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Traces of Singular Moduli and the Fourier Coefficients of the Elliptic Modular Function $j(\tau)$

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ABSTRACT. We give a closed formula for the Fourier coefficients of the elliptic modular function $j(\tau)$ expressed in terms of singular moduli, i.e. the values at imaginary quadratic arguments. The formula is a consequence of a theorem of Zagier.

The aim of the present article is to give a brief overview of two results, one by Don Zagier on traces of singular moduli and the other, which rests upon the former, by the author on the Fourier coefficients of the elliptic modular invariant. For the full details, connection to other works, and generalizations, we refer to [**Z**] (in preparation) and [**K**].

Let $j(\tau)$ be the classical elliptic modular invariant, which is a holomorphic function in the upper half-plane $\mathfrak{H}$, is invariant under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and has a simple pole at infinity. Zagier defines for each natural number $d > 0$, $d \equiv 0, 3 \pmod 4$, a quantity $\mathbf{t}(d)$ by

$$\mathbf{t}(d) = \sum_{\mathcal{O} \supseteq \mathcal{O}_d} \frac{2}{w_{\mathcal{O}}} \sum_{[\mathfrak{a}_{\mathcal{O}}]} (j(\mathfrak{a}_{\mathcal{O}}) - 744),$$

where the first sum runs over all imaginary quadratic orders $\mathcal{O}$ that contain the order $\mathcal{O}_d$ of discriminant $-d$, $w_{\mathcal{O}}$ is the number of units in $\mathcal{O}$, and the second sum is over a representative of the proper $\mathcal{O}$-ideal class. Note that here $j(\tau)$ is viewed in the standard manner as a function on the equivalence classes of lattices in $\mathbb{C}$. When $-d$ is a fundamental discriminant which is different from -3 and -4, the $\mathbf{t}(d)$ is the absolute trace of an algebraic integer $j(\mathcal{O}_d) - 744$, from which follows that the $\mathbf{t}(d)$ is a rational integer in that case; this turns out to be true also for general d. One might recall that the Hurwitz-Kronecker class number $H(d)$ (which, on the other hand, is not necessarily an integer) is defined by the similar sum, replacing $j(\mathfrak{a}_{\mathcal{O}}) - 744$ by 1. Values of $\mathbf{t}(d)$ and $H(d)$ up to $d = 48$ are given in the table at

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the end of the paper. In addition, we set

$$\mathbf{t}(0)=2,\ \mathbf{t}(-1)=-1 \text{ and } \mathbf{t}(d)=0 \text{ for } d<-1 \text{ or } d\equiv 1,2 \pmod 4.$$

Zagier [**Z**] established the following:

THEOREM A. *The series*

$$g(\tau)=\sum_{\substack{d\geq -1\\ d\equiv 0,3\ (4)}} \mathbf{t}(d)q^d \quad (q=e^{2\pi i\tau})$$

is a modular form of weight $\frac{3}{2}$ *on* $\Gamma_0(4)=\left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z}),\ 4|c\right\}$, *holomorphic in* $\mathfrak{H}$ *and meromorphic at cusps. Specifically,*

$$g(\tau)=-\frac{E_4(4\tau)\theta_1(\tau)}{\eta(4\tau)^6}, \tag{1}$$

where $E_4(\tau)=1+240\sum_{n=1}^{\infty}\left(\sum_{d|n}d^3\right)q^n$ *is the normalized Eisenstein series of weight 4,* $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ *is the Dedekind eta function, and* $\theta_1(\tau)=\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$ *is one of the standard theta series of Jacobi.*

He proves this theorem by calculating in two ways the Fourier expansion of the logarithmic derivative of the diagonal restriction of the classical modular polynomial, and also by a similar calculation of a suitable generalization of this logarithmic derivative. In fact, what he actually proves is the following formulas for $\mathbf{t}(d)$ which uniquely determine all the $\mathbf{t}(d)$ and provide the equality (1):

$$\sum_{r\in\mathbb{Z}}\mathbf{t}(4n-r^2)=0,\quad \sum_{r\in\mathbb{Z}}r^2\mathbf{t}(4n-r^2)=-2a_n \qquad (n\geq 0), \tag{2}$$

where $a_0=1$, $a_n=240\sum_{d|n}d^3$ $(n\geq 1)$ (the Fourier coefficients of $E_4(\tau)$). Note in particular that the quantity $\mathbf{t}(d)$ can be calculated by (2) recursively and in an elementary way, without knowing anything about complex multiplication. (The formula (2) also displays that the $\mathbf{t}(d)$ is an integer.) We also mention the similar formula classically known for $H(d)$ (cf. Hurwitz [**H**], Dickson [**D** , Chapter VI], Eichler [**E**]):

$$\sum_{r\in\mathbb{Z}}H(4n-r^2)=\sum_{d|n}\max\left(d,\frac{n}{d}\right),\quad \sum_{r\in\mathbb{Z}}(n-r^2)H(4n-r^2)=\sum_{d|n}\min\left(d,\frac{n}{d}\right)^3 \qquad (n\geq 1).$$

Zagier's proof, incidentally, gives at the same time these formulas.

Now consider the modular form

$$g(\tau)\theta_0(\tau)-\frac{1}{4}((g\theta_1)\mid U_4)\left(\tau+\frac{1}{2}\right)+\frac{1}{4}((g\theta_1)\mid U_4^2)(\tau),$$

where $\theta_0(\tau)=\sum_{n\in\mathbb{Z}}q^{n^2}$, and U_4 is the operator $\sum b_n q^n\mapsto\sum b_{4n}q^n$, which, as well as the translation $\tau\mapsto\tau+\frac{1}{2}$, sends a modular form to a modular form of the same weight (but possibly on a different group). This form is of weight 2, and, as can be detected from the calculation of the first several Fourier coefficients (and this is sufficient for the rigorous proof), is identical to $(1/2\pi i)dj(\tau)/dt$. Hence, equating coefficients of both sides, we obtain the following:

THEOREM B. *For any $n \geq 1$,*

$$c_n = \frac{1}{n}\sum_{r\in\mathbb{Z}}\left\{\mathbf{t}(n-r^2) - \frac{(-1)^{n+r}}{4}\mathbf{t}(4n-r^2) + \frac{(-1)^r}{4}\mathbf{t}(16n-r^2)\right\},$$

where c_n is the n-th Fourier coefficient of $j(\tau)$,

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n.$$

Note that the formula is a finite sum.

Furthermore, using the aforementioned recurrence relation (2) for $\mathbf{t}(d)$, we can reduce the number of the terms in the formula and thus obtain another form of Theorem B:

THEOREM B$'$. *For any $n \geq 1$,*

$$c_n = \frac{1}{n}\left\{\sum_{r\in\mathbb{Z}}\mathbf{t}(n-r^2) + \sum_{r\geq 1,\text{odd}}\left((-1)^n\mathbf{t}(4n-r^2) - \mathbf{t}(16n-r^2)\right)\right\}.$$

These are the formulas for the Fourier coefficients of the elliptic modular function $j(\tau)$ expressed in terms of singular moduli. Goro Shimura established in his series of works the general principle that, the "arithmeticity" of modular forms (in far general setting) defined by the algebraicity of Fourier coefficients, and the one deifned by the algebraicity of values at CM (complex multiplication) points, are equivalent. The classical proof of the algebraicity of singular moduli using the diagonal restriction of the modular equation gives a concrete example of one direction of this equivalence, and our formula is, at least, regarded as giving explicitly the converse direction.

Examples and Table

$$\begin{aligned}
c_1 &= 2\mathbf{t}(0) - \mathbf{t}(3) - \mathbf{t}(15) - \mathbf{t}(7)\\
&= 2\times 2 - (-248) - (-192513) - (-4119)\\
&= 196884.\\
c_2 &= \frac{1}{2}\big(\mathbf{t}(7) + \mathbf{t}(-1) - \mathbf{t}(31) - \mathbf{t}(23) - \mathbf{t}(7)\big)\\
&= \big(\mathbf{t}(-1) - \mathbf{t}(31) - \mathbf{t}(23)\big)/2\\
&= \big(-1 - (-39493539) - (-3493982)\big)/2\\
&= 21493760\\
c_3 &= \frac{1}{3}\big(\mathbf{t}(3) + 2\mathbf{t}(-1) - \mathbf{t}(11) - \mathbf{t}(3) - \mathbf{t}(47) - \mathbf{t}(39) - \mathbf{t}(23) - \mathbf{t}(-1)\big)\\
&= \big(\mathbf{t}(-1) - \mathbf{t}(11) - \mathbf{t}(47) - \mathbf{t}(39) - \mathbf{t}(23)\big)/3\\
&= \big(-1 - (-33512) - (-2257837845) - (-331534572) - (-3493982)\big)/3\\
&= 864299970.
\end{aligned}$$

d	$H(d)$	$\mathbf{t}(d)$	d	$H(d)$	$\mathbf{t}(d)$
-1		-1	24	2	4833456
0	$-1/12$	2	27	$4/3$	-12288992
3	$1/3$	-248	28	2	16576512
4	$1/2$	492	31	3	-39493539
7	1	-4119	32	3	52255768
8	1	7256	35	2	-117966288
11	1	-33512	36	$5/2$	153541020
12	$4/3$	53008	39	4	-331534572
15	2	-192513	40	2	425691312
16	$3/2$	287244	43	1	-884736744
19	1	-885480	44	4	1122626864
20	2	1262512	47	5	-2257837845
23	3	-3493982	48	$10/3$	2835861520

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