J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
36 (1989), 363-372.

*Certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces**)

By Masanobu KANEKO

(Communicated by Y. Ihara)

Introduction.

In this paper we present some results on certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces.

Let l be a prime, $g\geq 1$, $r\geq 0$ be integers and $G=G_{g,r}$ be the pro-l completion of the fundamental group of Riemann surface of genus g with r-points deleted. Assume $r\geq 1$. Then G is a pro-l free group of rank $2g+r-1$ having a standard presentation:

$$G=\left\langle \begin{matrix} x_1, x_2, \cdots, x_{2g} \\ z_1, \cdots, z_r \end{matrix} \middle| [x_1, x_{g+1}][x_2, x_{g+2}] \cdots [x_g, x_{2g}]z_1 \cdots z_r=1 \right\rangle_{\text{pro-}l}.$$

We give G a central filtration $\{G(m)\}_{m\geq 1}$ such that the elements $x_1, \cdots, x_{2g}$ are of degree 1, the elements $z_1, \cdots, z_r$ are of degree 2 and generally the degree of a commutator $[x, y]$ is the sum of degrees of x and y. For this filtration let $grG=\bigoplus_{m\geq 1} G(m)/G(m+1)$. Then, by a standard method, grG turns out to be a free Lie algebra generated by the classes of $x_1, \cdots, x_{2g}, z_1, \cdots, z_{r-1}$. By using this we first establish a "successive approximation lemma" to construct automorphisms of G. Then we study some basic properties of the subgroup

$$\tilde{\Gamma}=\tilde{\Gamma}_{g,r}=\{\sigma\in \operatorname{Aut} G \mid z_j^{\sigma}\sim z_j^{\alpha_j},\ \exists\alpha_j\in \mathbf{Z}_l^{\times},\ 1\leq j\leq r\}$$

of the automorphism group of G. Such type of groups arise naturally in the context of "large Galois representations" (cf. [2] [3]). These studies are viewed as a continuation of our previous study [2] in which we treated exclusively the case of $r=0$. A new ingredient is the filtration of G explained above. The author owes the idea of introducing such filtration to study the group $\tilde{\Gamma}$ to Professor Takayuki Oda. It seems

*) This paper is a part of the author's doctoral dissertation submitted to the University of Tokyo (1988).

that the lower central series used in [2] does not work well to study $\tilde{\Gamma}$ when $r\geq 2$.

As a consequence of these studies, we establish the following

THEOREM. *Suppose $r>s\geq 0$. The naturally induced homomorphism*

$$\tilde{\Gamma}_{g,r}\longrightarrow\tilde{\Gamma}_{g,s}$$

is surjective.

This is a pro-l analogue of the classical theorem of Dehn-Nielsen. We can derive from this a result on conjugacy classes of $\tilde{\Gamma}_{g,r}/\mathrm{Int}\,G$ as in [2].

The author would like to express his sincere gratitude to Professors Y. Ihara and Takayuki Oda for their helpful comments and advice without which this work would not be completed at all.

1. Filtration of the fundamental group and Lie algebra.

We fix a prime number l throughout the paper. Let g and r be two integers greater than or equal to 1. We denote by $G_{g,r}$ the pro-l completion of the fundamental group of r-punctured Riemann surface of genus g,

$$G_{g,r}=\left\langle \begin{matrix} x_1, x_2, \cdots, x_{2g} \\ z_1, \cdots, z_r \end{matrix} \middle| [x_1, x_{g+1}][x_2, x_{g+2}]\cdots[x_g, x_{2g}]z_1\cdots z_r=1 \right\rangle_{\text{pro-}l}.$$

We fix g and r throughout the sections 1, 2 and 3, so we write G for $G_{g,r}$ in these sections. In this section we define a certain central filtration of G and study its associated Lie algebra.

Let $N=N_{g,r}$ be the closed subgroup of G normally generated by the parabolic elements $z_1, \cdots, z_r$. For each $m\geq 1$, we define inductively a subset Σ_m of the set of all closed normal subgroups of G as follows:

$$\Sigma_1=\{G\},\ \Sigma_2=\{[G,G], N\},$$
$$\Sigma_m=\{[H_i, H_j]\mid H_i\in\Sigma_i,\ H_j\in\Sigma_j,\ i+j=m\}\quad (m\geq 3).$$

Here, [,] denotes the closure of algebraic commutator. Then the sequence $\{G(m)\}_{m\geq 1}$ of closed normal subgroups of G is defined by putting

$G(m)$ = the minimal closed normal subgroup of G containing all elements in Σ_m.

It is easy to check that the sequence $\{G(m)\}_{m\geq 1}$ has the properties

$$G=G(1)\supset G(2)\supset\cdots\supset G(m)\supset G(m+1)\supset\cdots$$

and

$$[G(m), G(n)]\subset G(m+n) \qquad (m, n\geq 1).$$

In particular, we have $G(m+1)\supset[G, G(m)]\supset[G(m), G(m)]$, i.e., the quotient $gr^m G=G(m)/G(m+1)$ is abelian, hence a $\boldsymbol{Z}_l$-module.

PROPOSITION 1. *Equipped with the bracket operator* [,], *the* $\boldsymbol{Z}_l$*-module* $grG=\bigoplus_{m\geq 1} gr^m G$ *is a free Lie algebra over* $\boldsymbol{Z}_l$ *generated by the elements* x_i $\bmod G(2)$ $(1\leq i\leq 2g)$ *and* z_j $\bmod G(3)$ $(1\leq j\leq r-1)$. *The module* $gr^m G$ *is a finitely generated free* $\boldsymbol{Z}_l$*-module whose rank* $\rho(m)$ *is given by the formula*

$$\prod_{m=1}^{\infty}(1-t^m)^{\rho(m)}=1-2gt-(r-1)t^2.$$

PROOF. As was pointed out by J. Labute in an abstract case ([4, Proposition 1]), this can be shown by a standard argument which, in the case of lower central series and pro-l, was indicated in [3, p. 58]. Likewise, the point is to show that there exists a representation of the Lie algebra grG into the free associative $\boldsymbol{Z}_l$-algebra generated by $2g+r-1$ elements $X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}$, which maps x_i to X_i $(1\leq i\leq 2g)$ and z_j to Z_j $(1\leq j\leq r-1)$. Here, we regard the associative algebra as given the graduation which assign X_i degree 1 and Z_j degree 2. Such a representation is obtained by the Magnus embedding

$$G \longrightarrow \boldsymbol{Z}_l[[X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}]]_{n.c}=\Lambda$$

of G into the non-commutative formal power series algebra $(x_i\mapsto 1+X_i$, $z_j\mapsto 1+Z_j)$. Here again the degree of each X_i $(1\leq i\leq 2g)$ is 1 and that of each Z_j $(1\leq j\leq r-1)$ is 2. Let I_m be the ideal of Λ consisting of all power series whose lowest degree is greater than or equal to m. Then $G(m)$ is mapped into $1+I_m$ $(m\geq 1)$ and we can associate to each element of $gr^m G$ a homogeneous polynomial of degree m in $X_1, \cdots, X_{2g}, Z_1, \cdots, Z_{r-1}$ $(\deg(X_i)=1$, $\deg(Z_j)=2)$. This gives the desired representation of grG. Calculation of the rank is also carried out in the similar manner as that in [7]. ∎

2. Filtration of "braid type" automorphism group.

Put

$$\tilde{\Gamma}=\tilde{\Gamma}_{g,r}=\{\sigma\in \mathrm{Aut}\, G_{g,r} \mid z_j^\sigma \sim z_j^{\alpha_j},\ \exists\alpha_j\in Z_l^\times,\ 1\leq j\leq r\},$$

where $\sim$ denotes conjugacy in $G=G_{g,r}$. Since each element in $\tilde{\Gamma}$ stabilizes $N=N_{g,r}$, $\tilde{\Gamma}$ acts on $G/G(2)\simeq Z_l^{2g}$. Taking the class of x_i $(1\leq i\leq 2g)$ in $G/G(2)$ as coordinates, we get a representation

$$\tilde{\lambda}:\tilde{\Gamma}\longrightarrow \mathrm{GL}(2g;\ Z_l).$$

PROPOSITION 2. *The representation $\tilde{\lambda}$ induces an exact sequence*

$$1\longrightarrow \tilde{\Gamma}(1)\longrightarrow \tilde{\Gamma}\xrightarrow{\tilde{\lambda}} \mathrm{GS_p}(2g;\ Z_l)\longrightarrow 1,$$

where $\tilde{\Gamma}(1)=\{\sigma\in\tilde{\Gamma}\mid x_i^\sigma\cdot x_i^{-1}\in G(2),\ 1\leq i\leq 2g\}$ *and*

$$\mathrm{GS_p}(2g;\ Z_l)=\left\{A\in \mathrm{GL}(2g;\ Z_l)\ \middle|\ {}^tAJ_gA=\mu(A)J_g,\ \mu(A)\in Z_l^\times,\ J_g=\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix}\right\}.$$

Moreover, for $\sigma\in\tilde{\Gamma}$, we have $z_j^\sigma\sim z_j^{\mu(\tilde{\lambda}(\sigma))}$ $(1\leq j\leq r)$.

PROOF. The fact that the image of $\tilde{\lambda}$ is contained in $\mathrm{GS_p}(2g;\ Z_l)$ and the relation $z_j^\sigma\sim z_j^{\mu(\tilde{\lambda}(\sigma))}$ are easily seen by a calculation modulo $G(3)$ of the effect of σ on the relation

$$[x_1,\ x_{g+1}][x_2,\ x_{g+2}]\cdots[x_g,\ x_{2g}]z_1\cdots z_r=1.$$

The crucial part is to show that the image of $\tilde{\lambda}$ coincides with $\mathrm{GS_p}(2g;\ Z_l)$. As in the proof of Proposition 1 in [2], the essential tool for that is the "successive approximation lemma" presented below. Once established the lemma, the proof of Proposition 2 is totally the same as that of Proposition 1 in [2]. ■

For $A\in \mathrm{GS_p}(2g;\ Z_l)$, let $\boldsymbol{a}_i$ denote the i-th column vector of A $(1\leq i\leq 2g)$ and $x^{\boldsymbol{a}_i}$ denote $x_1^{a_{1i}}x_2^{a_{2i}}\cdots x_{2g}^{a_{2gi}}$, where $\boldsymbol{a}_i={}^t(a_{1i},\ a_{2i},\ \cdots,\ a_{2gi})\in Z_l^{2g}$.

LEMMA 3 (Successive approximation). *Let $m\geq 1$ and $A=(\boldsymbol{a}_i)_{1\leq i\leq 2g}\in \mathrm{GS_p}(2g;\ Z_l)$. Suppose the elements $s_1^{(m)},\ \cdots,\ s_{2g}^{(m)}\in G(2)$ and $t_1^{(m)},\ \cdots,\ t_r^{(m)}\in G$ satisfy a congruence*

$$(\sharp_m)\qquad [s_1^{(m)}x^{\boldsymbol{a}_1},\ s_{g+1}^{(m)}x^{\boldsymbol{a}_{g+1}}]\cdots[s_g^{(m)}x^{\boldsymbol{a}_g},\ s_{2g}^{(m)}x^{\boldsymbol{a}_{2g}}]t_1^{(m)}z_1t_1^{(m)^{-1}}\cdots t_r^{(m)}z_rt_r^{(m)^{-1}}\equiv 1 \mod G(m+2).$$

Then, there exist $s_1, \cdots, s_{2g} \in G(2)$ *and* $t_1, \cdots, t_r \in G$ *such that*

$$s_i \equiv s_i^{(m)} \quad \bmod G(m+1) \ (1 \leq i \leq 2g),$$
$$t_j \equiv t_j^{(m)} \quad \bmod G(m) \ (1 \leq j \leq r),$$

and

$$[s_1 x^{a_1}, s_{g+1} x^{a_{g+1}}] \cdots [s_g x^{a_g}, s_{2g} x^{a_{2g}}] t_1 z_1 t_1^{-1} \cdots t_r z_r t_r^{-1} = 1.$$

PROOF. The proof is similar to that of Lemma 1 in [2]. So consult [2] as for the detail of the following calculation. Now, it suffices to prove that there exist $s_i^{(m+1)} \equiv s_i^{(m)} \bmod G(m+1)$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} \equiv t_j^{(m)} \bmod G(m)$ $(1 \leq j \leq r)$ satisfying the next higher congruence $(\sharp_{m+1})$. Put $s_i^{(m+1)} = S_i s_i^{(m)}$ with $S_i \in G(m+1)$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} = T_j t_j^{(m)}$ with $T_j \in G(m)$ $(1 \leq j \leq r)$. We shall show that we can choose S_i and T_j suitably so that $s_i^{(m+1)}$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)}$ $(1 \leq j \leq r)$ satisfy the congruence $(\sharp_{m+1})$. By the same calculation as in [2], we obtain

$$[s_i^{(m+1)} x^{a_i}, s_{g+i}^{(m+1)} x^{a_{g+i}}] \equiv [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}][s_i^{(m)} x^{a_i}, s_{g+i}^{(m)} x^{a_{g+i}}] \quad \bmod G(m+3),$$
$$t_j^{(m+1)} z_j t_j^{(m+1)^{-1}} \equiv [T_j, z_j] t_j^{(m)} z_j t_j^{(m)^{-1}} \quad \bmod G(m+3).$$

Put

$$\rho = [s_1^{(m)} x^{a_1}, s_{g+1}^{(m)} x^{a_{g+1}}] \cdots [s_g^{(m)} x^{a_g}, s_{2g}^{(m)} x^{a_{2g}}] \times t_1^{(m)} z_1 t_1^{(m)^{-1}} \cdots t_r^{(m)} z_r t_r^{(m)^{-1}} \in G(m+2).$$

Then the left hand side of $(\sharp_{m+1})$ is congruent modulo $G(m+3)$ to

$$\rho \cdot \prod_{i=1}^{g} [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}] \cdot \prod_{j=1}^{r} [T_j, z_j].$$

Since $x_i \bmod G(2)$ $(1 \leq i \leq 2g)$ and $z_j \bmod G(3)$ $(1 \leq j \leq r-1)$ are the generators of grG and since A is invertible, we have

$$gr^{m+2}G = \sum_{i=1}^{g} [x^{a_i} \bmod G(2), gr^{m+1}G] + \sum_{i=1}^{g} [gr^{m+1}G, x^{a_{g+i}} \bmod G(2)]$$
$$+ \sum_{j=1}^{r-1} [gr^m G, z_j \bmod G(3)].$$

Therefore, we can choose $S_1, \cdots, S_{2g}, T_1, \cdots, T_r$ such that the congruence

$$\rho^{-1} \equiv \prod_{i=1}^{g} [x^{a_i}, S_{g+i}][S_i, x^{a_{g+i}}] \cdot \prod_{j=1}^{r} [T_j, z_j] \quad \bmod G(m+3)$$

holds. (Actually we can take $T_r = 1$.) Then, $s_i^{(m+1)} = S_i s_i^{(m)}$ $(1 \leq i \leq 2g)$ and $t_j^{(m+1)} = T_j t_j^{(m)}$ $(1 \leq j \leq r)$ satisfy the congruence $(\sharp_{m+1})$. ■

For $m\geq 1$, put

$$\tilde{\Gamma}_{g,r}(m)=\tilde{\Gamma}(m)=\{\sigma\in\tilde{\Gamma}\mid x_i^{\sigma}\cdot x_i^{-1}\in G(m+1)\ (1\leq i\leq 2g),\ z_j^{\sigma}\overset{m}{\sim} z_j\ (1\leq j\leq r)\}$$

where $\overset{m}{\sim}$ denotes conjugacy by an element in $G(m)$. Let $\tilde{f}_m$ denote the following surjective Z_l-linear homomorphism

$$\tilde{f}_m:(gr^{m+1}G)^{2g}\times(gr^mG)^r\ni(s_i)_{1\leq i\leq 2g}\times(t_j)_{1\leq j\leq r}$$
$$\longmapsto\sum_{i=1}^{g}([\bar{x}_i,s_{g+i}]+[s_i,\bar{x}_{g+i}])+\sum_{j=1}^{r}[t_j,\bar{z}_j]\in gr^{m+2}G$$

where $\bar{x}_i=x_i \bmod G(2)$, $\bar{z}_j=z_j \bmod G(3)$. Assume $m\neq 2$. We can define an injective homomorphism from $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ to $(gr^{m+1}G)^{2g}\times(gr^mG)^r$ by

$$\sigma\longmapsto(x_i^{\sigma}\cdot x_i^{-1})_{1\leq i\leq 2g}\times(t_j)_{1\leq j\leq r},$$

where $z_j^{\sigma}=t_jz_jt_j^{-1}$, $t_j\in G(m)$ $(1\leq j\leq r)$. At this point we use, to confirm that this is well defined, the fact that the centralizer of z_j in G is $\langle z_j\rangle$, the (topologically) cyclic group generated by z_j $(1\leq j\leq r)$. (See [3, p. 55].)

PROPOSITION 4. (1) $[\tilde{\Gamma}(m),\tilde{\Gamma}(n)]\subset\tilde{\Gamma}(m+n)$ $m,n\geq 1$. (2) *Assume* $m\neq 2$. *The* Z_l-*module* $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ *is identified with the kernel of* $\tilde{f}_m$.

COROLLARY. *For* $m\geq 1$, $m\neq 2$, $\tilde{\Gamma}(m)/\tilde{\Gamma}(m+1)$ *is a finitely generated free* Z_l-*module of rank* $2g\rho(m+1)+r\rho(m)-\rho(m+2)$ $(\rho(m)=\mathrm{rank}(gr^mG))$.

PROOF. The proof is essentially the same as that of Theorem 1 in [2]. Successive approximation (Lemma 3) will play the crucial role. We omit the details here. ■

REMARK. With slight modification, the case of $m=2$ can be described similarly and $\tilde{\Gamma}(2)/\tilde{\Gamma}(3)$ turns out to be a finitely generated free Z_l-module whose rank can be given explicitly.

3. Outer automorphism group.

Put

$$\Gamma=\Gamma_{g,r}=\tilde{\Gamma}_{g,r}/\mathrm{Int}\,G_{g,r},\qquad \Gamma(1)=\Gamma_{g,r}(1)=\tilde{\Gamma}_{g,r}(1)/\mathrm{Int}\,G_{g,r},$$

where Int denotes inner automorphism group.

LEMMA 5. $\mathrm{Int}\,G\cap\tilde{\Gamma}(m)=\mathrm{Int}_G\,G(m)$, *where* $\mathrm{Int}_G\,G(m)=\{\sigma\in\mathrm{Int}\,G\mid \exists g\in G(m),\ x^{\sigma}=gxg^{-1},\ \forall x\in G\}$.

PROOF. The inclusion $\supset$ is obvious. Conversely, let $g \in G$ satisfy $[g, x_i] \in G(m+1)$, $[g, z_j] \in G(m+2)$. When g belongs to $G(k)$ for $k \leq m-1$, put $g_k = g \bmod G(k+1)$. Then in grG, $[\bar{x}_i, g_k] = [\bar{z}_j, g_k] = 0$. Since grG is a free Lie algebra generated by $\bar{x}_i$ $(1 \leq i \leq 2g)$ and $\bar{z}_j$ $(1 \leq j \leq r-1)$, we have $g_k = 0$, i.e., $g \in G(k+1)$ ([5, Theorem 5.10]). Hence $g \in G(m)$. ∎

For $m \geq 1$, put

$$\Gamma(m) = \Gamma_{g,r}(m) = (\tilde{\Gamma}_{g,r}(m) \cdot \mathrm{Int}\, G)/\mathrm{Int}\, G.$$

According to Lemma 5, we have

$$\Gamma(m) = \tilde{\Gamma}_{g,r}(m)/\mathrm{Int}_G\, G(m).$$

There is an exact sequence induced from that in Proposition 2;

$$1 \longrightarrow \Gamma(1) \longrightarrow \Gamma \longrightarrow \mathrm{GS_p}(2g;\ \boldsymbol{Z}_l) \longrightarrow 1,$$

and we have the following proposition analogous to Proposition 4.

PROPOSITION 6. (1) $[\Gamma(m), \Gamma(n)] \subset \Gamma(m+n)$ $(m, n \geq 1)$. (2) *Assume* $m \neq 2$. *The* $\boldsymbol{Z}_l$*-module* $\Gamma(m)/\Gamma(m+1)$ *is identified with the kernel of* f_m, *where* f_m *is a homomorphism from* $\{(gr^{m+1}G)^{2g} \times (gr^m G)^r\}/gr^m G$ *to* $gr^{m+2}G$ *induced by* $\tilde{f}_m$. *Here,* $gr^m G$ *is embedded in* $(gr^{m+1}G)^{2g} \times (gr^m G)^r$ *by the map*

$$g \bmod G(m+1) \longmapsto ([g, x_i])_{1 \leq i \leq 2g} \times (g, g, \cdots, g).$$

PROOF. This is a consequence of Proposition 4 and Lemma 5. Again the proof is essentially the same as that of Theorem 2 in [2]. Since G is free, present case is a little easier. ∎

COROLLARY. *For* $m \geq 1$, $m \neq 2$, $\Gamma(m)/\Gamma(m+1)$ *is a finitely generated free* $\boldsymbol{Z}_l$*-module of rank* $2g\rho(m+1) + (r-1)\rho(m) - \rho(m+2)$ $(\rho(m) = \mathrm{rank}(gr^m G))$.

PROOF. It suffices to show that the module $\{(gr^{m+1}G)^{2g} \times (gr^m G)^r\}/gr^m G$ is l-torsion free. Take an element $(s_i)_{1 \leq i \leq 2g} \times (t_j)_{1 \leq j \leq r}$ in $(gr^{m+1}G)^{2g} \times (gr^m G)^r$ such that

$$(ls_i)_{1 \leq i \leq 2g} \times (lt_j)_{1 \leq j \leq r} = ([g, x_i])_{1 \leq i \leq 2g} \times (g, g, \cdots, g)$$

for some $g \in gr^m G$. As $gr^m G$ is torsion free, we must have $t_1 = t_2 = \cdots = t_r$ and $g = lt_j$. Then $ls_i = [lt_j, x_i] = l[t_j, x_i]$. Hence $s_i = [t_j, x_i]$. ∎

4. Pro-l version of the theorem of Dehn-Nielsen.

In this section, we prove the surjectivity of the natural homomorphism

$$\tilde{\Gamma}_{g,r} \longrightarrow \tilde{\Gamma}_{g,s} \qquad (r>s\geq 0).$$

Here, we understand by $\tilde{\Gamma}_{g,0}$ the full automorphism group of the pro-l fundamental group of Riemann surface of genus g, which was studied in [2]. In the classical case, corresponding statement is known as the theorem of Dehn-Nielsen (cf. e.g. [8, 5.6]).

First we define the above homomorphism. Consider the homomorphism $G_{g,r}\to G_{g,s}$ defined by $x_i\mapsto x_i$ $(1\leq i\leq 2g)$, $z_j\mapsto z_j$ $(1\leq j\leq s)$ and $z_j\mapsto 1$ $(s+1\leq j\leq r)$. As each element of $\tilde{\Gamma}_{g,r}$ stabilizes the closed normal subgroup generated normally by $z_{s+1},\cdots,z_r$, the above homomorphism induces a homomorphism $\phi=\phi_{r,s}$;

$$\phi:\tilde{\Gamma}_{g,r}\longrightarrow\tilde{\Gamma}_{g,s}.$$

THEOREM 7 (Pro-l analogue of the theorem of Dehn-Nielsen). *For each $r>s\geq 0$, the homomorphism*

$$\phi:\tilde{\Gamma}_{g,r}\longrightarrow\tilde{\Gamma}_{g,s}$$

is surjective.

PROOF. First we prove the theorem in the case where $r>s\geq 1$. Next, by using some results in [2] and [4], we prove the surjectivity of $\tilde{\Gamma}_{g,1}\to\tilde{\Gamma}_{g,0}$. We follow the argument in [6] where the corresponding result in the case of $g=0$ is proved.

So first assume $r>s\geq 1$. In view of the commutative diagram below, it suffices to show that the induced homomorphism $\phi_1:\tilde{\Gamma}_{g,r}(1)\to\tilde{\Gamma}_{g,s}(1)$ is surjective.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_{g,r}(1) & \longrightarrow & \tilde{\Gamma}_{g,r} & \longrightarrow & GS_p(2g;\ Z_l) & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\mathrm{id.}} & & \\
1 & \longrightarrow & \tilde{\Gamma}_{g,s}(1) & \longrightarrow & \tilde{\Gamma}_{g,s} & \longrightarrow & GS_p(2g;\ Z_l) & \longrightarrow & 1.
\end{array}$$

For that purpose, we only need to check that the induced homomorphisms

$$gr^m\phi_1:\tilde{\Gamma}_{g,r}(m)/\tilde{\Gamma}_{g,r}(m+1)\longrightarrow\tilde{\Gamma}_{g,s}(m)/\tilde{\Gamma}_{g,s}(m+1)$$

are surjective for all $m\geq 1$. Consider the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Gamma}_{g,r}(m)/\tilde{\Gamma}_{g,r}(m+1) & \longrightarrow & (gr^{m+1}G_{g,r})^{2g}\times(gr^mG_{g,r})^r & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,r} & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle gr^m\phi_1} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\
1 & \longrightarrow & \tilde{\Gamma}_{g,s}(m)/\tilde{\Gamma}_{g,s}(m+1) & \longrightarrow & (gr^{m+1}G_{g,s})^{2g}\times(gr^mG_{g,s})^s & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,s} & \longrightarrow & 1
\end{array}$$

where α and β are naturally induced homomorphisms. By the snake lemma, the sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \operatorname{coker}(gr^m\phi_1) \longrightarrow 1$$

is exact. Hence our task is to show that $\ker\alpha \to \ker\beta$ is surjective. We let denote by $\mathfrak{A}_m$ the kernel of the surjective homomorphism $gr^m G_{g,r} \to gr^m G_{g,s}$ $(m\geq 1)$. Then $\ker\alpha$ (resp. $\ker\beta$) is equal to $\mathfrak{A}_{m+1}^{2g}\oplus\mathfrak{A}_m^s\oplus(gr^m G_{g,r})^{r-s}$ (resp. $\mathfrak{A}_{m+2}$). Since the ideal $\mathfrak{A}=\bigoplus_{m\geq 1}\mathfrak{A}_m$ is generated by $z_{s+1},\cdots,z_r$ and grG is generated by x_i $(1\leq i\leq 2g)$ and z_j $(1\leq j\leq r-1)$, we have

$$\mathfrak{A}_{m+2}=[\mathfrak{A}_{m+1}, gr^1 G]+\sum_{j=1}^{s}[\mathfrak{A}_m, z_j]+\sum_{j=s+1}^{r}[gr^m G, z_j].$$

From this and the definition of the map $\tilde{f}_m$, we conclude that $\ker\alpha\to\ker\beta$ is surjective.

Next we consider the case where $r=1$, $s=0$. As similar as the case treated above, we need to look at the following diagram and to show that $\ker\alpha\to\ker\beta$ is surjective:

$$\begin{array}{ccccccccc}
1 \longrightarrow & \tilde{\Gamma}_{g,1}(m)/\tilde{\Gamma}_{g,1}(m+1) & \longrightarrow & (gr^{m+1}G_{g,1})^{2g}\times gr^m G_{g,1} & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,1} & \longrightarrow 1 \\
& \Big\downarrow{\scriptstyle gr^m\phi_1} & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & \\
1 \longrightarrow & \tilde{\Gamma}_{g,0}(m)/\tilde{\Gamma}_{g,0}(m+1) & \longrightarrow & (gr^{m+1}G_{g,0})^{2g} & \xrightarrow{\tilde{f}_m} & gr^{m+2}G_{g,0} & \longrightarrow 1.
\end{array}$$

Let $\mathfrak{A}_m$ be the kernel of the homomorphism $gr^m G_{g,1}\to gr^m G_{g,0}$. Also in this case, owing to a theorem of J. Labute [4], the ideal $\mathfrak{A}=\bigoplus_{m\geq 1}\mathfrak{A}_m$ is generated by $z_1=[x_1, x_{g+1}]+\cdots+[x_g, x_{2g}]$. Hence the argument as above also works in this case. ■

As a direct consequence of the above theorem and Theorem 3 in [2], we get the following

COROLLARY. *Suppose $g\geq 3$. Let A be an element of* $\mathrm{GS_p}(2g; \mathbf{Z}_l)$ *satisfying the following conditions:*

$$A\equiv 1_{2g}\begin{cases}\bmod l & l\neq 2\\ \bmod l^2 & l=2,\end{cases}$$

and C be the $\mathrm{GS_p}(2g; \mathbf{Z}_l)$*-conjugacy class of A. Then $\lambda^{-1}(C)$ contains more than one $\Gamma_{g,r}$-conjugacy class. Here, λ is the map induced from the action of $\Gamma_{g,r}$ on $G_{g,r}/G_{g,r}(2)\simeq\mathbf{Z}_l^{2g}$.*

REMARK. By a result of M. Asada [1], the above corollary holds for $g=2$ in a slightly weaker form.

References

[1] Asada, M., Indistinguishability of conjugacy classes of the pro-l mapping class group, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 256-259.
[2] Asada, M. and M. Kaneko, On the automorphism group of some pro-l fundamental groups, Adv. Studies in Pure Math. **12** (1987), 137-159.
[3] Ihara, Y., Profinite braid groups, Galois representations and complex multiplications, Ann. of Math. **123** (1986), 43-106.
[4] Labute, J., On the descending central series of groups with a single defining relation, J. Algebra **14** (1970), 16-23.
[5] Magnus, W., Karrass, A. and D. Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
[6] Oda, T., Two propositions on pro-l braid groups, preprint.
[7] Witt, E., Treue Darstellung Liescher Ringe, J. Reine Angew. Math. **177** (1937), 152-160.
[8] Ziescang, H., Vogt, E. and H. D. Coldewey, Surfaces and planar discontinuous groups, Lecture Notes in Math. vol. 835, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

(Received November 10, 1988)

Department of Mathematics
Osaka University
Toyonaka, Osaka
560 Japan