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Poly-Bernoulli numbers

par Masanobu KANEKO

Résumé. Par le biais des séries logarithmiques multiples, nous définissons l'analogue en plusieurs variables des nombres de Bernoulli. Nous démontrons une formule explicite ainsi qu'un théorème de dualité pour ces nombres. Nous donnons aussi un théorème de type von Staudt et une nouvelle preuve d'un théorème de Vandiver.

Abstract. By using polylogarithm series, we define "poly-Bernoulli numbers" which generalize classical Bernoulli numbers. We derive an explicit formula and a duality theorem for these numbers, together with a von Staudt-type theorem for di-Bernoulli numbers and another proof of a theorem of Vandiver.

For every integer k, we define a sequence of rational numbers $B_n^{(k)}$ ($n = 0, 1, 2, \cdots$), which we refer to as poly-Bernoulli numbers, by

$$\frac{1}{z}\mathrm{Li}_k(z)\Big|_{z=1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}. \tag{1}$$

Here, for any integer k, $\mathrm{Li}_k(z)$ denotes the formal power series (for the k-th polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$) $\sum_{m=1}^{\infty} z^m/m^k$. When $k = 1$, $B_n^{(1)}$ is the usual Bernoulli number (with $B_1^{(1)} = 1/2$):

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!} ,$$

and when $k \geq 1$, the lefthand side of (1) can be written in the form of "iterated integrals":

(2)
$$e^x \cdot \underbrace{\frac{1}{e^x - 1}\int_0^x \frac{1}{e^t - 1}\int_0^t \cdots \frac{1}{e^t - 1}\int_0^t}_{(k-1)\text{-times}} \frac{t}{e^t - 1} dt\, dt \cdots dt = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

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In this paper, we give both an explicit formula for $B_n^{(k)}$ in terms of the Stirling numbers of the second kind and a sort of duality for negative index poly-Bernoulli numbers. Both formulas are elementary, and in fact almost direct consequences of the definition and properties of the Stirling numbers. As applications, we prove a von Staudt-type theorem for di-Bernoulli numbers ($k=2$) and give an alternative proof of a theorem due to Vandiver on a congruence for $B_n^{(1)}$.

1. Explicit formula and duality

An explicit formula for $B_n^{(k)}$ is given by the following:
THEOREM 1.

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! S(n,m)}{(m+1)^k} \quad (n \geq 0,\ \forall k),$$

where

$$S(n,m) = \frac{(-1)^m}{m!} \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} \ell^n$$

is the Stirling number of the second kind.
REMARK. When $k=1$, the theorem and its many variants are classical results in the study of Bernoulli numbers (cf. [1]).

Because the Stirling numbers are integers, we see from the formula that $B_n^{(k)}$ for $k \leq 0$ is an integer (actually positive, as demonstrated in the remark at the end of this section).
THEOREM 2. *For any $n, k \geq 0$, we have*

$$B_n^{(-k)} = B_k^{(-n)}.$$

PROOF OF THEOREMS 1 AND 2. One way to define the Stirling numbers of the second kind $S(n,m)$ $(n \geq 0, 0 \leq m \leq n)$ is via the formula

$$x^n = \sum_{m=0}^{n} S(n,m)(x)_m,$$

where, for each integer $m \geq 0$, we denote by $(x)_m$ the polynomial $x(x-1)(x-2)\cdots(x-m+1)$ ($(x)_0 = 1$). Then they satisfy the following formulas (when $n=0$ in (3), the identity $0^0 = 1$ is understood):

$$S(n,m) = \frac{(-1)^m}{m!}\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}\ell^n, \tag{3}$$

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m)\frac{t^n}{n!}. \tag{4}$$

For other definitions, further properties and proofs, we refer to [3].

Now Theorem 1 is readily derived from the definition (1) and the formula (4). In fact,

$$\begin{aligned}
\frac{1}{z}\mathrm{Li}_k(z)\Big|_{z=1-e^{-x}} &= \sum_{m=0}^{\infty}\frac{(1-e^{-x})^m}{(m+1)^k} \\
&= \sum_{m=0}^{\infty}\frac{m!}{(m+1)^k}\sum_{n=m}^{\infty}(-1)^m S(n,m)\frac{(-x)^n}{n!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\frac{(-1)^m m! S(n,m)}{(m+1)^k}\Big)\frac{(-x)^n}{n!}.
\end{aligned}$$

Hence the theorem follows.

To prove Theorem 2, we calculate the generating function of $B_n^{(-k)}$:

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(1-e^{-x})^m(m+1)^k\frac{y^k}{k!} \\
&= \sum_{m=0}^{\infty}(1-e^{-x})^m e^{(m+1)y} \\
&= \frac{e^{x+y}}{e^x+e^y-e^{x+y}}.
\end{aligned}$$

The last expression being symmetric in x and y yields Theorem 2.

REMARK. Since

$$\begin{aligned}
\frac{e^{x+y}}{e^x+e^y-e^{x+y}} &= \frac{e^{x+y}}{1-(e^x-1)(e^y-1)} \\
&= e^{x+y}(1+(e^x-1)(e^y-1)+((e^x-1)(e^y-1))^2+\cdots),
\end{aligned}$$

the number $B_n^{(-k)}$ $(k \geq 0)$ is always positive.

2. Denominators of di-Bernoulli numbers

Using Theorem 1, we can completely determine the denominator of di-Bernoulli numbers as follows.

THEOREM 3.

(1) *When n is odd, $B_n^{(2)} = -\frac{(n-2)}{4} B_{n-1}^{(1)}$ $(n \geq 1)$. (Hence the description of the denominator reduces to the classical Clausen-von Staudt theorem.)*

(2) *When n is even (≥ 2), the p-order $ord(p,n)$ of $B_n^{(2)}$ for a prime number p is given as follows.*

  (a) *$ord(p,n) \geq 0$ if $p > n+1$.*

  (b) *For $5 \leq p \leq n+1$, we have:*

    (i) *$ord(p,n) = -2$ if $p-1 | n$.*

    (ii) *If $p-1 \not| n$, then:*

      (A) *$ord(p,n) \geq 0$ if $p | \frac{B_n^{(1)}}{n}$, or $n \equiv n' \bmod p(p-1)$ for some $1 < n' < p-1$.*

      (B) *$ord(p,n) = -1$ otherwise.*

  (c) *$ord(3,n) \geq 0$ if $n \equiv 2 \bmod 3$ and $n > 2$. Otherwise $ord(3,n) = -2$.*

  (d) *$ord(2,n) \geq 0$ if $n \equiv 2 \bmod 4$ and $n > 2$. $ord(2,n) = -1$ if $n \equiv 0 \bmod 4$. $ord(2,2) = -2$.*

Before proving the theorem, we establish the following lemma, which will be needed in the proof.

LEMMA 1. *Assume $n \geq 2$ is even and $p \geq 5$ is a prime number such that $m+1 = 2p$. Then*

$$(-1)^m m! S(n,m) \equiv 0 \bmod p^2,$$

and hence $(-1)^m m! S(n,m)/(m+1)^2$ is p-integral.

PROOF. By (3),

$$\begin{aligned}
(-1)^m m! S(n,m) &= \sum_{\ell=1}^{2p-1} (-1)^\ell \binom{2p-1}{\ell} \ell^n \\
&= \sum_{\ell=1}^{p-1} \{(-1)^\ell \binom{2p-1}{\ell} \ell^n + (-1)^{2p-\ell} \binom{2p-1}{2p-\ell} (2p-\ell)^n\} \\
&\quad +(-1)^p \binom{2p-1}{p} p^n
\end{aligned}$$

$$\equiv \sum_{\ell=1}^{p-1} \{(-1)^\ell \binom{2p-1}{\ell} \ell^n + (-1)^\ell \binom{2p-1}{\ell-1} (-2np\ell^{n-1} + \ell^n)\} \quad \bmod p^2.$$

Since

$$\binom{2p-1}{\ell}+\binom{2p-1}{\ell-1}=\frac{2p}{\ell}\binom{2p-1}{\ell-1},$$

the last sum is equal to

$$2p(1-n)\sum_{\ell=1}^{p-1}(-1)^{\ell}\binom{2p-1}{\ell-1}\ell^{n-1}.$$

Noting that

$$\binom{2p-1}{\ell-1}\equiv(-1)^{\ell-1}\mod p,$$

we see that

$$\sum_{\ell=1}^{p-1}(-1)^{\ell}\binom{2p-1}{\ell-1}\ell^{n-1}\equiv-\sum_{\ell=1}^{p-1}\ell^{n-1}\equiv 0\mod p,$$

because $p-1\nmid n-1$ (recall that n is even and p is odd). This proves the lemma.

PROOF OF THEOREM 3. 1. Let $B_n=B_n^{(1)}$ for $n\neq 1$ and $B_1=-1/2$. Then $\sum_{n=0}^{\infty}B_nx^n/n!=x/(e^x-1)$. By (2) in the introduction, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(2)}\frac{x^n}{n!} &= \frac{e^x}{e^x-1}\int_0^x\sum_{\ell=0}^{\infty}B_\ell\frac{t^\ell}{\ell!}dt\\ &= \sum_{m=0}^{\infty}B_m^{(1)}\frac{x^m}{m!}\cdot\sum_{\ell=0}^{\infty}B_\ell\frac{x^\ell}{(\ell+1)!}.\end{aligned}$$

From this we see that

$$B_n^{(2)}=\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{B_{n-\ell}^{(1)}B_\ell}{\ell+1}.$$

Since $B_\ell^{(1)}=B_\ell=0$ for odd $\ell\geq 3$, we have for odd n

$$B_n^{(2)}=\frac{n}{2}B_{n-1}^{(1)}B_1+B_1^{(1)}B_{n-1}=-\frac{(n-2)}{4}B_{n-1}^{(1)}.$$

2. We make use of Theorem 1. Part (a) is obvious because the Stirling numbers in the formula in Theorem 1 are integers. For the remainder of the proof, first we note that the expression $m!/(m+1)^2$ in the summand of the formula is an integer except when $m+1=8,9$, *a prime number*, or $2\times$ *a prime number*, as can be checked in an elementary way. Now, Lemma 1 tells us that any prime number $p\geq 5$ satisfying $m+1=2p$ does not appear in the denominator of $B_n^{(2)}$.

Our next task is to consider the case $m+1=p$, where p is a prime number ≥ 5. In this case

$$(-1)^m m! S(n,m) = \sum_{\ell=1}^{p-1} (-1)^\ell \binom{p-1}{\ell} \ell^n.$$

The righthand side is congruent modulo p to -1 if $p-1|n$ and to 0 if $p-1 \not| n$. Thus if $p-1|n$, the p-order of $(-1)^m m! S(n,m)/(m+1)^2$ is -2. Since the other summands in the formula in Theorem 1 are p-integral, we have shown part (b)-i. Suppose $p-1 \not| n$ and calculate modulo p^2. Using

$$\binom{p-1}{\ell} \equiv (-1)^\ell + (-1)^{\ell-1} p \sum_{i=1}^{\ell} \frac{1}{i} \mod p^2,$$

we see that

$$\sum_{\ell=1}^{p-1} (-1)^\ell \binom{p-1}{\ell} \ell^n \equiv \sum_{\ell=1}^{p-1} \ell^n - p \sum_{\ell=1}^{p-1} \ell^n \sum_{i=1}^{\ell} \frac{1}{i} \mod p^2.$$

It is known that (cf. Cor. of Prop. 15.2.2 in [2]) if n is even and $p-1 \not| n$, then

$$\sum_{\ell=1}^{p-1} \ell^n \equiv p B_n^{(1)} \mod p^2.$$

On the other hand, when we put $n \bmod p-1 = n', 1 < n' < p-1$ (since both n and $p-1$ are even, n' is also even), we find

$$\begin{aligned} \sum_{\ell=1}^{p-1} \ell^n \sum_{i=1}^{\ell} \frac{1}{i} &\equiv \sum_{\ell=1}^{p-1} \ell^{n'} \sum_{i=1}^{\ell} \frac{1}{i} \mod p \\ &\equiv B_{n'}^{(1)} \mod p \quad \text{(see (63) of Vandiver [4] and Section 3 below).} \end{aligned}$$

We therefore have

$$(-1)^m m! S(n,m) \equiv p(B_n^{(1)} - B_{n'}^{(1)}) \mod p^2,$$

where $m+1=p$ and $n' \equiv n \mod p-1, 1 < n' < p-1$. Since $p-1 \not| n$, the number $B_n^{(1)}/n$ is p-integral and $B_{n'}^{(1)} \equiv n' B_n^{(1)}/n \mod p$ (Prop. 15.2.4 and Th.5 following it in [2]). Thus

$$(-1)^m m! S(n,m) \equiv p(n-n') \frac{B_n^{(1)}}{n} \mod p^2.$$

This readily gives part (b)-ii of the theorem.

The only summands in Theorem 1 which may not be 3-integral are $2!S(n,2)/3^2$, $-5!S(n,5)/6^2$, and $8!S(n,8)/9^2$. By direct calculation using the formula (3), we obtain part (c). In a similar manner, we can determine the 2-order as well, but we omit the details here.

3. A theorem of Vandiver

As an application of Theorems 1 and 2, we prove the following proposition originally due to Vandiver.

PROPOSITION. *Let p be an odd prime number. For $1 \leq i \leq p-2$,*

$$B_i^{(1)} \equiv \sum_{m=1}^{p-2}(1+\frac{1}{2}+\cdots+\frac{1}{m})(m+1)^i \mod p.$$

PROOF. By Theorem 1 and Fermat's little theorem, we see that

$$B_i^{(1)} \equiv B_i^{(2-p)} \mod p.$$

Theorem 2 says that the righthand side is equal to $B_{p-2}^{(-i)}$, which by Theorem 1 is equal to $-\sum_{m=0}^{p-2}(-1)^m m!S(p-2,m)(m+1)^i$.

LEMMA 2. *Suppose p is an odd prime, and $1 \leq m \leq p-2$. Then*

$$(-1)^{m-1}m!S(p-2,m) \equiv 1+\frac{1}{2}+\cdots+\frac{1}{m} \mod p.$$

PROOF. The Stirling numbers satisfy the recurrence formula

$$S(n,m) = S(n-1,m-1)+mS(n-1,m) \ (n \geq 1) \ \text{(see [3])}.$$

Thus if we put $(-1)^{m-1}m!S(p-2,m) = b_m$, we get

$$(-1)^{m-1}m!S(p-1,m) = m(-b_{m-1}+b_m) \ (m \geq 2).$$

But by (3),

$$\begin{aligned} (-1)^{m-1}m!S(p-1,m) &= -\sum_{\ell=1}^{m}(-1)^\ell\binom{m}{\ell}\ell^{p-1} \\ &\equiv -\sum_{\ell=1}^{m}(-1)^\ell\binom{m}{\ell} \mod p \\ &\equiv 1 \mod p, \end{aligned}$$

and we thus conclude that

$$b_m \equiv b_{m-1}+\frac{1}{m} \mod p.$$

This together with the relation $b_1 = S(p-2,1) = 1$ gives the lemma and hence completes the proof of the proposition.

REMARK. If $i > 1$, the righthand side of the proposition is congruent modulo p to

$$\sum_{m=1}^{p-1}(1+\frac{1}{2}+\cdots+\frac{1}{m})m^i,$$

and this being congruent to B_i (even when $i = 1$) is a special case of Vandiver's congruence (63) in [4].

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Masanobu KANEKO
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-81, Japan
e-mail: mkaneko@math.kyushu-u.ac.jp