bmod p\mathbb{Z}_p$. If $n \equiv 0 \bmod (p-1)$, then $\left\{ {n \atop p-1} \right\} \equiv 1 \bmod p$ and $p^k b_n^{(k)}(m) \equiv -1 \bmod p\mathbb{Z}_p$. Summing up, and noting the factor $(-1)^n$ before the summation in (1) (also note n is odd if $n \equiv 1$ $\bmod (p-1)$), we obtain the theorem.

3. Proof of the closed formula for negative index poly-Bernoulli numbers

In this section we prove Theorem 2. In the course of our proof, we obtain

PROPOSITION. *For all* $n>0$,

$$\sum_{l=0}^{n}(-1)^l B_{n-l}^{(-l)}=0 .$$

EXAMPLE. $B_2^{(0)}-B_1^{(-1)}+B_0^{(-2)}=1-2+1=0$, $B_4^{(0)}-B_3^{(-2)}+B_2^{(-2)}-B_1^{(-3)}+B_0^{(-4)}=1-8+14-8+1=0$, etc.

This is trivial when n is odd because of the symmetry mentioned in the remark after the theorem.

In order to prove the theorem, we calculate the generating function $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B_n^{(-k)}x^n y^k$ of $B_n^{(-k)}$ in the following form:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k = \sum_{j=0}^{\infty} p_j(x) p_j(y), \tag{3}$$

where

$$p_j(x) = \frac{j!x^j}{(1-x)(1-2x)\cdots(1-(j+1)x)}.$$

Once we establish this, the theorem follows by equating the coefficients of both sides, because we have by the formula (2) in §2

$$p_j(x) = j! \sum_{n=j}^{\infty} \left\{ {n+1 \atop j+1} \right\} x^n. \tag{4}$$

Put the left-hand side of (3) $= B(x, y)$. Using (1) we have

$$\begin{aligned} B(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((-1)^n \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} (m+1)^k\right) x^n y^k \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} x^n \sum_{k=0}^{\infty} (m+1)^k y^k \\ &= \sum_{m=0}^{\infty} (-1)^m m! \sum_{n=m}^{\infty} (-1)^n \left\{ {n \atop m} \right\} x^n \frac{1}{1-(m+1)y}. \end{aligned} \tag{5}$$

Here we use (2) to get

$$B(x,y) = \sum_{m=0}^{\infty} \frac{m!x^m}{(1+x)(1+2x)\cdots(1+mx)(1-(m+1)y)}.$$

The proposition follows from this. Namely, putting $y = -x$ gives

$$\begin{aligned} B(x,-x) &= \sum_{m=0}^{\infty} \frac{m!x^m}{(1+x)(1+2x)\cdots(1+mx)(1+(m+1)x)} \\ &= \sum_{m=1}^{\infty} \frac{(m-1)!x^{m-1}}{(1+x)\cdots(1+mx)} \\ &= \sum_{m=1}^{\infty} (-1)^m (m-1)! \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} (-1)^n x^{n-1} \qquad \text{(by (2))} \\ &= \sum_{n=1}^{\infty} (-1)^n \left(\sum_{m=1}^{n} (-1)^m (m-1)! \left\{ {n \atop m} \right\} \right) x^{n-1} \\ &= 1 \qquad \text{(by [4, (6.16)])}, \end{aligned}$$

while by definition

$$B(x,-x)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k B_n^{(-k)}x^{n+k}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^l B_{n-l}^{(-l)}\right)x^n \qquad (n+k\to n,\ k\to l)\,,$$

and hence the proposition.

Let us return to the proof of (3). We need the following lemma.

LEMMA. (i) $\dfrac{1}{1-(m+1)y}=\sum_{j=0}^{m}\binom{m}{j}p_j(y).$

(ii) $\sum_{m=j}^{n}(-1)^m m!\binom{m}{j}\left\{{n \atop m}\right\}=(-1)^n j!\left\{{n+1 \atop j+1}\right\} \qquad (n\geq j\geq 0).$

Proof will be given later. From (5),

$$B(x,y)=\sum_{m=0}^{\infty}(-1)^m m!\sum_{n=m}^{\infty}(-1)^n\left\{{n \atop m}\right\}x^n\frac{1}{1-(m+1)y}$$

$$=\sum_{m=0}^{\infty}\left((-1)^m m!\sum_{n=m}^{\infty}(-1)^n\left\{{n \atop m}\right\}x^n\right)\sum_{j=0}^{m}\binom{m}{j}p_j(y) \qquad \text{(by Lemma (i))}$$

$$=\sum_{j=0}^{\infty}p_j(y)\left(\sum_{m=j}^{\infty}(-1)^m m!\binom{m}{j}\sum_{n=m}^{\infty}(-1)^n\left\{{n \atop m}\right\}x^n\right)$$

$$=\sum_{j=0}^{\infty}p_j(y)\sum_{n=j}^{\infty}(-1)^n x^n\left(\sum_{m=j}^{n}(-1)^m m!\binom{m}{j}\left\{{n \atop m}\right\}\right)$$

$$=\sum_{j=0}^{\infty}p_j(y)\sum_{n=j}^{\infty}j!\left\{{n+1 \atop j+1}\right\}x^n \qquad \text{(by Lemma (ii))}$$

$$=\sum_{j=0}^{\infty}p_j(x)p_j(y) \qquad \text{(by (4))}\,.$$

This is (3) and thus the theorem is proved

Proof of Lemma. (i) The following partial fraction expansion is easily established by residue calculation:

$$\frac{1}{z(z-1)\cdots(z-m)}=\frac{(-1)^m}{m!}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{z-l}\,.$$

From this we have

$$yp_j(y)=\frac{j!y^{j+1}}{(1-y)(1-2y)\cdots(1-(j+1)y)}=\frac{j!}{\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\cdots\left(\frac{1}{y}-(j+1)\right)}$$

$$=\frac{j!(-1)^{j+1}}{y(j+1)!}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{\frac{1}{y}-l}=\frac{(-1)^{j+1}}{j+1}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{1-ly},$$

and therefore,

$$y\sum_{j=0}^{m}\binom{m}{j}p_j(y)=\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{1-ly}$$
$$=\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1}+\sum_{l=1}^{m+1}\frac{(-1)^l}{1-ly}\sum_{j=l-1}^{m}\frac{(-1)^{j+1}}{j+1}\binom{m}{j}\binom{j+1}{l}.$$

Now, since

$$\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1}=-\frac{1}{m+1}$$

$\left(\text{take}\int_0^1 dx \text{ of } (1-x)^m=\sum_{j=0}^m(-1)^j\binom{m}{j}x^j\right)$, and

$$\sum_{j=l-1}^{m}\frac{(-1)^{j+1}}{j+1}\binom{m}{j}\binom{j+1}{l}=\frac{1}{l}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m}{j}\binom{j}{l-1}$$
$$=\frac{1}{l}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m}{l-1}\binom{m-l+1}{j-l+1}$$
$$=\frac{1}{l}\binom{m}{l-1}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m-l+1}{j-l+1}$$
$$=\begin{cases}\dfrac{(-1)^{m+1}}{m+1} & l=m+1\\ 0 & l<m+1\end{cases}$$

$\left(\text{the second equality is by }\binom{p}{q}\binom{q}{r}=\binom{p}{r}\binom{p-r}{q-r}\right)$, we have

$$y\sum_{j=0}^{m}\binom{m}{j}p_j(y)=-\frac{1}{m+1}+\frac{(-1)^{m+1}}{1-(m+1)y}\cdot\frac{(-1)^{m+1}}{m+1}$$
$$=\frac{y}{1-(m+1)y}.$$

This gives Lemma (i).

(ii) First we show

$$\sum_{n=j}^{\infty}(-1)^n j!\left\{{n+1 \atop j+1}\right\}\frac{t^n}{n!}=(e^{-t}-1)^j\cdot e^{-t}.$$

For this, we start with

$$\frac{(e^t-1)^j}{j!}=\sum_{n=j}^{\infty}\left\{{n \atop j}\right\}\frac{t^n}{n!} \tag{6}$$

(this is [4, (7.49)]). Replacing j by $j+1$,

$$\begin{aligned}\frac{(e^t-1)^{j+1}}{(j+1)!}&=\sum_{n=j+1}^{\infty}\left\{{n \atop j+1}\right\}\frac{t^n}{n!}\\&=\sum_{n=j}^{\infty}\left\{{n+1 \atop j+1}\right\}\frac{t^{n+1}}{(n+1)!} \qquad (n\to n+1).\end{aligned}$$

Differentiation by t gives

$$\frac{(e^t-1)^j\cdot e^t}{j!}=\sum_{n=j}^{\infty}\left\{{n+1 \atop j+1}\right\}\frac{t^n}{n!}.$$

From this we have

$$(e^{-t}-1)^j\cdot e^{-t}=\sum_{n=j}^{\infty}(-1)^n j!\left\{{n+1 \atop j+1}\right\}\frac{t^n}{n!}.$$

The proof will be finished if we show

$$\sum_{n=j}^{\infty}\left(\sum_{m=j}^{n}(-1)^m m!\binom{m}{j}\left\{{n \atop m}\right\}\right)\frac{t^n}{n!}=(e^{-t}-1)^j\cdot e^{-t}.$$

The left-hand side is equal to

$$\begin{aligned}&\sum_{m=j}^{\infty}(-1)^m m!\binom{m}{j}\sum_{n=m}^{\infty}\left\{{n \atop m}\right\}\frac{t^n}{n!}\\&=\sum_{m=j}^{\infty}(-1)^m m!\binom{m}{j}\frac{(e^t-1)^m}{m!} \qquad \text{(by (6))}\\&=\sum_{m=j}^{\infty}(-1)^m\binom{m}{j}(e^t-1)^m.\end{aligned}$$

Since $\sum_{m=j}^{\infty}\binom{m}{j}X^m=\frac{X^j}{(1-X)^{j+1}}$ $\left(\text{replace } m \text{ by } m-j \text{ in } \sum_{m=0}^{\infty}\binom{m+j}{j}X^m=(1-X)^{-(j+1)}\right)$ we obtain

$$\sum_{m=j}^{\infty}(-1)^m\binom{m}{j}(e^t-1)^m=\sum_{m=j}^{\infty}\binom{m}{j}(1-e^t)^m=\frac{(1-e^t)^j}{(1-(1-e^t))^{j+1}}=(e^{-t}-1)^j\cdot e^{-t}.$$

This completes the proof of the lemma and Theorem 2 is thus established.

Table 1. $B_n^{(k)}$ $(-5 \leq k \leq 5,\ 0 \leq n \leq 7)$.

k \ n	0	1	2	3	4	5	6	7
−5	1	32	454	4718	41506	329462	2441314	17234438
−4	1	16	146	1066	6902	41506	237686	1315666
−3	1	8	46	230	1066	4718	20266	85310
−2	1	4	14	46	146	454	1394	4246
−1	1	2	4	8	16	32	64	128
0	1	1	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$-\frac{38}{2205}$	$-\frac{5}{168}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$	$-\frac{75613}{3704400}$	$\frac{599}{35280}$
4	1	$\frac{1}{16}$	$-\frac{49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$-\frac{1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$
5	1	$\frac{1}{32}$	$-\frac{179}{7776}$	$\frac{515}{41472}$	$-\frac{216383}{194400000}$	$-\frac{183781}{25920000}$	$\frac{4644828197}{653456160000}$	$\frac{153375307}{49787136000}$

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