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Selfsimilarity in a Class of Quadratic-Quasiperiodic Chains

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We prove that quasiperiodic chains associated with a class of quadratic irrational numbers have an inflation symmetry and can be generated from a regular chain by a hyperinflation. We devise the explicit method to find the hyperinflation symmetry and discuss the properties of such a class of quasiperiodic sequences.

§1. Introduction

Quasiperiodic systems[1,2] provide exotic structures between periodic crystals and amorphous systems. Symmetry of periodic crystals is used to classify the crystals, and lack of symmetry in amorphous systems makes it difficult to classify the structure. For quasiperiodic systems, there are infinitely many variety of the structure and its classification is one of the most urgent problems. As an example, let us consider a one-dimensional lattice produced by a projection method shown in Fig. 1.[3] Lattice points in a strip with slope α in the first quadrant of a square lattice are projected onto a line with slope ρ to produce, after adjusting the length scale, a one-dimensional lattice

$$x_n = n + \rho[\alpha n] \quad (n=0, 1, \cdots), \tag{1}$$

where $[x]$ denotes the largest integer less than or equal to x. It is apparent that there are two lattice spacings, 1 and $1+\rho$, which are placed in an order determined by α. When α is rational, a periodic lattice is produced and when α is irrational, the sequence becomes quasiperiodic. Every irrational $\alpha (0 \le \alpha \le 1)$ produces a distinct quasiperiodic sequence. The sequence of two lattice spacings in the lattice is isomorphic to the sequence of 0 and 1 determined by a relation

$$F_n(\alpha) = [(n+1)\alpha] - [n\alpha], \tag{2}$$

and α represents the fraction of 1 in the sequence.

There have been many works on the physical properties of quasiperiodic systems which consist of two components in the order isomorphic to the sequence of 0 and 1 in eq. (2). It has been expected without rigorous proofs that all the quasiperiodic systems exhibit the same properties. Therefore most of the works on quasiperiodic chains have been carried out for the Fibonacci chain corresponding to $\alpha = (\sqrt{5}-1)/2$ in eq. (2) and its derivatives.[2] There have been several works which introduced a method to generate quasiperiodic systems corresponding to an arbitrary irrational number, utilizing its continued fraction expansion.[4,5]

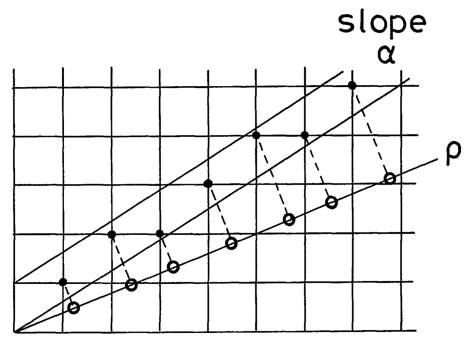

Fig. 1. The projection method to produce quasiperiodic lattices. Lattice points in a strip with slope α in a square lattice are projected onto a line with slope ρ. When α is irrational, the lattice is quasiperiodic with two lattice spacings 1 and $1+\rho$ and the fraction of the longer spacing is given by α.

In this paper, we show that the quasiperiodic system generated by eq. (2) has a selfsimilarity (inflation) transformation when and only when α is a quadratic irrational number in (0,1) whose conjugate is *not* in (0, 1). We call such numbers quasi-reduced quadratic (QRQ) numbers (with an analogous notion in the number theory in mind). We present an explicit method to find the inflation rule for the quasiperiodic chains generated by eq. (2) for $\alpha \in$ {QRQ numbers}. For any QRQ number α, we can find a unimodular fractional linear transformation (FLT) with integral coefficients which fixes α. The inflation rule is derived as the hyperinflation[6] at the fixed point of the FLT. The physical properties of the quasiperiodic chain can be obtained once the transformation of the physical properties is known along the hyperinflation path. Therefore, we conclude that quasiperiodic chains generated by eq. (2) with QRQ numbers show similar physical properties.

We organize this paper as follows. In §2, we discuss the relation between the sequence generated by eq. (2) and the Farey series. We consider an integral fractional linear transformation in §3 and derive a corresponding hyperinflation rule, and subsequently inflation rules for quasiperiodic chains. The hyperinflation symmetry was first found by Odagaki and Aoyama[6] for certain periodic and quasiperiodic sequences given by eq. (2). In §4, we present an explicit method to derive the hyperinflation rule in the sequence (2) for an arbitrary QRQ number. We discuss properties of quasiperiodic chains characterized by QRQ numbers and their approximants in §5. Some concluding remarks are also given in §5. Appendices A and B contain the sketch of proofs for the theorems given in §2 and §3, while in Appendix C we give a detailed explanation of finding an FLT for a given QRQ number.

§2. The Farey Series and Periodic Units

First, we call a series of 0 and 1, $F(\alpha)=\{F_n(\alpha)\}_{n\geq 1}$, a Farey expansion of α ($\in [0, 1)$). When $\alpha=b/a$ ($a>b\geq 0$) is a rational number in its lowest term, then the Farey expansion $F(b/a)$ is a periodic sequence whose unit consists of b of 1 and $a-b$ of 0. We denote the periodic unit by Per (b/a), namely Per (b/a) $=F_1(b/a)F_2(b/a)\cdots F_a(b/a)$. Note that if $a\geq 2$ Per (b/a) always ends with 10. We denote by Per′ (b/a) the sequence identical to Per (b/a) except for 10 at the right end changed to 01.

The series $F(\alpha)$ is closely related to the Farey series;[7] the Farey series of order m, denoted by $\mathbf{F_m}$, is the ascending series of irreducible fractions in (0, 1) whose denominators do not exceed m ($0\equiv 0/1$ and $1\equiv 1/1$). For example, the Farey series of order 3 is 0, 1/3, 1/2, 2/3, 1. If b/a and d/c are two successive terms of $\mathbf{F_m}$, we call them a Farey pair and the semi-open interval $[b/a, d/c)$ a Farey interval (of order m). Then, we can prove the following theorem.

Theorem 1 *Let $[b/a, d/c)$ be a Farey interval of order m. For $\alpha\in[b/a, d/c)$, the first $m-1$ entries of $F(\alpha)$ are identical to those of $F(b/a)$.*

A brief proof is given in Appendix A. In other words, the first $m-1$ terms of $F(\alpha)$ are determined by the Farey interval of order m to which the number α belongs.

§3. Fractional Linear Transformation

We introduce a fractional linear transformation (FLT) belonging to $PSL_2(\mathbf{Z})$

$$\gamma(\alpha)=\frac{p\alpha+q}{r\alpha+s}, \tag{3}$$

where p, q, r and s are integers satisfying

$$ps-rq=1, \quad s(r+s)>0, \quad q/s\geq 0, \\ (p+q)/(r+s)\leq 1. \tag{4}$$

The first condition implies that the transformation is a simple rotation and the latter three conditions ensure $0\leq\gamma(0)<\gamma(1)\leq 1$.

Our main observation can be stated as the following theorem.

Theorem 2 *For any $\alpha\in[0, 1)$, the Farey expansion, $F(\gamma(\alpha))$, for $\gamma(\alpha)$ is obtained from $F(\alpha)$ by a hyperinflation*

$$0\to \mathrm{Per}\,(\gamma(0)), \\ 1\to \mathrm{Per}'\,(\gamma(1)). \tag{5}$$

Consequently, when α is a fixed point of FLT (3), namely $\gamma(\alpha)=\alpha$, the Farey expansion $F(\alpha)$ has the symmetry given by inflation (5). We give a sketch of the proof in Appendix B.

A similar hyperinflation rule can be shown to exist when p, q, r, s in FLT (3) satisfy

$$ps-rq=-1, \tag{6}$$

instead of $ps-rq=1$. For this FLT, we suppose that $\gamma(\alpha)$ satisfies

$$0 \leq \gamma(1) < \gamma(0) \leq 1. \tag{7}$$

When $\alpha \in (0, 1)$ is irrational, $F(\gamma(\alpha))$ is obtained from $F(\alpha)$ by

$$0 \rightarrow \mathrm{Per}'\,(\gamma(0)),$$
$$1 \rightarrow \mathrm{Per}\,(\gamma(1)). \tag{8}$$

If α is rational, then inflation (8) applied to Per (α) yields Per$'$ $(\gamma(\alpha))$.

§4. The Quasi-Reduced Quadratic Numbers

We now present an explicit algorithm to find the selfsimilar transformation for an arbitrary QRQ number. As we have explained in Introduction, the QRQ numbers are defined to be a set of irrational numbers in (0, 1), each of which is a real solution to a quadtratic equation over integers and its conjugate is not in (0, 1). Now, let α be an QRQ number and satisfy

$$A\alpha^2 + B\alpha + C = 0, \tag{9}$$

where A, B, C are integers without common divisor. It is straightforward to show that α is a fixed point of the fractional linear transformation corresponding to the matrix

$$\Gamma \equiv \begin{pmatrix} \dfrac{x-yB}{2} & -yC \\ yA & \dfrac{x+yB}{2} \end{pmatrix}, \tag{10}$$

where x and y $(y \neq 0)$ are integer solution to

$$x^2 - (B^2 - 4AC)y^2 = \pm 4. \tag{11}$$

We take x to be positive and the sign of y is chosen so as to the condition (4) is satisfied. The sign in eq. (11) coincides with that of det (Γ). Equation (11) is classically known as the Pell equation.[8] It has a unique "minimal" solution and other solutions are exhausted, up to sign, by its "powers". In Appendix C we give an explicit algorithm for solving eq. (11). From the matrix $\Gamma = (\Gamma_{ij})$, we find the hyperinflation rule for the Farey expansion of α as

$$0 \rightarrow \mathrm{Per}\,(\Gamma_{12}/\Gamma_{22}),$$
$$1 \rightarrow \mathrm{Per}'\,((\Gamma_{11}+\Gamma_{12})/(\Gamma_{21}+\Gamma_{22})), \tag{12}$$

when det $(\Gamma) = 1$ and

$$0 \rightarrow \mathrm{Per}'\,(\Gamma_{12}/\Gamma_{22}),$$
$$1 \rightarrow \mathrm{Per}\,((\Gamma_{11}+\Gamma_{12})/(\Gamma_{21}+\Gamma_{22})), \tag{13}$$

when det $(\Gamma) = -1$.

As an example, we consider $\alpha = 2 - \sqrt{3}$ which satisfies $\alpha^2 - 4\alpha + 1 = 0$. Equation (11) becomes $x^2 - 12y^2 = 4$, one of whose solution is given by $x = 4$ and $y = -1$. Corresponding matrix Γ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 4 \end{smallmatrix}\right)$, and the hyperinflation rule is given by

$$\gamma(\alpha) = \frac{1}{4-\alpha}, \tag{14}$$

and

$$0 \rightarrow \mathrm{Per}\,(1/4) = 0010,$$
$$1 \rightarrow \mathrm{Per}'\,(1/3) = 001. \tag{15}$$

In general, $\alpha = (n - \sqrt{n^2-4})/2$ $(n \geq 3)$ satisfying $\alpha^2 - n\alpha + 1 = 0$ is a fixed point of FLT

$$\gamma(\alpha) = \frac{1}{n-\alpha}, \tag{16}$$

whose hyperinflation rule is

$$0 \rightarrow \mathrm{Per}\,(1/n) = \underbrace{0 \cdots 0}_{n-2}\,10,$$
$$1 \rightarrow \mathrm{Per}'\,(1/(n-1)) = \underbrace{0 \cdots 0}_{n-2}\,1. \tag{17}$$

As the second example, we consider $\alpha = 2 - \sqrt{2}$ which is a solution to $\alpha^2 - 4\alpha + 2 = 0$. We find that the hyperinflation (of determinant 1) is given by

$$\gamma(\alpha) = \frac{4-\alpha}{7-2\alpha}, \tag{18}$$

and

$$0 \rightarrow \mathrm{Per}\,(4/7) = 1010110,$$
$$1 \rightarrow \mathrm{Per}'\,(3/5) = 10101. \tag{19}$$

Finally, we consider $\alpha = (\sqrt{n^2+4} - n)/2$ $(n \geq 1)$ which is a solution to $\alpha^2 + n\alpha - 1 = 0$. The irrational number $1/\alpha$ is known to be the golden, silver and bronze mean for $n = 1$, 2 and 3, respectively. It is a simple task to find the hyperinflation rule for these α's:

$$\Gamma = \begin{pmatrix} 1 & n \\ n & n^2+1 \end{pmatrix}, \tag{20}$$

that is,

$$\gamma(\alpha)=\frac{\alpha+n}{n\alpha+n^2+1}, \quad (21)$$

and

$$0\to\mathrm{Per}\left(\frac{n}{n^2+1}\right),$$
$$1\to\mathrm{Per}'\left(\frac{n+1}{n^2+n+1}\right). \quad (22)$$

It is clear that this gives the well-known inflation rule for the Fibonacci chain when $n=1$.

When eq. (11) with minus sign has a solution which yields $\bar{\Gamma}$ ($\det(\bar{\Gamma})=-1$) for the FLT, then $\Gamma=\bar{\Gamma}^2$ is an FLT with $\det(\Gamma)=1$ with the hyperinflation given by repeated application of the hyperinflation for $\bar{\Gamma}$. A necessary and sufficient condition for this case is discussed in Appendix C. For example, we find for $\alpha=2-\sqrt{2}$ $\bar{\Gamma}=\left(\begin{smallmatrix}-1 & 2\\ -1 & 3\end{smallmatrix}\right)$ with hyperinflation $0\to\mathrm{Per}'(2/3)=101$ and $1\to\mathrm{Per}(1/2)=10$. The hyperinflation in eq. (19) is the square transformation of this rule. Similarly, the matrix Γ in eq. (20) can be written as $\Gamma=\bar{\Gamma}^2$ where

$$\bar{\Gamma}=\begin{pmatrix}0 & 1\\ 1 & n\end{pmatrix}. \quad (23)$$

Noting that $\det(\bar{\Gamma})=-1$, we apply the inflation rule (8) and find the hyperinflation rule

$$\bar{\gamma}(\alpha)=\frac{1}{n+\alpha}, \quad (24)$$

and

$$0\to\mathrm{Per}'\left(\frac{1}{n}\right)=\underbrace{0\cdots0}_{n-1}1,$$
$$1\to\mathrm{Per}\left(\frac{1}{n+1}\right)=\underbrace{0\cdots0}_{n-1}10. \quad (25)$$

The hyperinflation rule (21) and (22) is the square transformation of rule (24) and (25). This type of hyperinflations has been discussed by Suzuki.[9)]

§5. Hyperinflation and Approximant Crystals

Let us consider a hyperinflation

$$\gamma(\alpha)=\frac{p\alpha+q}{r\alpha+s}, \quad (26)$$

with conditions (4) and

$$0\to\mathrm{Per}(q/s),$$
$$1\to\mathrm{Per}'((p+q)/(r+s)). \quad (27)$$

When we start the transformation from $\alpha^{(1)}=0$, we generate a series of periodic sequences converging to a fixed point of the transformation. These periodic sequences are considered to be approximants[3)] to the quasiperiodic sequence at the fixed point. After $n-1$ hyperinflation, we arrive at a rational number $\alpha^{(n)}\equiv b_n/a_n$. We can show that a_n satisfies the recurrence relation

$$a_{n+2}=(p+s)a_{n+1}-a_n, \quad (28)$$

with initial conditions $a_1=1$, $a_2=s$, and b_n satisfisfies the same recurrence relation with $b_1=0$, $b_2=q$. Therfore, we find

$$a_n=\frac{1}{\lambda_+-\lambda_-}[(1-p\lambda_-)\lambda_+^n-(1-p\lambda_+)\lambda_-^n],$$
$$b_n=\frac{q}{\lambda_+-\lambda_-}[\lambda_+^{n-1}-\lambda_-^{n-1}], \quad (29)$$

where $\lambda_\pm$ ($\lambda_+>\lambda_-$) are the solutions to

$$\lambda^2-(p+s)\lambda+1=0. \quad (30)$$

Since $p+s>2$ must be satisfied from eqs. (10) and (11), $\lambda_-\in(0,1)$ and $\lambda_+>1$. It is noted that for FLT (16) λ_+ coincides with the conjugate of their fixed point. The periodic unit for the n-th approximant consists of a_n elements. Therefore, as n gets larger, the number of elements in the unit increases exponentially as $\sim\lambda_+^n$.

The physical properties of quasiperiodic chains and their approximants can be obtained if the transformation of the properties is known along the hyperinflation path. As an example, let us consider a tight-binding electron on the quasiperiodic chain and its approximants, where two kinds of the nearest neighbor transfer energies t_0 and t_1 are placed in the order of the Farey expansion of $\alpha\in$ QRQ numbers and the site energies are set to zero. We follow a hyperinflation converging to α from $\alpha^{(1)}=0$. After $n-1$ transformation, we arrive $\alpha^{(n)}\equiv b_n/a_n$ whose periodic unit consists of a_n sites. Therefore, the number of energy bands increases exponentially as the order of the approximant is increased and the energy spectrum becomes point-like at the fixed point.[10] Precise determination of the band structure can be acheived by the trace mapping of the transfer matrices proposed first by Kohmoto *et al.*[11] In fact, the trace mapping for an arbitrary irrational quasiperiodic lattice has been reported.[5,12]

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Appendix

A. Proof of Theorem 1.

First we note that $F_n(\alpha)=0$ or 1 according to $\alpha\in[i/n,(i+1)/(n+1))$ or $\alpha\in[i/(n+1), i/n)$, i being the unique integer such that $\alpha\in[i/(n+1), \quad (i+1)/(n+1))(i/(n+1)<i/n<(i+1)/(n+1))$. For k with $1\le k\le m-1$, let i be the unique integer such that $b/a\in[i/(k+1),(i+1)/(k+1))$. Then $F_k(b/a)=0$ if and only if $b/a\in[i/k,(i+1)/(k+1))$. Since the denominators of i/k and $(i+1)/(k+1)$ do not exceed m and $[b/a, d/c)$ is a Farey interval of order m, we must have either $[b/a, d/c)\subset[i/k,(i+1)/(k+1))$ or $[b/a, d/c)\cap[i/k,(i+1)/(k+1))=\emptyset$. Hence $F_k(\alpha)=0$ if and only if $F_k(b/a)=0$.

B Proof of Theorem 2.

Because rational numbers are dense in [0, 1), it is sufficient to prove the hyperinflation rule for rational α's. Let b/a and d/c be a Farey pair of certain order. Then b/a, $(b+d)/(a+c)$ and d/c are three successive terms of a Farey series of higher order and so are $\gamma(b/a)$, $\gamma((b+d)/(a+c))$ and $\gamma(d/c)$. It is straightforward to show the following two propositions:

1. Suppose $b/a<d/c$ are a Farey pair. When $d/c<1$, Per $((b+d)/(a+c))=$ Per (b/a) Per $(d/c)=$ Per$'$ (d/c) Per (b/a) (just concatenate the two periods). When $d/c=1$, $b=a-1$ and Per $(a/(a+1))=1$ Per $((a-1)/a)$.
2. For $b/a\in(0,1)$, Per$'$ $(1-b/a)$ is obtained from Per (b/a) by transformation $0\to1$ and $1\to0$.

These proposition can be shown by direct calculation using the criterion noted in the beginning of Appendix A.[13] The proof of the theorem is completed by induction on the order of the Farey series. For example, when $d/c\neq1$, Per $(\gamma((b+d)/(a+c)))=$ Per $(\gamma(b/a))$ Per $(\gamma(d/c))$. Thus if Per $(\gamma(b/a))$ and Per $(\gamma(d/c))$ are obtained from Per (b/a) and Per (d/c) by inflation $0\to$ Per $(\gamma(0))$ and $1\to$ Per$'$ $(\gamma(1))$, then Per $(\gamma((b+d)/(a+c)))$ is obtained from Per $((b+d)/(a+c))$ by the same inflation rule.

C Solution to the Pell equation

First we set $r=[\sqrt{D/4}]$ when D is even and $r=$ "the largest odd number less than $\sqrt{D}$" when D is odd, where $D=B^2-4AC$. We further set $\theta=r+\sqrt{D/4}$ or $\theta=(r+\sqrt{D})/2$ according as D is even or odd. Since the quadratic irrational θ is reduced in the sense of Gauss (i.e. $\theta\ge1$, $0\ge\theta'\ge-1$), its continued fraction expantion is purely periodic;

$$\theta=k_0+\frac{1}{k_1+}\frac{1}{k_2+}\cdots\frac{1}{k_{n-1}+}\frac{1}{\theta}. \tag{31}$$

If we write the right hand side of eq. (31) as $(a\theta+b)/(c\theta+d)$, our solution (x,y) is given by $x\pm y\sqrt{D}=2(c\theta+d)$.

We can show that $x^2-Dy^2=-4$ has a solution if and only if the length n of the period of the continued fraction in eq. (31) is odd. If this is the case $x\pm y\sqrt{D}=2(c\theta+d)^2$ gives a solution of $x^2-Dy^2=4$.

If we know *a priori* that $x^2-Dy^2=-4$ has no solution, for example if D has a prime factor congruent to 3 modulo 4, we can apply a more simple algorithm as follows.

Case 1. $\alpha\in(0,1)$, $\alpha'\equiv$ conjugate of $\alpha\ge1$. In this case α' has the following purely periodic continued fraction expansion

$$\alpha'=k_0-\frac{1}{k_1-}\frac{1}{k_2-}\cdots\frac{1}{k_{n-1}-}\frac{1}{\alpha'}. \tag{32}$$

We determine p', q', r' and s' by writing the right hand side of eq. (32) as $(p'\alpha'+q')/(r'\alpha'+s')$, and then the FLT corresponding to $\Gamma\equiv\begin{pmatrix}p&q\\r&s\end{pmatrix}=\begin{pmatrix}p'&q'\\r'&s'\end{pmatrix}^{-1}$ is the desired transformation which fixes α.

Case 2. $\alpha\in(0,1)$, $\alpha'\le0$.

In this case $1-\alpha$ satisfies Case 1, thus we compute $\begin{pmatrix}p&q\\r&s\end{pmatrix}$ for $1-\alpha$ using the method explained above, and we obtain the desired matrix by $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

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