L-S categories of simply-connected compact simple Lie groups of low rank

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Abstract. We determine the L-S category of $Sp(3)$ by showing that the 5-fold reduced diagonal $\overline{\Delta}_5$ is given by ν^2, using a Toda bracket and a generalised cohomology theory h^* given by $h^*(X, A) = \{X/A, \mathcal{S}[0, 2]\}$, where $\mathcal{S}[0, 2]$ is the 3-stage Postnikov piece of the sphere spectrum $\mathcal{S}$. This method also yields a general result that $\mathrm{cat}(Sp(n)) \geq n + 2$ for $n \geq 3$, which improves the result of Singhof [**21**].

1. Introduction

In this paper, we firstly discuss the L-S category of G_2 as in Theorem 1.1 to illustrate the methods to be used later in the argument for $Sp(3)$. Secondly, we prove $\mathrm{cat}(Sp(3)) = 5$ as in Theorem 1.2, although an alternative proof of it can be deduced from public sources by Lucía Fernández-Suárez, Antonio Gómez-Tato, Jeffrey Strom and Daniel Tanré [**6**]; the earlier version, however, appeared to the authors to contain an error ([**5**]). In fact, this is our starting point and motivation to write the present paper with a short and clear proof for $\mathrm{cat}(Sp(3)) = 5$. Finally we show that this argument for $Sp(3)$ partially extends to the general case as in Theorem 1.4.

From now on, each space is assumed to have the homotopy type of a CW complex. The (normalised) L-S category of X is the least number m such that there is a covering of X by $(m+1)$ open subsets each of which is contractible in X. Hence $\mathrm{cat}\,\{*\} = 0$. By Lusternik and Schnirelmann [**14**], the number of critical points of a smooth function on a manifold M is bounded below by $\mathrm{cat}\, M + 1$.

G. Whitehead showed that $\mathrm{cat}(X)$ coincides with the least number m such that the diagonal map $\Delta_{m+1} : X \to \prod^{m+1} X$ can be compressed into the 'fat wedge' $\mathrm{T}^{m+1}(X)$ (see Chapter X of [**24**]). Since $\prod^{m+1} X/\,\mathrm{T}^{m+1}(X)$ is the $(m+1)$-fold smash product $\wedge^{m+1}X$, we have a weaker invariant $w\mathrm{cat}\, X$, *the weak L-S category of* X, given by the least number m such that the reduced diagonal map $\overline{\Delta}_{m+1} : X \to \wedge^{m+1}X$ is trivial. Hence $w\mathrm{cat}\, X \leq \mathrm{cat}\, X$.

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T. Ganea has also introduced a stronger invariant $\mathrm{Cat}\,X$, *the strong L-S category of* X, by the least number m such that there is a space Y homotopy equivalent to X and a covering of Y by $(m+1)$ open subsets each of which is contractible in itself. Thus $w\mathrm{cat}\,X \le \mathrm{cat}\,X \le \mathrm{Cat}\,X$. The weak and strong L-S categories usually give nice estimates of L-S category especially for manifolds. Actually, we do not know any example of a closed manifold whose strong L-S, L-S and weak L-S categories are not the same. The following problems are posed by Ganea [**7**]:

i) (Problem 1) Determine the L-S category of a manifold.
ii) (Problem 4) Describe the L-S category of a sphere-bundle over a sphere in terms of homotopy invariants of the characteristic map of the bundle.

Problem 1 has been studied by many authors, such as Singhof [**20, 21, 22**], Montejano [**16**], Schweizer [**19**], Gomez-Larrañaga and Gonzalez-Acuña [**8**], James and Singhof [**13**] and Rudyak [**17, 18**]. In particular for compact simply-connected simple Lie groups, $\mathrm{cat}(SU(n+1)) = n$ for $n \ge 1$ by [**20**], $\mathrm{cat}(Sp(2)) = 3$ by [**19**] and $\mathrm{cat}(Sp(n)) \ge n+1$ for $n \ge 2$ by [**21**]. It was also announced recently that Problem 4 was solved by the first-named author [**10**].

The method in the present paper also provides a result for G_2, and thus we have the following result.

Theorem 1.1. *The following is the complete list of L-S categories of a simply-connected compact simple Lie group of rank* ≤ 2*:*

Lie groups	$Sp(1)=SU(2)=Spin(3)$	$SU(3)$	$Sp(2)=Spin(5)$	G_2
$w\mathrm{cat}$	1	2	3	4
cat	1	2	3	4
Cat	1	2	3	4

Although the above result is known for experts, we give a short proof for G_2. In fact, the result for G_2 has never been published and is obtained in a similar but easier manner than the following result for $Sp(3)$:

Theorem 1.2. $w\mathrm{cat}(Sp(3)) = \mathrm{cat}(Sp(3)) = \mathrm{Cat}(Sp(3)) = 5.$

Remark 1.3. *The argument given to prove Theorem 1.2 provides an alternative proof of Schweizer's result*

$$w\mathrm{cat}(Sp(2)) = \mathrm{cat}(Sp(2)) = \mathrm{Cat}(Sp(2)) = 3.$$

The authors know that a similar result to Theorem 1.2 is obtained by Lucía Fernández-Suárez, Antonio Gómez-Tato, Jeffrey Strom and Daniel Tanré [**6**]. Our method is, however, much simpler and provides the following general result:

Theorem 1.4. $n+2 \le w\mathrm{cat}(Sp(n)) \le \mathrm{cat}(Sp(n)) \le \mathrm{Cat}(Sp(n))$ *for* $n \ge 3$.

This improves Singhof's result: $\mathrm{cat}(Sp(n)) \ge n+1$ for $n \ge 2$. We propose the following conjecture.

Conjecture 1.5. *Let* G *be a simply-connected compact Lie group with* $G = \prod_{i=1}^{n} H_i$ *where* H_i *is a simple Lie group. Then* $w\mathrm{cat}(G) = \mathrm{cat}(G) = \mathrm{Cat}(G)$ *and* $\mathrm{cat}(G) = \sum_{i=1}^{n} \mathrm{cat}(H_i)$.

It might be difficult to say something about $\mathrm{cat}\,Sp(n)$, but an old conjecture says the following.

Conjecture 1.6. $\mathrm{cat}\,Sp(n) = 2n-1$ *for all* $n \ge 1$.

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2. Proof of Theorem 1.1

Let us recall a CW decomposition of G_2 from [**15**]:

$$G_2 = e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

On the other hand, we have the following cone-decomposition.

THEOREM 2.1. *There is a cone-decomposition of* G_2 *as follows:*

$$G_2^{(5)} = \Sigma\mathbb{C}P^2, \quad S^5 \cup e^7 \to G_2^{(5)} \hookrightarrow G_2^{(8)},$$
$$S^8 \cup e^{10} \to G_2^{(8)} \hookrightarrow G_2^{(11)}, \quad S^{13} \to G_2^{(11)} \hookrightarrow G_2.$$

Proof. The first and the last formulae are obvious. So we show the 2nd and 3rd formulae: By taking the homotopy fibre F_1 of $G_2^{(5)} \hookrightarrow G_2$, we can easily observe using the Serre spectral sequence that the fibre has a CW structure given by $S^5 \cup e^7 \cup$ (cells in dimensions ≥ 7), where the cohomology generators corresponding to S^5 and e^7 are transgressive. Thus the mapping cone of $S^5 \cup e^7 \subset F_1 \to G_2^{(5)}$ has the homotopy type of $G_2^{(8)}$. Similarly, the homotopy fibre F_2 of $G_2^{(8)} \hookrightarrow G_2$ has a CW structure given by $S^8 \cup e^{10} \cup$ (cells in dimensions ≥ 10), where the cohomology generators corresponding to S^8 and e^{10} are transgressive. Thus the mapping cone of $S^8 \cup e^{10} \subset F_2 \to G_2^{(8)}$ has the homotopy type of $G_2^{(11)}$. *QED.*

COROLLARY 2.1.1. $1 \geq \mathrm{Cat}(G_2^{(5)}) \geq \mathrm{Cat}(G_2^{(3)})$, $2 \geq \mathrm{Cat}(G_2^{(8)}) \geq \mathrm{Cat}(G_2^{(6)})$, $3 \geq \mathrm{Cat}(G_2^{(11)}) \geq \mathrm{Cat}(G_2^{(9)})$ *and* $4 \geq \mathrm{Cat}(G_2)$.

Let us recall the following well-known fact due to Borel.

FACT 2.2. $H^*(G_2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x_3, x_5]/(x_3^4, x_5^2)$.

COROLLARY 2.2.1. $w\mathrm{cat}(G_2^{(5)}) \geq w\mathrm{cat}(G_2^{(3)}) \geq 1$, $w\mathrm{cat}(G_2^{(8)}) \geq w\mathrm{cat}(G_2^{(6)}) \geq 2$, $w\mathrm{cat}(G_2^{(11)}) \geq w\mathrm{cat}(G_2^{(9)}) \geq 3$ *and* $w\mathrm{cat}(G_2) \geq 4$.

Corollaries 2.1.1 and 2.2.1 yield the following.

THEOREM 2.3.

Skeleta	$G_2^{(3)}$	$G_2^{(5)}$	$G_2^{(6)}$	$G_2^{(8)}$	$G_2^{(9)}$	$G_2^{(11)}$	G_2
$w$cat	1	1	2	2	3	3	4
cat	1	1	2	2	3	3	4
Cat	1	1	2	2	3	3	4

This completes the proof of Theorem 1.1.

REMARK 2.4. *If we disregard the information of L-S categories of CW filtrations of* G_2 *and if we want only to deduce the equation* $w\mathrm{cat}(G_2) = \mathrm{cat}(G_2) = \mathrm{Cat}(G_2) = 4$, *we have an alternative short proof of it rather than the above elementary homotopy-theoretical argument: Since the manifold* G_2 *is* 2*-connected and of dimension* 14, *we know that* $\mathrm{cat}(G_2) \leq \dfrac{14}{3}$ *by James* [**11**]. *On the other hand, the cohomology algebra of* G_2 *with coefficients in* $\mathbb{F}_2$ *is well-known by Borel as in Fact 2.2, and hence its cup-length is* 4 *and we get immediately that* $w\mathrm{cat}(G_2) = \mathrm{cat}(G_2) = 4$. *Concerning on the strong L-S category* $\mathrm{Cat}(G_2)$ *of a manifold* G_2, *we are in the range of validity of Corollary 5.9 of Clapp and Puppe* [**3**] *which implies immediately that* $\mathrm{cat}(G_2) = \mathrm{Cat}(G_2)$.

3. The ring structure of $h^*(Sp(3))$

To show Theorem 1.2, we introduce a cohomology theory $h^*(-)$ such that $h^*(X, A) = \{X/A, \mathcal{S}[0, 2]\}$, where $\mathcal{S}[0, 2]$ is the spectrum obtained from the sphere spectrum $\mathcal{S}$ by killing all homotopy groups of dimensions bigger than 2. Then $\mathcal{S}[0, 2]$ is a ring spectrum with $\pi^S_*(\mathcal{S}[0, 2]) \cong \mathbb{Z}[\eta]/(\eta^3, 2\eta)$, where η is the Hopf element in $\pi^S_1(\mathcal{S}) = \pi^S_1(\mathcal{S}[0, 2])$. Thus h^* is an additive and multiplicative cohomology theory with $h^* = h^*(pt) \cong \mathbb{Z}[\varepsilon]/(\varepsilon^3, 2\varepsilon)$, $\deg \varepsilon = -1$, where $\varepsilon \in h^{-1} = \pi^S_0(\Sigma^{-1}\mathcal{S}) \cong \pi^S_1(\mathcal{S})$ corresponds to η.

The characteristic map of the principal $Sp(1)$-bundle

$$Sp(1) \hookrightarrow Sp(2) \to S^7$$

is given by $\omega = \langle \iota_3, \iota_3\rangle : S^6 \to Sp(1) \approx S^3$ the Samelson product of two copies of the identity $\iota_3 : S^3 \to S^3$, which is a generator of $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$. We state the following well-known fact (see Whitehead [**24**]).

Fact 3.1. *Let $\mu : S^3{\times}S^3 \to S^3$ be the multiplication of $Sp(1) \approx S^3$. Then we have*

$$Sp(2) \simeq S^3 \cup_{\mu\circ(1\times\omega)} S^3{\times}C(S^6) = S^3 \cup_\omega C(S^6) \cup_{\hat\mu\circ[\iota_3,\omega]^r} C(S^9),$$

where $\hat\mu : S^3{\times}S^3 \cup_{*\times\omega} \{*\}{\times}C(S^6) \to S^3 \cup_\omega C(S^6)$ is given by $\hat\mu|_{S^3\times S^3} = \mu$ and $\hat\mu|_{S^3\cup_\omega C(S^6)} = 1$ the identity and $[\iota_3, \chi_\omega]^r : S^9 \to S^3{\times}S^3 \cup_{*\times\omega} \{*\}{\times}C(S^6)$ is the relative Whitehead product of the identity $\iota_3 : S^3 \to S^3$ and the characteristic map $\chi_\omega : (C(S^6), S^6) \to (S^3 \cup e^7, S^3)$ of the 7-cell. Thus we have $1 \geq \mathrm{Cat}(Sp(2)^{(3)})$, $2 \geq \mathrm{Cat}(Sp(2)^{(7)})$ and $3 \geq \mathrm{Cat}(Sp(2))$.

Let $\nu : S^7 \to S^4$ be the Hopf element whose suspension $\nu_n = \Sigma^{n-4}\nu$ $(n \geq 4)$ gives a generator of $\pi_{n+3}(S^n) \cong \mathbb{Z}/24\mathbb{Z}$ for $n \geq 5$. Then we remark that $\omega_n = \Sigma^{n-3}\omega$ $(n \geq 3)$ satisfies the formula $\omega_n = 2\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 5$. By Zabrodsky [**25**], there is a natural splitting

$$\Sigma(S^3{\times}S^3 \cup \{*\}{\times}(S^3 \cup_\omega e^7)) \simeq \Sigma S^3 \vee \Sigma(S^3 \cup_\omega e^7) \vee \Sigma S^3{\wedge}S^3.$$

Then by the definition of a relative Whitehead product, the composition of $[\iota_3, \omega]^r$ with the projections to S^3 and $S^3 \cup_\omega e^7$ are trivial and the composition with the projection to $S^3{\wedge}S^3$ is given by $\iota_3{\wedge}\omega$. Thus we have

$$\Sigma(\hat\mu\circ[\iota_3, \omega]^r) = H(\mu)\circ\Sigma(\iota_3{\wedge}\omega) = \pm\nu\circ\omega_7 = 2\nu\circ\nu_7 \neq 0$$

in $\pi_{10}(S^4) \cong \mathbb{Z}/24\mathbb{Z}\langle\nu\circ\nu_7\rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle\omega_4\circ\nu_7\rangle$, and hence we have

$$\Sigma^2(\hat\mu\circ[\iota_3, \omega]^r) = \nu_5\circ\omega_8 = 2\nu_5^2 = 0 \in \pi_{11}(S^5) \cong \mathbb{Z}/2\mathbb{Z}$$

by Proposition 5.11 of Toda [**23**]. Thus we have the following well-known facts.

Fact 3.2. *We have the following homotopy equivalences:*

$$Sp(2)/S^3 \simeq (S^3{\times}C(S^6))/(S^3{\times}S^6) = S^3_+{\wedge}\Sigma(S^6) = S^7 \vee S^{10},$$
$$\Sigma^2 Sp(2) \simeq \Sigma^2(S^3 \cup_\omega C(S^6)) \vee \Sigma^2 S^{10} = S^5 \cup_{\omega_5} C(S^8) \vee S^{12}.$$

Fact 3.3. *The 11-skeleton $X^{(11)}_{3,2}$ of $X_{3,2} = Sp(3)/Sp(1)$ has the homotopy type of $S^7 \cup_{\nu_7} e^{11}$.*

Restricting the principal $Sp(1)$-bundle $Sp(1) \hookrightarrow Sp(3) \xrightarrow{q} X_{3,2}$ to the subspace $X^{(11)}_{3,2} = S^7 \cup_{\nu_7} e^{11}$ of $X_{3,2}$, we obtain the subspace $q^{-1}(X^{(11)}_{3,2}) = Sp(3)^{(14)}$ of $Sp(3)$ as the total space of the principal $Sp(1)$-bundle $Sp(1) \hookrightarrow Sp(3)^{(14)} \xrightarrow{q} \Sigma(S^6 \cup_{\nu_6} e^{10})$

with a characteristic map $\phi : S^6 \cup_{\nu_6} e^{10} \to Sp(1) \approx S^3$, which is an extension of $\omega : S^6 \to S^3$.

Proposition 3.4. *We have the following homotopy equivalences:*

$$\begin{aligned}
Sp(3)^{(14)} &\simeq S^3 \cup_{\mu\circ(1\times\phi)} S^3{\times}C(S^6 \cup_{\nu_6} e^{10})\\
&= S^3 \cup_\phi C(S^6 \cup_{\nu_6} e^{10}) \cup C(S^9 \cup_{\nu_9} e^{13}),\\
Sp(3)^{(14)}/S^3 &\simeq (S^3{\times}C(S^6 \cup_{\nu_6} e^{10}))/(S^3{\times}(S^6 \cup_{\nu_6} e^{10}))\\
&= S^3_+{\wedge}\Sigma(S^6 \cup_{\nu_6} e^{10}) = (S^7 \cup_{\nu_7} e^{11}) \vee (S^{10} \cup_{\nu_{10}} e^{14}),\\
Sp(n) &\simeq Sp(n-1) \cup Sp(n-1){\times}C(S^{4n-2}),
\end{aligned}$$

where $Sp(n-1) \subset Sp(n)^{((2n+1)n-11)}$ *for* $n \geq 3$*, and hence*

$$\begin{aligned}
&Sp(n)/Sp(n)^{((2n+1)n-11)}\\
&\quad\simeq (Sp(n-1){\times}C(S^{4n-2}))/(Sp(n-1){\times}S^{4n-2}\\
&\qquad\qquad \cup Sp(n-1)^{((2n-1)(n-1)-11)}{\times}C(S^{4n-2}))\\
&\quad= (Sp(n-1)/Sp(n-1)^{((2n-1)(n-1)-11)}){\wedge}\Sigma S^{4n-2}\\
&\quad= \cdots = (Sp(2)/\emptyset){\wedge}\Sigma S^{10}{\wedge}\cdots{\wedge}\Sigma S^{4n-2} = (Sp(2)_+){\wedge}S^{(2n+1)n-10}\\
&\quad= S^{(2n+1)n-10} \vee S^{(2n+1)n-10}{\wedge}Sp(2)\\
&\quad= S^{(2n+1)n-10} \vee (S^{(2n+1)n-7} \cup_{\omega_{(2n+1)n-7}} e^{(2n+1)n-3}) \vee S^{(2n+1)n}, \quad \text{for } n \geq 3.
\end{aligned}$$

This yields the following result.

Proposition 3.5. *Let* $\hat{\mu} : S^3{\times}S^3 \cup_{*\times\phi} \{*\}{\times}(S^3 \cup_\phi C(S^6 \cup_{\nu_6} e^{10})) \to S^3 \cup_\phi C(S^6 \cup_{\nu_6} e^{10})$ *be the map given by* $\hat{\mu}|_{S^3\times S^3} = \mu$ *and* $\hat{\mu}|_{S^3\cup_\phi C(S^6\cup_{\nu_6}e^{10})} = 1$ *the identity. Then we have the following cone decomposition of* $Sp(3)$*:*

$$Sp(3) \simeq S^3 \cup_\phi C(S^6 \cup_{\nu_6} e^{10}) \cup_{\hat{\mu}\circ\hat{\phi}} C(S^9 \cup_{\nu_9} e^{13}) \cup C(S^{17}) \cup C(S^{20}).$$

Corollary 3.5.1. $1 \geq \mathrm{Cat}(Sp(3)^{(3)})$, $2 \geq \mathrm{Cat}(Sp(3)^{(7)})$, $3 \geq \mathrm{Cat}(Sp(3)^{(14)}) \geq \mathrm{Cat}(Sp(3)^{(11)}) \geq \mathrm{Cat}(Sp(3)^{(10)})$, $4 \geq \mathrm{Cat}(Sp(3)^{(18)})$ *and* $5 \geq \mathrm{Cat}(Sp(3))$.

To determine the ring structures of $h^*(Sp(2))$ and $h^*(Sp(3))$, we show the following lemma.

Lemma 3.6. *Let* h^* *be any multiplicative generalised cohomology theory and let* $Q = S^r \cup_f e^q$ *for a given map* $f : S^{q-1} \to S^r$ *with* $h^*(Q) \cong h^*\langle 1, x, y\rangle$*, where* x *and* y *correspond to the generators of* $h^*(S^r) \cong h^*\langle x_0\rangle$ *and* $h^*(S^q) \cong h^*\langle y_0\rangle$*. Then*

$$x^2 = \pm\bar{H}^h_1(f){\cdot}y \quad in \quad h^*(Q),$$

where $\bar{H}^h_1$ *is the composition* $\rho^h{\circ}\lambda_2$ *of the Boardman-Steer Hopf invariant* $\lambda_2 : \pi_{q-1}(S^r) \to \pi_q(S^{2r})$ *(see Boardman and Steer* [**2**]*) with the Hurewicz homomorphism* $\rho^h : \pi_q(S^{2r}) \to h^{2r}(S^q) \cong h^{2r-q}$ *given by* $\rho^h(g) = \Sigma^{-q}_* g^*(x_0{\otimes}x_0)$*.*

Remark 3.7. *By* [**2**], $\lambda_2(f)$ *is equal to* $\Sigma h^J_2(f)$ *the suspension of the 2nd James Hopf invariant* $h^J_2(f)$*. Hence by Remarks 2.5 and 4.3 of* [**9**], $\lambda_2(f) = \Sigma h_2(f)$ *gives the Berstein-Hilton crude Hopf invariant* $\bar{H}_1(f)$ *(see Berstein-Hilton* [**1**] *or* [**9**]*).*

Proof. By [**2**], $\overline{\Delta} : Q = S^r \cup_f e^q \to Q \wedge Q$ equals the composition $(i_Q \wedge i_Q) \circ \lambda_2(f) \circ q_Q$, where $q_Q : Q \to Q/S^r = S^q$ is the collapsing map and $i_Q : S^r \hookrightarrow Q$ is the bottom-cell inclusion. Then we have $i_Q^*(x) = x_0$ and $q_Q^*(y_0) = y$, and hence we obtain

$$\begin{aligned} x^2 &= \overline{\Delta}^*(x \otimes x) = ((i_Q \wedge i_Q) \circ \lambda_2(f) \circ q_Q)^*(x \otimes x) \\ &= q_Q^*(\lambda_2(f)^*(i_Q^*(x) \otimes i_Q^*(x))) = q_Q^*(\lambda_2(f)^*(x_0 \otimes x_0)) = q_Q^*(\Sigma_*^q \circ \rho^h(\lambda_2(f))). \end{aligned}$$

Since $\Sigma_*^q \circ \rho^h(\lambda_2(f))$ is $\bar{H}_1^h(f) \cdot y_0 \in h^{2r}(S^q)$ up to sign, we proceed as

$$x^2 = q_Q^*(\pm \bar{H}_1^h(f) \cdot y_0) = \pm \bar{H}_1^h(f) \cdot q_Q^*(y_0) = \pm \bar{H}_1^h(f) \cdot y.$$

This completes the proof of the lemma. *QED.*

Using cohomology long exact sequences derived from the cell structure of $Sp(3)$ and a direct calculation using Proposition 3.4 and Lemma 3.6 with the fact that $\lambda_2(\omega) = \eta_6$, we deduce the following result for the cohomology theory h^* considered at the beginning of this section.

THEOREM 3.8. *The ring structures of $h^*(Sp(2))$ and $h^*(Sp(3))$ are as follows:*

$$\begin{aligned} h^*(Sp(2)) &\cong h^*\{1, x_3, x_7, y_{10}\}, \\ h^*(Sp(3)) &\cong h^*\{1, x_3, x_7, x_{11}, y_{10}, y_{14}, y_{18}, z_{21}\} \end{aligned}$$

where the suffix of each additive generator indicates its degree in the graded algebras $h^*(Sp(2))$ and $h^*(Sp(3))$. Moreover we have $x_3^2 = \varepsilon \cdot x_7$, $x_7^2 = 0$, $x_{11}^2 = 0$, $x_3 x_7 = y_{10}$, $x_3 x_{11} = y_{14}$, $x_7 x_{11} = y_{18}$ and $x_3 x_7 x_{11} = z_{21}$.

REMARK 3.9. *The two possible attaching maps : $S^{10} \to S^3 \cup_\omega e^7$ of e^{11} found by Lucía Fernández-Suárez, Antonio Gómez-Tato and Daniel Tanré* [**4**] *are homotopic in $Sp(2)$. So, we can not make any effective difference in the ring structure of $h^*(Sp(3))$ by altering, as is performed in* [**6**]*, the attaching map of e^{11}.*

COROLLARY 3.9.1. *$\mathrm{wcat}(Sp(3)^{(3)}) \geq 1$, $\mathrm{wcat}(Sp(3)^{(7)}) \geq 2$, $\mathrm{wcat}(Sp(3)^{(18)}) \geq \mathrm{wcat}(Sp(3)^{(14)}) \geq \mathrm{wcat}(Sp(3)^{(11)}) \geq \mathrm{wcat}(Sp(3)^{(10)}) \geq 3$ and $\mathrm{wcat}(Sp(3)) \geq 4$, together with $\mathrm{wcat}(Sp(2)^{(3)}) \geq 1$, $\mathrm{wcat}(Sp(2)^{(7)}) \geq 2$ and $\mathrm{wcat}(Sp(2)) \geq 3$.*

COROLLARY 3.9.2.

Skeleta	$Sp(2)^{(3)}$	$Sp(2)^{(7)}$	$Sp(2)$
$w$cat	1	2	3
cat	1	2	3
Cat	1	2	3

4. Proof of Theorem 1.2

By Facts 3.1 and 3.2, the smash products $\wedge^4 Sp(3)$ and $\wedge^5 Sp(3)$ satisfy

$$(\wedge^4 Sp(3))^{(19)} \simeq S^{12} \cup_{\omega_{12}} e^{16} \vee (S^{16} \vee S^{16} \vee S^{16}) \vee (S^{19} \vee S^{19} \vee S^{19} \vee S^{19}),$$

$$(\wedge^5 Sp(3))^{(22)} \simeq S^{15} \cup_{\omega_{15}} e^{19} \vee (S^{19} \vee S^{19} \vee S^{19}) \vee (S^{22} \vee S^{22} \vee S^{22} \vee S^{22}).$$

Then we have the following two propositions.

PROPOSITION 4.1. *The bottom-cell inclusions $i : S^{12} \hookrightarrow \wedge^4 Sp(3)^{(18)}$ and $i' : S^{15} \hookrightarrow \wedge^5 Sp(3)$ induce injective homomorphisms*

$$i_* : \pi_{18}(S^{12}) \to \pi_{18}(\wedge^4 Sp(3)^{(18)}) \quad \text{and} \quad i'_* : \pi_{21}(S^{15}) \to \pi_{21}(\wedge^5 Sp(3)),$$

respectively.

Proof. We have the following two exact sequences

$$\pi_{18}(S^{15}) \xrightarrow{\psi} \pi_{18}(S^{12}) \xrightarrow{i_*} \pi_{18}(\wedge^4 Sp(3)^{(18)}) \to \pi_{18}(S^{16}\vee S^{16}\vee S^{16}\vee S^{16}),$$

$$\pi_{21}(S^{18}) \xrightarrow{\psi'} \pi_{21}(S^{15}) \xrightarrow{i'_*} \pi_{21}(\wedge^5 Sp(3)) \to \pi_{21}(S^{19}\vee S^{19}\vee S^{19}\vee S^{19}\vee S^{19}),$$

where $\pi_{18}(S^{12}) \cong \pi_{21}(S^{15}) \cong \mathbb{Z}/2\mathbb{Z}\nu_{15}^2$ and ψ and ψ' are induced from $\omega_{12} = 2\nu_{12}$ and $\omega_{15} = 2\nu_{15}$. Thus ψ and ψ' are trivial, and hence i_* and i'_* are injective. *QED.*

Proposition 4.2. *The collapsing maps $q : Sp(3)^{(18)} \to Sp(3)^{(18)}/Sp(3)^{(14)} = S^{18}$ and $q' : Sp(3) \to Sp(3)/Sp(3)^{(18)} = S^{21}$ induce injective homomorphisms*

$q^* : \pi_{18}(\wedge^4 Sp(3)^{(18)}) \to [Sp(3)^{(18)}, \wedge^4 Sp(3)^{(18)}]$ *and*
$q'^* : \pi_{21}(\wedge^5 Sp(3)) \to [Sp(3), \wedge^5 Sp(3)],$

respectively.

Proof. Firstly, we show that q'^* is injective: Since we have $[Sp(3), \wedge^5 Sp(3)] = [(S^{14}\cup_{\omega_{14}} e^{18})\vee S^{21}, \wedge^5 Sp(3)] = [S^{14}\cup_{\omega_{14}} e^{18}, \wedge^5 Sp(3)]\oplus\pi_{21}(\wedge^5 Sp(3))$ by Proposition 3.4, q'^* is clearly injective.

Secondly, we show that q^* is injective: Similarly we have $[Sp(3)^{(18)}, \wedge^4 Sp(3)^{(18)}] = [S^{14} \cup_{\omega_{14}} e^{18}, \wedge^4 Sp(3)^{(18)}]$ by Proposition 3.4. Thus it is sufficient to show that $\bar{q}^* : \pi_{18}(\wedge^4 Sp(3)^{(18)}) \to [S^{14} \cup_{\omega_{14}} e^{18}, \wedge^4 Sp(3)^{(18)}]$ is injective, where $\bar{q} : S^{14} \cup_{\omega_{14}} e^{18} \to S^{18}$ is the collapsing map. In the exact sequence

$$\pi_{15}(\wedge^4 Sp(3)^{(18)}) \xrightarrow{\omega_{15}{}^*} \pi_{18}(\wedge^4 Sp(3)^{(18)}) \xrightarrow{\bar{q}^*} [S^{14} \cup_{\omega_{14}} e^{18}, \wedge^4 Sp(3)^{(18)}],$$

we know that $\pi_{15}(\wedge^4 Sp(3)^{(18)}) \cong \pi_{15}(S^{12} \cup_{\omega_{12}} e^{16}) = \mathbb{Z}/2\mathbb{Z}$ is generated by the composition of ν_{12} and the bottom-cell inclusion. Since $\nu_{12}\circ\omega_{15} = 0 \in \pi_{18}(S^{12})$, the homomorphism $\omega_{15}{}^*$ is trivial, and hence $\bar{q}^*$ is injective. *QED.*

Then the following lemma implies that $\overline{\Delta}_4$ and $\overline{\Delta}_5$ are non-trivial by Propositions 4.1 and 4.2.

Lemma 4.3. *We obtain that $\overline{\Delta}_4 = i\circ\nu_{12}^2\circ q : Sp(3)^{(18)} \to \wedge^4 Sp(3)^{(18)}$ and that $\overline{\Delta}_5 = i'\circ\nu_{15}^2\circ q' : Sp(3) \to \wedge^5 Sp(3)$.*

Proof. Firstly, we show that $\overline{\Delta}_4 = i\circ\nu_{12}^2\circ q$ implies $\overline{\Delta}_5 = i'\circ\nu_{15}^2\circ q'$. For dimensional reasons, the image of $\overline{\Delta} : Sp(3) \to Sp(3)\wedge Sp(3)$ is in $Sp(3)^{(18)}\wedge Sp(3)^{(14)} \cup S^3\wedge Sp(3)^{(18)}$. Since $Sp(3)^{(14)}$ is of cone-length 3 by Corollary 3.5.1, the restriction of the map $1\wedge\overline{\Delta}_4$ to $Sp(3)^{(18)}\wedge Sp(3)^{(14)}$ is trivial. Thus $\overline{\Delta}_5$ is given as

$$\overline{\Delta}_5 : Sp(3) \to S^3\wedge(Sp(3)^{(18)}/Sp(3)^{(14)}) \xrightarrow{1\wedge(i\circ\nu_{12}^2)} \wedge^5 Sp(3)^{(18)} \subset \wedge^5 Sp(3),$$

since $\overline{\Delta}_4 = i\circ\nu_{12}^2\circ q$. Thus we observe that $\overline{\Delta}_5 = i'\circ(\iota_3\wedge\nu_{12}^2)\circ q' = i'\circ\nu_{15}^2\circ q'$.

So, we are left to show $\overline{\Delta}_4 = i\circ\nu_{12}^2\circ q$. For dimensional reasons, the image of $\overline{\Delta} : Sp(3)^{(18)} \to Sp(3)^{(18)}\wedge Sp(3)^{(18)}$ is in $Sp(3)^{(14)}\wedge S^3 \cup Sp(3)^{(11)}\wedge Sp(3)^{(7)} \cup Sp(3)^{(7)}\wedge Sp(3)^{(11)} \cup S^3\wedge Sp(3)^{(14)}$. Since $S^3 \cup_\phi C(S^6 \cup_{\nu_6} e^{10})$ is of cone-length 2 by Corollary 3.5.1, the restriction of $\overline{\Delta}_3 : Sp(3)^{(18)} \to \wedge^3 Sp(3)^{(18)}$ to $S^3\cup_\phi C(S^6\cup_{\nu_6} e^{10})$ is trivial. Hence $1\wedge\overline{\Delta}_3 : Sp(3)^{(14)}\wedge S^3 \cup Sp(3)^{(11)}\wedge Sp(3)^{(7)} \cup Sp(3)^{(7)}\wedge Sp(3)^{(11)} \cup S^3\wedge Sp(3)^{(14)} \to \wedge^4 Sp(3)^{(18)}$ is given as

$$1\wedge\overline{\Delta}_3 : (Sp(3)\wedge Sp(3))^{(18)} \xrightarrow{\alpha} (S^3 \cup_\omega e^7)\wedge S^{10} \cup S^3\wedge(S^{10} \cup_{\nu_{10}} e^{14}) \xrightarrow{1\wedge\beta} \wedge^4(S^3 \cup_\omega e^7).$$

The map $\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \to (S^3 \cup_\omega e^7)\wedge S^{10} \cup S^3\wedge(S^{10} \cup_{\nu_{10}} e^{14})$ is given as

$$\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \to S^{14} \cup_{\omega_{14}} e^{18} \to (S^3 \cup_\omega e^7)\wedge S^{10} \cup S^3\wedge(S^{10} \cup_{\nu_{10}} e^{14}).$$

Collapsing the subspace $S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14})$ of $(S^3\cup_\omega e^7)\wedge S^{10}\cup S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14})$, we obtain a map

$$q'\circ\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \to S^7\wedge S^{10},$$

where $q' : (S^3\cup_\omega e^7)\wedge S^{10}\cup S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}) \to S^3\wedge S^{10}$ is the collapsing map. For dimensional reasons, $q'\circ\alpha\circ\overline{\Delta}$ is as follows:

$$q'\circ\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \to Sp(3)^{(18)}/Sp(3)^{(14)} = S^{18} \xrightarrow{\gamma} S^7\wedge S^{10}.$$

If γ were non-trivial, then γ would be $\eta_{17} : S^{18} \to S^{17}$, and hence we should have $x_7y_{10} = \varepsilon\cdot y_{18} \neq 0$. However, from the ring structure of $h^*(Sp(3))$ given in Theorem 3.8, we know $x_7y_{10} = 0$, and hence we obtain $\gamma = 0$. Then the image of $\alpha\circ\overline{\Delta}$ is in the subspace $S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14})$ of $(S^3\cup_\omega e^7)\wedge S^{10}\cup S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14})$, since they are 12-connected. Hence $\overline{\Delta}_4 = (1\wedge\overline{\Delta}_3)\circ\overline{\Delta}$ is given as

$$\overline{\Delta}_4 : Sp(3)^{(18)} \xrightarrow{\alpha\circ\overline{\Delta}} S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}) \xrightarrow{1\wedge\beta} S^3\wedge(\wedge^3(S^3\cup_\omega e^7))^{(15)} \subset \wedge^4 Sp(3)^{(18)},$$

where $(\wedge^3(S^3\cup_\omega e^7))^{(15)}$ is $(S^3\cup_\omega e^7)\wedge S^3\wedge S^3 \cup S^3\wedge(S^3\cup_\omega e^7)\wedge S^3 \cup S^3\wedge S^3\wedge(S^3\cup_\omega e^7)$. Collapsing the subspace $\wedge^3 S^3$ of $(\wedge^3(S^3\cup_\omega e^7))^{(15)}$, we obtain a map

$$q''\circ\beta : S^{10}\cup_{\nu_{10}}e^{14} \to S^7\wedge S^3\wedge S^3 \,\cup\, S^3\wedge S^7\wedge S^3 \,\cup\, S^3\wedge S^3\wedge S^7,$$

where $q'' : (\wedge^3(S^3\cup_\omega e^7))^{(15)} \to S^7\wedge S^3\wedge S^3 \,\cup\, S^3\wedge S^7\wedge S^3 \,\cup\, S^3\wedge S^3\wedge S^7$ is the collapsing map. For dimensional reasons, $q''\circ\beta$ is given as

$$q''\circ\beta : S^{10}\cup_{\nu_{10}}e^{14} \to S^{14} \xrightarrow{\gamma'} S^7\wedge S^3\wedge S^3 \vee S^3\wedge S^7\wedge S^3 \vee S^3\wedge S^3\wedge S^7.$$

If γ' were non-trivial, then its projection to S^{13} would be $\eta_{13} : S^{14} \to S^{13}$, and hence we should have $x_3^2x_7 = \varepsilon\cdot y_{14} \neq 0$. However, from the ring structure of $h^*(Sp(3))$ given in Theorem 3.8, we know $x_3^2x_7 = \varepsilon\cdot x_7^2 = 0$, and hence we obtain $\gamma' = 0$. Hence the image of β lies in the subspace $\wedge^3S^3$ of $\wedge^3Sp(3)^{(18)}$.

On the other hand, for dimensional reasons, $\alpha\circ\overline{\Delta}$ is given as

$$\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \to S^{14}\cup_{\omega_{14}}e^{18} \xrightarrow{\alpha'} S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}),$$

where the restriction $\alpha'|_{S^{14}}$ is given as

$$\alpha'|_{S^{14}} : S^{14} \xrightarrow{\gamma''} S^{13} \hookrightarrow S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}).$$

If it were non-trivial, then γ'' would be $\eta_{13} : S^{14} \to S^{13}$, and hence we should have $x_3y_{10} = \varepsilon\cdot y_{14} \neq 0$. However, from the ring structure of $h^*(Sp(3))$ given in Theorem 3.8, we know $x_3y_{10} = x_3^2x_7 = \varepsilon\cdot x_7^2 = 0$, and hence $\gamma'' = 0$. Hence $\alpha\circ\overline{\Delta}$ is given as

$$\alpha\circ\overline{\Delta} : Sp(3)^{(18)} \xrightarrow{q} S^{18} \xrightarrow{\alpha''} S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}),$$

and hence $\overline{\Delta}_4$ is given as

$$\overline{\Delta}_4 : Sp(3)^{(18)} \xrightarrow{q} S^{18} \xrightarrow{\alpha''} S^3\wedge(S^{10}\cup_{\nu_{10}}e^{14}) \xrightarrow{1\wedge\beta} S^3\wedge(\wedge^3S^3) \xrightarrow{i} \wedge^4Sp(3)^{(18)}.$$

Now, we are ready to determine $\overline{\Delta}_4$: By Theorem 3.8, we know $x_3^2x_{11} = \varepsilon\cdot z_{18}$ and $x_3^2 = \varepsilon\cdot x_7$, hence $\alpha'' : S^{18} \to S^{13}\cup_{\nu_{13}}e^{17}$ is a co-extension of $\eta_{16} : S^{17} \to S^{16}$ on $S^{13}\cup_{\nu_{13}}e^{17}$ and $1\wedge\beta : S^{13}\cup_{\nu_{13}}e^{17} \to S^{12}$ is an extension of $\eta_{12} : S^{13} \to S^{12}$. Thus the composition $(1\wedge\beta)\circ\alpha''$ is an element of the Toda bracket $\{\eta_{12}, \nu_{13}, \eta_{16}\}$ which contains a single element ν_{12}^2 by Lemma 5.12 of [**23**], and hence $\overline{\Delta}_4 = i\circ\nu_{12}^2\circ q$. *QED.*

COROLLARY 4.3.1. *$w\mathrm{cat}(Sp(3)^{(18)}) \geq 4$ and $w\mathrm{cat}(Sp(3)) \geq 5$.*

This yields the following result.

THEOREM 4.4.

Skeleta	$Sp(3)^{(3)}$	$Sp(3)^{(7)}$	$Sp(3)^{(10)}$	$Sp(3)^{(11)}$	$Sp(3)^{(14)}$	$Sp(3)^{(18)}$	$Sp(3)$
$w$cat	1	2	3	3	3	4	5
cat	1	2	3	3	3	4	5
Cat	1	2	3	3	3	4	5

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.4

We know that for $n \geq 4$,

$$Sp(n)^{(16)} = Sp(4)^{(15)} = Sp(3)^{(14)} \cup e^{15},$$

$$Sp(n)^{(19)} = \begin{cases} Sp(4)^{(15)} \cup (e^{18} \vee e^{18}) & n = 4, \\ Sp(4)^{(15)} \cup (e^{18} \vee e^{18}) \cup e^{19} & n \geq 5, \end{cases}$$

$$Sp(n)^{(21)} = Sp(n)^{(19)} \cup e^{21}$$

and that $w\mathrm{cat}(Sp(3)^{(14)}) = \mathrm{cat}(Sp(3)^{(14)}) = \mathrm{Cat}(Sp(3)^{(14)}) = 3$. Firstly, we show the following.

PROPOSITION 5.1. $w\mathrm{cat}(Sp(4)^{(15)}) = 3$.

Proof. Since the pair $(Sp(4)^{(15)}, Sp(3)^{(11)})$ is 13-connected, $w\mathrm{cat}(Sp(3)^{(11)}) = 3$ implies that $\overline{\Delta}_3 : Sp(4)^{(15)} \to \wedge^3 Sp(4)^{(15)}$ is non-trivial, and hence $w\mathrm{cat}(Sp(4)^{(15)}) \geq 3$. Thus we are left to show $w\mathrm{cat}(Sp(4)^{(15)}) \leq 3$: For dimensional reasons, $\overline{\Delta}_4 = (\overline{\Delta}\wedge\overline{\Delta})\circ\overline{\Delta} : Sp(4)^{(15)} \to \wedge^4 Sp(4)^{(15)}$ is given as

$$\overline{\Delta}_4 : Sp(4)^{(15)} \xrightarrow{\alpha_0} Sp(4)^{(11)}\wedge Sp(4)^{(11)} \xrightarrow{\overline{\Delta}\wedge\overline{\Delta}} \wedge^4 Sp(4)^{(11)} \hookrightarrow \wedge^4 Sp(4)^{(15)},$$

for some α_0. By Fact 3.2, $\overline{\Delta} : Sp(4)^{(11)} \to \wedge^2 Sp(4)^{(11)}$ is given as

$$\overline{\Delta} : Sp(4)^{(11)} \xrightarrow{\beta_0} (S^7 \vee S^{10}) \cup e^{11} \xrightarrow{\gamma_0} \wedge^2(S^3 \cup_\omega e^7) \hookrightarrow \wedge^2 Sp(4)^{(11)},$$

for some β_0 and γ_0. Then for dimensional reasons, $(\beta_0\wedge\beta_0)\circ\alpha_0 : Sp(4)^{(15)} \to ((S^7 \vee S^{10}) \cup e^{11})\wedge((S^7 \vee S^{10}) \cup e^{11})$ and $(\gamma_0\wedge\gamma_0)|_{S^7\wedge S^7} : S^7\wedge S^7 \to \wedge^4(S^3 \cup_\omega e^7)$ are respectively equal to the compositions

$$(\beta_0\wedge\beta_0)\circ\alpha_0 : Sp(4)^{(15)} \xrightarrow{\alpha_0'} S^7\wedge S^7 \hookrightarrow ((S^7 \vee S^{10}) \cup e^{11})\wedge((S^7 \vee S^{10}) \cup e^{11}),$$

$$(\gamma_0\wedge\gamma_0)|_{S^7\wedge S^7} : S^7\wedge S^7 \xrightarrow{\gamma_0'} \wedge^4 S^3 \hookrightarrow \wedge^4(S^3 \cup_\omega e^7),$$

for some α_0' and γ_0'. Hence $\overline{\Delta}_4 : Sp(4)^{(15)} \to \wedge^4 Sp(4)^{(15)}$ is given as

$$\overline{\Delta}_4 : Sp(4)^{(15)} \xrightarrow{\alpha_0'} S^7\wedge S^7 \xrightarrow{\gamma_0'} \wedge^4 S^3 \hookrightarrow \wedge^4 Sp(4)^{(15)},$$

where $Sp(4)^{(15)} = Sp(3)^{(14)} \cup e^{15}$. By Theorem 3.8, $x_7^2 = 0$ in $h^*(Sp(3))$, and hence α_0' annihilates $Sp(3)^{(14)}$. Thus $\overline{\Delta}_4 : Sp(4)^{(15)} \to \wedge^4 Sp(4)^{(15)}$ is given as

$$\overline{\Delta}_4 : Sp(4)^{(15)} \xrightarrow{q''} S^{15} \xrightarrow{\beta_0'} S^{14} \xrightarrow{\gamma_0'} S^{12} \xhookrightarrow{i''} \wedge^4 Sp(4)^{(15)}$$

for some β_0', where $q'' : Sp(4)^{(15)} \to Sp(4)^{(15)}/Sp(4)^{(14)} = S^{15}$ is the projection and $i'' : S^{12} = S^3\wedge S^3\wedge S^3\wedge S^3 \hookrightarrow \wedge^4 Sp(4)^{(15)}$ is the inclusion. Hence the non-triviality of $\overline{\Delta}_4$ implies the non-triviality of β_0' and γ_0'. Therefore $\overline{\Delta}_4$ should be $i''\circ\eta_{12}^3\circ q''$, if it were non-trivial. However, we also know from (5.5) of [**23**] that η_{12}^3 is $12\nu_{12} = 6\omega_{12}$ and that $i''\circ\omega_{12}$ is trivial by Fact 3.1. Therefore, $\overline{\Delta}_4 : Sp(4)^{(15)} \to \wedge^4 Sp(4)^{(15)}$ is trivial, and hence $w\mathrm{cat}\, Sp(4)^{(15)} \leq 3$. This implies that $w\mathrm{cat}\, Sp(4)^{(15)} = 3$. *QED.*

Secondly, we show the following.

PROPOSITION 5.2. *$w\mathrm{cat}(Sp(n)^{(19)}) \leq 4$ for $n \geq 4$.*

Proof. Let $n \geq 4$. Since $\overline{\Delta}_5 = ((1_{Sp(n)})\wedge\overline{\Delta}_4)\circ\overline{\Delta} : Sp(n)^{(19)} \to \wedge^5 Sp(n)^{(19)}$, it is given as

$$\overline{\Delta}_5 : Sp(n)^{(19)} \overset{\overline{\Delta}}{\to} Sp(n)^{(16)}\wedge Sp(n)^{(16)} = Sp(4)^{(15)}\wedge Sp(4)^{(15)}$$
$$\overset{(1_{Sp(4)^{(15)}})\wedge\overline{\Delta}_4}{\to} \wedge^5 Sp(4)^{(15)} \hookrightarrow \wedge^5 Sp(n)^{(19)},$$

which is trivial, since $\overline{\Delta}_4 : Sp(4)^{(15)} \to \wedge^4 Sp(4)^{(15)}$ is trivial by Proposition 5.1. Thus $w\mathrm{cat}(Sp(n)^{(19)}) \leq 4$ when $n \geq 4$. *QED.*

Let $p_j : Sp(n) \to X_{n,j} = Sp(n)/Sp(n-j)$ be the projection for $j \geq 1$. Then we have the following.

PROPOSITION 5.3. *Let $q''' : Sp(n) \to Sp(n)/Sp(n)^{((2n+1)n-3)} = S^{(2n+1)n}$ be the collapsing map and $i''' : S^{(2n+1)n-6} \hookrightarrow (\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}$ the inclusion. Then*

$$q'''^{*}\circ i'''_{*} : \pi_{(2n+1)n}(S^{(2n+1)n-6}) \to [Sp(n), (\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}]$$

is injective.

Proof. Firstly, we have the following exact sequence

$$\pi_{(2n+1)n}(S^{(2n+1)n-3}) \overset{\psi'''}{\to} \pi_{(2n+1)n}(S^{(2n+1)n-6})$$
$$\overset{i'''_*}{\to} \pi_{(2n+1)n}((\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}) \to \pi_{(2n+1)n}(\vee_5 S^{(2n+1)n-2}),$$

where $\pi_{(2n+1)n}(S^{(2n+1)n-6}) \cong \mathbb{Z}/2\mathbb{Z}\nu^2_{(2n+1)n-6}$ and ψ''' is induced from $\omega_{(2n+1)n-6} = 2\nu_{(2n+1)n-6}$. Thus ψ''' is trivial, and hence i'''_* is injective.

Secondly, since $(\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1})$ is $(n(2n+1)-11)$-connected, we have

$$\begin{aligned}
&[Sp(n), (\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}]\\
&= [(S^{(2n+1)n-7} \cup_{\omega_{(2n+1)n-7}} e^{(2n+1)n-3}) \vee S^{(2n+1)n}, (\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}]\\
&= [S^{(2n+1)n-7} \cup_{\omega_{(2n+1)n-7}} e^{(2n+1)n-3}, (\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}]\\
&\qquad \oplus\pi_{(2n+1)n}((\wedge^5 Sp(n))\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1})
\end{aligned}$$

by Proposition 3.4, and hence q'''^{*} is injective. Thus $q'''^{*}\circ i'''_{*}$ is injective. *QED.*

Then the following lemma implies that $((1_{\wedge^5 Sp(n)})\wedge p_{n-3}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n+2}$ is non-trivial by Proposition 5.3, and hence we obtain Theorem 1.4.

LEMMA 5.4. $((1_{\wedge^5 Sp(n)})\wedge p_{n-3}\wedge p_{n-4}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n+2} = i'''\circ\nu^2_{(2n+1)n-6}\circ q'''$.

Proof. We have

$$\begin{aligned}
((1_{\wedge^5 Sp(n)})\wedge p_{n-3}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n+2} &= (\overline{\Delta}_5\wedge p_{n-3}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n-2}\\
&= (\overline{\Delta}_5\wedge(1_{\wedge^{n-3} Sp(n)}))\circ((1_{Sp(n)})\wedge p_{n-3}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n-2}.
\end{aligned}$$

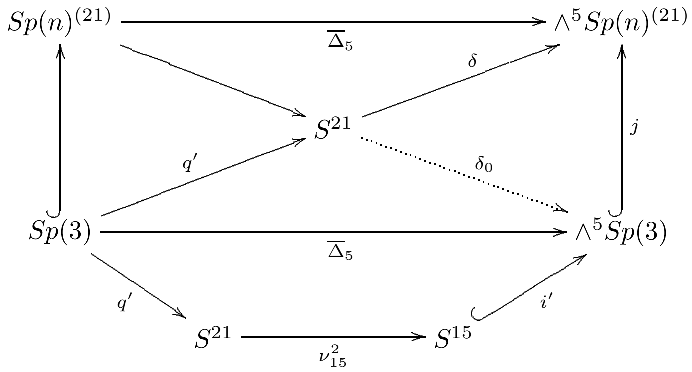
For dimensional reasons, the image of $((1_{Sp(n)})\wedge p_{n-3}\wedge\cdots\wedge p_1)\circ\overline{\Delta}_{n-2}$ lies in

$$Sp(n)^{(21)}\wedge S^{15}\wedge\cdots\wedge S^{4n-1} \cup Sp(n)^{(19)}\wedge X_{n,n-3}\wedge\cdots\wedge X_{n,1}.$$

From Proposition 5.2, it follows that $\overline{\Delta}_5$ annihilates $Sp(n)^{(19)}$, and hence $\overline{\Delta}_5$ is given as

$$\overline{\Delta}_5 : Sp(n)^{(21)} \to S^{21} \xrightarrow{\delta} \wedge^5 Sp(n)^{(21)}$$

for some $\delta \in \pi_{21}(\wedge^5 Sp(3))$. Using Lemma 4.3, we obtain the following diagram except for the dotted arrow, which is commutative up to homotopy:



Since the pair $(\wedge^5 Sp(n), \wedge^5 Sp(3))$ is 26-connected for $n \geq 4$, we can compress δ into $\wedge^5 Sp(3)$ as $\delta \sim j \circ \delta_0$. Thus we have a homotopy relation

$$j \circ \delta_0 \circ q' \sim \delta \circ q' \sim j \circ \overline{\Delta}_5 \sim j \circ i' \circ \nu_{15}^2 \circ q'.$$

Now we know that $\dim Sp(3) = 21 < 26 - 1$, and hence we can drop j from the above homotopy relation and obtain

$$\delta_0 \circ q' \sim i' \circ \nu_{15}^2 \circ q'.$$

By Proposition 4.2, ${q'}^* : \pi_{21}(\wedge^5 Sp(3)) \to [Sp(3), \wedge^5 Sp(3)]$ is injective, and hence we obtain a homotopy relation

$$\delta_0 \sim i' \circ \nu_{15}^2.$$

Thus $\overline{\Delta}_5$ is given as

$$\overline{\Delta}_5 : Sp(n)^{(21)} \to S^{21} \xrightarrow{\nu_{15}^2} S^{15} \hookrightarrow \wedge^5 Sp(n)^{(21)}.$$

Thus $((1_{\wedge^5 Sp(n)}) \wedge p_{n-3} \wedge \cdots \wedge p_1) \circ \overline{\Delta}_{n+2}$ is given as

$$((1_{\wedge^5 Sp(n)}) \wedge p_{n-3} \wedge \cdots \wedge p_1) \circ \overline{\Delta}_{n+2} : Sp(n) \to S^{21} \wedge S^{(2n+7)(n-3)}$$
$$\xrightarrow{\nu_{(2n+1)n-6}^2} S^{15} \wedge S^{(2n+7)(n-3)} \hookrightarrow (\wedge^5 Sp(n)) \wedge X_{n,n-3} \wedge \cdots \wedge X_{n,1}.$$

This completes the proof of the lemma. *QED.*

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