Errata for : Topological Complexity is a Fibrewise L-S Category

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Abstract

There is a problem with the proof of Theorem 1.13 of [2] which states that for a fibrewise well-pointed space X over B, we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{cat}^{*}_{\mathrm{B}}(X)$ and that for a locally finite simplicial complex B, we have $\mathcal{TC}(B) = \mathcal{TC}^{\mathrm{M}}(B)$. While we still conjecture that Theorem 1.13 is true, this problem means that, at present, no proof is given to exist. Alternatively, we show the difference between two invariants $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is at most 1 and the conjecture is true for some cases. We give further corrections mainly in the proof of Theorem 1.12.

Key words: Toplogical complexity; Lusternik-Schnirelmann category.
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It was pointed out to the authors by Jose Calcines that there is a problem with the proof of Theorem 1.13 of [2] which states that for a fibrewise well-pointed space X over B, we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{cat}^{*}_{\mathrm{B}}(X)$ and that for a locally finite simplicial complex B, we have $\mathcal{TC}(B) = \mathcal{TC}^{\mathrm{M}}(B)$, where $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ and $\mathcal{TC}^{\mathrm{M}}(B)$ are new versions of a fibrewise L-S category and a topological complexity, respectively, which are introduced in [2].

While we still conjecture that Theorem 1.13 of [2] is true, this problem means that, at present, no proof is given to exist. It then results that "$\mathcal{TC}(B)$" in Corollary 8.7 of [2] must be replaced with "$\mathcal{TC}^{\mathrm{M}}(B)$" and the resulting

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inequality should be presented in the following form:

$$\mathcal{Z}_\pi(B) \leq \mathrm{wgt}_\pi(B) \leq \mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(d(B)) \leq \mathcal{TC}^{\mathrm{M}}(B){-}1 \leq \mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}(d(B)) \leq \mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)).$$

The problem in the argument occurs on page 14 where a homotopy

$$\hat{\Phi}_i : \hat{U}_i \times [0,1] \to \hat{X}$$

is given, while the definition of $\hat{\Phi}_i$ apparently is not well-defined. Alternatively, we show here the difference between two invariants $\mathrm{cat}^*_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is at most 1 and the conjecture is true for some cases.

Theorem 1 *For a fibrewise well-pointed space X over B, we have $\mathrm{cat}^*_{\mathrm{B}}(X) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{cat}^*_{\mathrm{B}}(X) + 1$ which implies that, for a locally finite simplicial complex B, we have $\mathcal{TC}(B) \leq \mathcal{TC}^{\mathrm{M}}(B) \leq \mathcal{TC}(B) + 1$.*

Proof: The inequality of $\mathcal{TC}(B)$ and $\mathcal{TC}^{\mathrm{M}}(B)$ in Theorem 1 for a locally finite simplicial complex B is, by Theorem 1.7 in [2], a special case of the inequality of $\mathrm{cat}^*_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ in Theorem 1 for a fibrewise well-pointed space X. So it is sufficient to show the inequality for X: because the inequality $\mathrm{cat}^*_{\mathrm{B}}(X) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is clear by definition, all we need to show is the inequality $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{cat}^*_{\mathrm{B}}(X) + 1$. Let X be a fibrewise well-pointed space over B with a projection $\mathrm{p}_X : X \to B$ and a section $\mathrm{s}_X : B \to X$. Let (u, h) be a fibrewise (strong) Strøm structure (see Crabb and James [1]) on $(X, \mathrm{s}_X(B))$, i.e., $u : X \to [0,1]$ is a map and $h : X \times [0,1] \to X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = \mathrm{s}_X(B)$, $h(x,0) = x$ for any $x \in X$ and $h(x,1) = \mathrm{s}_X \circ \mathrm{p}_X(x)$ for any $x \in X$ with $u(x) < 1$. Assume $\mathrm{cat}^*_{\mathrm{B}}(X) = m$ and the family $\{U_i\,;\, 0{\leq}i{\leq}m\}$ of open sets of X satisfies $X = \bigcup_{i=0}^{m} U_i$ and each open set U_i is fibrewise contractible (into $\mathrm{s}_X(B)$) by a fibrewise homotopy $H_i : U_i \times [0,1] \to X$. Let $V_i = U'_i \cup V$ for $0 \leq i \leq m$ and $V_{m+1} = u^{-1}([0, \frac{2}{3}))$ where $U'_i = U_i \smallsetminus u^{-1}([0, \frac{1}{2}])$ and $V = u^{-1}([0, \frac{1}{3}))$. Then the restriction $H_i|_{U'_i} : U'_i \times [0,1] \to X$ gives a fibrewise contraction of U'_i and the restriction of the fibrewise (strong) Strøm structure $h|_V : V \times [0,1] \to X$ gives a fibrewise pointed contraction of V. Since U'_i and V are obviously disjoint, we obtain that $V_i = U'_i \cup V \supset \Delta(B)$ is a fibrewise contractible open set by a fibrewise pointed homotopy. Similarly the restriction of the fibrewise (strong) Strøm structure $h|_{V_{m+1}} : V_{m+1} \times [0,1] \to X$ gives a fibrewise pointed contraction of $V_{m+1} \supset \Delta(B)$. Since $V_i \cup V_{m+1} = U'_i \cup V_{m+1} = U_i \cup V_{m+1} \supset U_i$, we obtain $\bigcup_{i=0}^{m+1} V_i = \bigcup_{i=0}^{m} (V_i \cup V_{m+1}) \supset \bigcup_{i=0}^{m} U_i = X$. This implies $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq m+1 = \mathrm{cat}^*_{\mathrm{B}}(X) + 1$ and it completes the proof of Theorem 1. □

Theorem 2 *Let X be a fibrewise well-pointed space over B with $\mathrm{cat}^*_{\mathrm{B}}(X) = m$ and $\{U_i\,;\, 0{\leq}i{\leq}m\}$ be an open cover of X, in which U_i is fibrewise contractible (into $\mathrm{s}_X(B)$) by a fibrewise homotopy $H_i : U_i{\times}[0,1] \to X$. Then we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = m = \mathrm{cat}^*_{\mathrm{B}}(X)$ if one of the following conditions is satisfied.*

(1) There exists i $(0 \le i \le m)$ such that U_i does not intersect with $\mathrm{s}_X(B)$.
(2) There exists i $(0 \le i \le m)$ and an open and closed subset O of U_i such that $U_i \cap \mathrm{s}_X(B) = O \cap \mathrm{s}_X(B)$ and O includes $\mathrm{s}_X \circ \mathrm{p}_X(O) \subset \mathrm{s}_X(B)$.
(3) There exists i $(0 \le i \le m)$, an open and closed subset O of U_i and a fibrewise compression $c : \mathrm{s}_X \circ \mathrm{p}_X(O) \times [0,1] \to X$ of $\mathrm{s}_X \circ \mathrm{p}_X(O)$ into O such that $U_i \cap \mathrm{s}_X(B) = O \cap \mathrm{s}_X(B)$ and $c((O \cap \mathrm{s}_X(B)) \times [0,1]) \subset O$.

Theorem 2 immediately implies the following corollary.

Corollary 3 *Let B be a locally finite simplicial complex with $\mathcal{TC}(B) = m$ and $\{U_i\,;\, 1 \le i \le m\}$ be an open cover of X, in which U_i is compressible into the image $\Delta(B)$ of diagonal map $\Delta : B \to B \times B$. Then we have $\mathcal{TC}^{\mathrm{M}}(B) = m = \mathcal{TC}(B)$ if one of the following conditions is satisfied.*

(1) There exists i $(1 \le i \le m)$ such that U_i does not intersect with $\Delta(B)$.
(2) There exists i $(1 \le i \le m)$ and an open and closed subset O of U_i such that $U_i \cap \Delta(B) = O \cap \Delta(B)$ and O includes $\Delta \circ \mathrm{pr}_2(U_i) \subset \Delta(B)$.
(3) There exists i $(1 \le i \le m)$, an open and closed subset O of U_i and a fibrewise compression $c : \Delta \circ \mathrm{pr}_2(O) \times [0,1] \to B \times B$ of $\Delta \circ \mathrm{pr}_2(O)$ into O such that $U_i \cap \Delta(B) = O \cap \Delta(B)$ and $c((O \cap \Delta(B)) \times [0,1]) \subset O$.

Proof of Theorem 2: For simplicity, we assume that $i = 0$ in each cases. Let X be a fibrewise well-pointed space over B with a projection $\mathrm{p}_X : X \to B$ and a section $\mathrm{s}_X : B \to X$. Let (u, h) be a fibrewise (strong) Strøm structure on $(X, \mathrm{s}_X(B))$, i.e., $u : X \to [0,1]$ is a map and $h : X \times [0,1] \to X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = \mathrm{s}_X(B)$, $h(x,0) = x$ for any $x \in X$ and $h(x,1) = \mathrm{s}_X \circ \mathrm{p}_X(x)$ for any $x \in X$ with $u(x) < 1$. Then the fibrewise map $r : X \to X$ given by $r(x) = h(x,1)$ satisfies the following.

i) $X = \bigcup_{i=0}^{m} r^{-1}(U_i)$, since $X = \bigcup_{i=0}^{m} U_i$.
ii) r is fibrewise homotopic to the identity by a fibrewise homotopy h.
iii) $r^{-1}(\mathrm{s}_X(B)) \supset U = u^{-1}([0,1))$, where U is fibrewise contractible by $h|_U$.
iv) Each $r^{-1}(U_i)$ is fibrewise contractible, since r is fibrewise homotopic to the identity by ii) and U_i is fibrewise contractible.

Case (1): let $V_0 = r^{-1}(U_0) \cup u^{-1}([0,\frac{2}{3}))$ and $V_i = (r^{-1}(U_i) \smallsetminus u^{-1}([0,\frac{1}{2}])) \cup u^{-1}([0,\frac{1}{3}))$, $1 \le i \le m$. Thus $\bigcup_{i=0}^{m} V_i = r^{-1}(U_0) \cup \bigcup_{i=1}^{m} (V_i \cup u^{-1}([0,\frac{2}{3}))) \supset r^{-1}(U_0) \cup \bigcup_{i=1}^{m} r^{-1}(U_i) = \bigcup_{i=0}^{m} r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \smallsetminus u^{-1}([0,\frac{1}{2}])$ for every $i \ge 0$, where $r^{-1}(U_0) \smallsetminus u^{-1}([0,\frac{1}{2}]) = r^{-1}(U_0)$ since U_0 does not intersect with $\mathrm{s}_X(B)$. On the other hand, $u^{-1}([0,\frac{t}{3}))$, $t = 1, 2$ are fibrewise contractible by fibrewise pointed homotopies by iii). Hence each V_i, $0 \le i \le m$ is fibrewise contractible by a fibrewise pointed homotopy, and hence $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \le m$. Thus $\mathrm{cat}^{*}_{\mathrm{B}}(X) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$.

Case (3) where Case (2) is a special case of Case (3): let $W_0 = r^{-1}(O) \cup u^{-1}([0, \frac{2}{3}))$, $V_0 = r^{-1}(U_0 \smallsetminus O) \cup W_0$ and $V_i = (r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])) \cup u^{-1}([0, \frac{1}{3}))$, $1 \leq i \leq m$. Thus $\bigcup\limits_{i=0}^{m} V_i = r^{-1}(U_0) \cup \bigcup\limits_{i=1}^{m} (V_i \cup u^{-1}([0, \frac{2}{3}))) \supset \bigcup\limits_{i=0}^{m} r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])$ which does not intersect with $u^{-1}([0, \frac{1}{3}))$, for every $i > 0$. On the other hand, each open set $u^{-1}([0, \frac{t}{3}))$, $t = 1, 2$ is fibrewise contractible by a fibrewise pointed homotopy by iii). Hence each open set V_i, $1 \leq i \leq m$ is fibrewise contractible by fibrewise pointed homotopy. For $i = 0$, we construct a fibrewise pointed homotopy $H : W_0 \times [0, 1] \to X$ using $c : \mathrm{s}_X \circ \mathrm{p}_X(U_i) \times [0, 1] \to$

X, $H_0 : U_0 \times [0,1] \to X$ and the Strøm structure (u,h) as follows:

$$H(x,t) = \begin{cases} \begin{Bmatrix} x, & t=0 \\ h(x,4t), & 0 \le t \le \frac{1}{4} \\ r(x), & t = \frac{1}{4} \\ H_0(r(x),4t-1), & \frac{1}{4} \le t \le \frac{1}{2} \\ \mathrm{s}_X \circ \mathrm{p}_X(r(x)) = \mathrm{s}_X \circ \mathrm{p}_X(x) = \mathrm{s}_X(b), & t = \frac{1}{2} \\ H_0(c(\mathrm{s}_X(b),1)), 3-4t), & \frac{1}{2} \le t \le \frac{3}{4} \\ c(\mathrm{s}_X(b), 4-4t), & \frac{3}{4} \le t \le 1 \\ \mathrm{s}_X(b), & t=1 \end{Bmatrix}, & x \in W_0 \smallsetminus U, \\ \begin{Bmatrix} x, & t=0 \\ h(x,4t), & 0 \le t \le \frac{1}{4} \\ r(x) = \mathrm{s}_X(b), & t = \frac{1}{4} \\ H_0(\mathrm{s}_X(b),4t-1), & \frac{1}{4} \le t \le \frac{1}{2} \\ \mathrm{s}_X \circ \mathrm{p}_X(\mathrm{s}_X(b)) = \mathrm{s}_X(b), & t = \frac{1}{2} \\ H_0(c(\mathrm{s}_X(b),4u(x)-3)), 3-4t), & \frac{1}{2} \le t \le \frac{3}{4} \\ c(\mathrm{s}_X(b), (4-4t)(4u(x)-3)), & \frac{3}{4} \le t \le 1 \\ \mathrm{s}_X(b), & t=1 \end{Bmatrix}, & \frac{3}{4} \le u(x) < 1, \\ \begin{Bmatrix} x, & t=0 \\ h(x,4t), & 0 \le t \le \frac{1}{4} \\ r(x) = \mathrm{s}_X(b), & t = \frac{1}{4} \\ H_0(\mathrm{s}_X(b),4t-1), & \frac{1}{4} \le t \le u(x)-\frac{1}{2} \\ H_0(\mathrm{s}_X(b),4u(x)-2), & u(x)-\frac{1}{4} \le t \le \frac{3}{2}-u(x) \\ H_0(\mathrm{s}_X(b),3-4t), & \frac{3}{2}-u(x) \le t \le \frac{3}{4} \\ \mathrm{s}_X(b), & \frac{3}{4} \le t \le 1 \end{Bmatrix}, & \frac{1}{2} \le u(x) < \frac{3}{4}, \\ \begin{Bmatrix} x, & t=0 \\ h(x,4t), & 0 \le t \le \frac{1}{4} \\ r(x) = \mathrm{s}_X(b), & \frac{1}{4} \le t \le 1 \end{Bmatrix}, & 0 \le u(x) < \frac{2}{3}, \\ \mathrm{s}_X(b), & x \in \mathrm{s}_X(B), \end{cases}$$

where $b = \mathrm{p}_X(x) = \mathrm{p}_X(r(x))$, and hence for $x \in W_0 \smallsetminus u^{-1}([0,\frac{2}{3})) \subset r^{-1}(O)$, we have $\mathrm{s}_X(b) = \mathrm{s}_X \circ \mathrm{p}_X(r(x)) \in O$ since $r(x) \in O$. Since an open set $U_0 \smallsetminus O$ does not intersect with $\mathrm{s}_X(B)$, $r^{-1}(U_0 \smallsetminus O)$ does not intersect with $u^{-1}([0,1)) \supset u^{-1}([0,\frac{2}{3}))$. Hence $V_0 = r^{-1}(U_0 \smallsetminus O) \cup W_0$ is fibrewise contractible by a fibrewise pointed homotopy, and hence $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \le m$. Thus $\mathrm{cat}^{*}_{\mathrm{B}}(X) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$, and it completes the proof of Theorem 2. □

The following are corrections in [2].

- The part of Proof of Theorem 1.12 from page 13 line -3 to page 14 line 12 is not clearly given and must be rewritten completely:

Proof: For each vertex β of B, let V_β be the star neighbourhood in B and $V = \bigcup_\beta V_\beta \times V_\beta \subset B \times B$. Then the closure $\bar{V} = \bigcup_\beta \bar{V}_\beta \times \bar{V}_\beta$ is a subcomplex of $B \times B$. For the barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ of x and y, resp, we see that $(x, y) \in V$ if and only if $\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta) > 0$ and that $\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta) = 1$ if and only if the barycentric coordinates of x and y are the same, or equivalently, $(x, y) \in \Delta(B)$. Hence we can define a continuous map $v : B \times B \to [0, 3]$ by the following formula.

$$v(x,y) = \begin{cases} 3 - 3\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta), & \text{if } (x,y) \in \bar{V}, \\ 3, & \text{if } (x,y) \notin V. \end{cases}$$

Since B is locally finite, v is well-defined on $B \times B$, and we have $v^{-1}(0) = \Delta(B)$ and $v^{-1}([0, 3)) = V$. Let $U = v^{-1}([0, 1))$ an open neighbourhood of $\Delta(B)$. In [3], Milnor defined a map $\mu : V \to B$ giving an 'average' of $(x, y) \in V$ as follows.

$$\mu(x,y) = (\zeta_\beta), \quad \zeta_\beta = \mathrm{Min}(\xi_\beta, \eta_\beta) / \sum_\gamma \mathrm{Min}(\xi_\gamma, \eta_\gamma),$$

where $\{\xi_\beta\}$ and $\{\eta_\beta\}$ are barycentric coodinates of x and y respectively, and γ runs over all vertices in B. Since B is locally finite, μ is well-defined on V and satisfies $\mu(x, x) = x$ for any $x \in B$. Using the map μ, Milnor introduced a map $\lambda : V \times [0, 1] \to B$ as follows.

$$\lambda(x,y,t) = \begin{cases} (1-2t)x + 2t\mu(x,y), & t \le \frac{1}{2}, \\ (2-2t)\mu(x,y) + (2t-1)y, & t \ge \frac{1}{2}. \end{cases}$$

Hence we have $\lambda(x, x, t) = x$ for any $x \in B$ and $t \in [0, 1]$. Using Milnor's map λ, we obtain a pair of maps (u, h) as follows:

$$u(x,y) = \mathrm{Min}\{1, v(x,y)\} \quad \text{and}$$
$$h(x,y,t) = \begin{cases} (\lambda(x,y,\mathrm{Min}\{t, w(x,y)\}), y), & \text{if } v(x,y) < 3, \\ (x,y), & \text{if } v(x,y) > 2, \end{cases}$$

where $w : B \times B \to [0, 1]$ is given by

$$w(x,y) = \begin{cases} 1, & v(x,y) \le 1, \\ 2 - v(x,y), & 1 \le v(x,y) \le 2, \\ 0, & v(x,y) \ge 2. \end{cases}$$

If $2 < v(x, y) < 3$, then, by definition, we have $w(x, y) = 0$ and

$$(\lambda(x,y,\mathrm{Min}\{t, w(x,y)\}), y) = (\lambda(x,y,0), y) = (x,y).$$

Thus h is also a well-defined continuous map. Then we have $u^{-1}(0) = \Delta(B)$, $u^{-1}([0,1)) = U$ and $h(x,y,0) = (x,y)$ for any $(x,y) \in B\times B$. If $(x,y) \in U$, we have $w(x,y) = 1$, $h(x,y,t) = (\lambda(x,y,t),y)$ and $h(x,y,1) = (y,y) \in \Delta(B)$. Moreover, we have $h(x,x,t) = (x,x)$ for any $x \in B$ and $t \in [0,1]$ and $\mathrm{pr}_2 \circ h(x,y,t) = y$ for any $(x,y,t) \in B\times B\times[0,1]$. This implies that h is a fibrewise pointed homotopy. Thus the data (u,h) gives the fibrewise (strong) Strøm structure on $(B\times B, \Delta(B))$. □

- In page 19, line 34, "$t = 0$" must be replaced by "$t = 1$".
- In page 20, line 17, "that" must be replaced by "that $H(s_Z(b),t) = s_W(b)$ for any $b \in B$ and".
- In page 20, line 28, the formula "$\check{H}(q(s_Z(b),t),s) = s_W(b)$," must be added.

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