TOPOLOGICAL COMPLEXITY IS A FIBREWISE L-S CATEGORY

NORIO IWASE AND MICHIHIRO SAKAI

Abstract. Topological complexity $\mathcal{TC}(B)$ of a space B is introduced by M. Farber to measure how much complex the space is, which is first considered on a configuration space of a motion planning of a robot arm. We also consider a stronger version $\mathcal{TC}^{\mathrm{M}}(B)$ of topological complexity with an additional condition: in a robot motion planning, a motion must be stasis if the initial and the terminal states are the same. Our main goal is to show the equalities $\mathcal{TC}(B) = \mathrm{cat}^{*}_{\mathrm{B}}(d(B))+1$ and $\mathcal{TC}^{\mathrm{M}}(B) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B))+1$, where $d(B) = B{\times}B$ is a fibrewise pointed space over B whose projection and section are given by $p_{d(B)} = \mathrm{pr}_2 : B{\times}B \to B$ the canonical projection to the second factor and $s_{d(B)} = \Delta_B : B \to B{\times}B$ the diagonal. In addition, our method in studying fibrewise L-S category is able to treat a fibrewise space with singular fibres.

1. Introduction

We say a pair of spaces (X, A) is an NDR pair or A is an NDR subset of X, if the inclusion map is a (closed) cofibration, in other words, the inclusion map has the (strong) Strøm structure (see page 22 in G. Whitehead [24]). When the set of the base point of a space is an NDR subset, the space is called well-pointed.

Let us recall the definition of a sectional category (see James [14]) which is originally defined and called by Schwarz 'genus'.

Definition 1.1 (Schwarz [21], James [15]). *For a fibration $p : E \to X$, the sectional cateory $\mathrm{secat}(p)$ (= one less than the Schwarz genus $\mathrm{Genus}(p)$) is the minimal number $m \geq 0$ such that there exists a cover of X by $(m{+}1)$ open subsets $U_i \subset X$ each of which admits a continuous section $s_i : U_i \to E$.*

The topological complexity of a robot motion planning is first introduced by M. Farber [2] in 2003 to measure the discontinuity of a robot motion planning algorithm searching also the way to minimise the discontinuity. At a more general view point, Farber defined a numerical invariant $\mathcal{TC}(B)$ of any topological space B: let $\mathcal{P}(B)$ be the space of all paths in B. Then there is a Serre path fibration $\pi : \mathcal{P}(B) \to B{\times}B$ given by $\pi(\ell) = (\ell(0), \ell(1))$ for $\ell \in \mathcal{P}(B)$.

Definition 1.2 (Farber). *For a space B, the topological complexity $\mathcal{TC}(B)$ is the minimal number $m \geq 1$ such that there exists a cover of $B{\times}B$ by m open subsets U_i each of which admits a continuous section $s_i : U_i \to \mathcal{P}(B)$ for $\pi : \mathcal{P}(B) \to B{\times}B$.*

By definition, we can observe that the topological complexity is nothing but the Schwartz genus or the sectional category.

Date: October 27, 2008, [First draft].
2000 *Mathematics Subject Classification.* Primary 55M30, Secondary 55Q25.
Key words and phrases. Toplogical complexity, Lusternik-Schnirelmann category.

Farber has further introduced a new invariant restricting motions by giving two additional conditions on the section $s : U \to \mathcal{P}(B)$.

(1) $s(b, b) = c_b$ the constant path at b for any $b \in B$,
(2) $s(b_1, b_2) = s(b_2, b_1)^{-1}$ if $(b_1, b_2) \in U$.

It gives a stronger invariant than the topological complexity, and the $\mathbb{Z}/2$-equivariant theory must be applied as in Farber-Grant [4]. This new topological invariant, in turn, suggests us another motion planning under the condition that a motion is stasis if the initial and the terminal states are the same. Let us state more precisely.

Definition 1.3. *For a space B, the 'monoidal' topological complexity $\mathcal{TC}^{\mathrm{M}}(B)$ is the minimal number $m \geq 1$ such that there exists a cover of $B{\times}B$ by m open subsets $U_i \supset \Delta(B)$ each of which admits a continuous section $s_i : U_i \to \mathcal{P}(B)$ for the Serre path fibration $\pi : \mathcal{P}(B) \to B{\times}B$ satisfying $s_i(b, b) = c_b$ for any $b \in B$.*

Remark 1.4. *This new topological complexity $\mathcal{TC}^{\mathcal{M}}$ is **not** a homotopy invariant, in general. However, it is a homotopy invariant if we restrict our working category to the category of a space B such that the pair $(B{\times}B, \Delta(B))$ is NDR.*

On the other hand, a fibrewise *pointed* L-S category of a fibrewise pointed space is introduced and studied by James-Morris [13]. Let us recall the definition:

Definition 1.5 (James-Morris [13])**.** (1) *Let X be a fibrewise pointed space over B. The fibrewise **pointed** L-S category $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is the minimal number $m \geq 0$ such that there exists a cover of X by $(m+1)$ open subsets $U_i \supset s_X(B)$ each of which is fibrewise null-homotopic in X by a fibrewise pointed homotopy. If there are no such m, we say $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \infty$.*

(2) *Let $f : Y \to X$ be a fibrewise pointed map over B. The fibrewise **pointed** L-S category $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f)$ is the minimal number $m \geq 0$ such that there exists a cover of Y by $(m+1)$ open subsets $U_i \supset s_Y(B)$, where the restriction $f|_{U_i}$ to each subset is fibrewise compressible into $s_X(B)$ in X by a fibrewise pointed homotopy. If there are no such m, we say $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f) = \infty$.*

To describe our main result, we further introduce a new unpointed version of fibrewise L-S category: the fibrewise L-S category $\mathrm{cat}_{\mathrm{B}}(\ )$ of an fibrewise *unpointed* space is also defined by James and Morris [13] as the minimum number (minus one) of open subsets which cover the given space and are fibrewise null-homotopic (see also James [14] and Crabb-James [1]). In this paper, we give a new version of a fibrewise *unpointed* L-S category of a fibrewise *pointed* space as follows:

Definition 1.6. (1) *Let X be a fibrewise pointed space over B. The fibrewise **unpointed** L-S category $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ is the minimal number $m \geq 0$ such that there exists a cover of X by $(m+1)$ open subsets U_i each of which is fibrewise compressible into $s_X(B)$ in X by a fibrewise homotopy. If there are no such m, we say $\mathrm{cat}^{*}_{\mathrm{B}}(X) = \infty$.*

(2) *Let $f : Y \to X$ be a fibrewise pointed map over B. The fibrewise **unpointed** L-S category $\mathrm{cat}^{*}_{\mathrm{B}}(f)$ is the minimal number $m \geq 0$ such that there exists a cover of Y by $(m+1)$ open subsets U_i, where the restriction $f|_{U_i}$ to each subset is fibrewise compressible into $s_X(B)$ in X by a fibrewise homotopy. If there are no such m, we say $\mathrm{cat}^{*}_{\mathrm{B}}(f) = \infty$.*

For a given space B, we define a fibrewise pointed space $d(B)$ by $d(B) = B{\times}B$ with $p_{d(B)} = \mathrm{pr}_2 : B{\times}B \to B$ and $s_{d(B)} = \Delta_B : B \to B{\times}B$ the diagonal. One of our main goals of this paper is to show the following theorem.

Theorem 1.7. *For a space B, we have the following equalities.*

1. $\mathcal{TC}(B) = \mathrm{cat}^*_{\mathrm{B}}(d(B)) + 1$.
2. $\mathcal{TC}^{\mathrm{M}}(B) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) + 1$.

Farber and Grant has also introduced lower bounds for the topological complexity by using the cup length and category weight (see Rudyak [17] for example) on the ideal of zero-divisors, i.e, the kernel of $\Delta^* : H^*(B{\times}B;R) \to H^*(B;R)$.

Definition 1.8 (Farber [2] and Farber-Grant [4])**.** *For a space B and a ring $R \ni 1$, the zero-divisors cup-length $\mathcal{Z}_{\mathrm{R}}(B)$ and the TC-weight $\mathrm{wgt}_\pi(u;R)$ for $u \in I = \ker\Delta^* : H^*(B{\times}B;R) \to H^*(B;R)$ is defined as follows.*

1. $\mathcal{Z}_{\mathrm{R}}(B) = \mathrm{Max}\,\{m{\geq}0 | H^*(B{\times}B;R) \supset I^m \neq 0\}$
2. $\mathrm{wgt}_\pi(u;R) = \mathrm{Max}\,\{m{\geq}0 | \forall f : Y \to B{\times}B\ (\mathrm{secat}(f^*\pi) < m),\ f^*(u) = 0\}$

In the category $\underline{\underline{\mathcal{T}}}^B_B$ of fibrewise pointed spaces with base space B and maps between them, we also have corresponding definitions.

Definition 1.9. *For a fibrewise pointed space X over B and a ring $R \ni 1$ and $u \in I = H^*(X,B;R) \subset H^*(X;R)$, we define*

1. $\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X;R) = \mathrm{Max}\,\{m{\geq}0 | \exists\{u_1,\cdots,u_m \in H^*(X,B;R)\}$ *s.t.* $u_1 \cdots u_m \neq 0\}$
2. $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R) = \mathrm{Max}\left\{m{\geq}0 \middle| \forall f : Y \to X \in \underline{\underline{\mathcal{T}}}^B_B\ (\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f) < m),\ f^*(u) = 0\right\}$

This immediately implies the following.

Theorem 1.10. *For a space B, we have $\mathcal{Z}_{\mathrm{R}}(B) = \mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(d(B);R)$ for a ring $R \ni 1$.*

Motivating by this equality, we proceed to obtain the following result.

Theorem 1.11. *For any space B, any element $u \in H^*(B{\times}B, \Delta(B);R)$ and a ring $R \ni 1$, we have $\mathrm{wgt}_\pi(u;R) = \mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R)$.*

Let us consider one technical condition on a fibrewise pointed space:

Theorem 1.12. *For any space B having the homotopy type of a locally finite simplicial complex, we may assume that $d(B)$ is fibrewise well-pointed up to homotopy.*

The following is the main result of our paper.

Theorem 1.13. *For any fibrewise well-pointed space X over B, we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{cat}^*_{\mathrm{B}}(X)$. So, if B is a locally finite simplicial complex, we have $\mathcal{TC}(B) = \mathcal{TC}^{\mathrm{M}}(B) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) + 1$.*

In [19], Sakai showed, in his study of the fibrewise *pointed* L-S category of a fibrewise well-pointed spaces, using Whitehead style definition, that we can utilise A_∞ methods used in the study of L-S category (see Iwase [7, 8]). Let us state the Whitehead style definitions of fibrewise L-S categories following [19].

Definition 1.14. *Let X be a fibrewise well-pointed space over B. The fibrewise **pointed** L-S category $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is the minimal number $m \geq 0$ such that the $(m{+}1)$-fold fibrewise diagonal $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi}_B X$ is compressible into the fibrewise fat wedge $\overset{m+1}{\mathrm{T}}_B X$ in $\underline{\underline{\mathcal{T}}}^B_B$. If there are no such m, we say $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \infty$.*

We remark that this new definition coincides with the ordinary one, if the total space X is a finite simplicial complex.

The above Whitehead-style definition allows us to define the module weight, cone length and categorical length, and moreover, to give their relationship as in Section 8. To show that, we need a criterion given by fibrewise A_∞ structure on the fibrewise loop space (see Sections 6–7).

2. Proof of Theorem 1.7

First, we show the equality $\mathcal{TC}^{\mathrm{M}}(B) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B))+1$: assume $\mathcal{TC}^{\mathrm{M}}(B) = m+1$, $m \geq 0$ and that there are an open cover $\bigcup_{i=0}^{m} U_i = B\times B$ and a series of sections $s_i : U_i \to \mathcal{P}(B)$ of $\pi : \mathcal{P}(B) \to d(B)$ satisfying $s_i(b,b) = c_b$ for $b \in B$, since we are considering monoidal topological complexity. Then each U_i is fibrewise compressible relative to $\Delta(B)$ into $\Delta(B) \subset B\times B = d(B)$ by a homotopy $H_i : U_i\times[0,1] \to B\times B$ given by the following:

$$H_i(a,b;t) = (s_i(a,b)(t), b), \quad (a,b) \in U_i,\ t \in [0,1],$$

where we can easily check that H_i gives a fibrewise compression of U_i relative to $\Delta(B)$ into $\Delta(B) \subset B\times B$. Since $\bigcup_{i=0} U_i = B\times B = d(B)$, we obtain $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) \leq m$, and hence we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) + 1 \leq \mathcal{TC}^{\mathrm{M}}(B)$.

Conversely assume that $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) = m$, $m \geq 0$ and there is an open cover $\bigcup_{i=0}^{m} U_i = d(B)$ of $d(B) = B\times B$ where U_i is fibrewise compressible relative to $\Delta(B)$ into $\Delta(B) \subset d(B) = B\times B$: let us denote the compression homotopy of U_i by $H_i(a,b;t) = (\sigma_i(a,b;t), b)$ for $(a,b) \in U_i$ and $t \in [0,1]$, where $\sigma_i(a,b;0) = a$ and $\sigma_i(a,b;1) = b$. Hence we can define a section $s_i : U_i \to \mathcal{P}(B)$ by the formula

$$s_i(a,b)(t) = \sigma_i(a,b;t) \quad t \in [0,1].$$

Since $\bigcup_{i=0} U_i = B\times B$, we obtain $\mathcal{TC}^{\mathrm{M}}(B) \leq m+1$ and hence we have $\mathcal{TC}^{\mathrm{M}}(B) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) + 1$. Thus we have $\mathcal{TC}^{\mathrm{M}}(B) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)) + 1$.

Second, we show the equality $\mathcal{TC}(B) = \mathrm{cat}^{*}_{\mathrm{B}}(d(B))+1$: assume $\mathcal{TC}(B) = m+1$, $m \geq 0$ and that there is a open cover $\bigcup_{i=0}^{m} U_i = B\times B$ and a section $s_i : U_i \to \mathcal{P}(B)$ of $\pi : \mathcal{P}(B) \to d(B)$. Then each U_i is fibrewise compressible into $\Delta(B) \subset B\times B = d(B)$ by a homotopy $H_i : U_i\times[0,1] \to B\times B$ which is given by

$$H_i(a,b;t) = (s(a,b)(t), b), \quad (a,b) \in U_i,\ t \in [0,1],$$

where we can easily check that H gives a fibrewise compression of U_i into $\Delta(B) \subset B\times B = d(B)$. Since $\bigcup_{i=0} U_i = B\times B = d(B)$, we obtain $\mathrm{cat}^{*}_{\mathrm{B}}(d(B)) \leq m$, and hence we have $\mathrm{cat}^{*}_{\mathrm{B}}(d(B)) + 1 \leq \mathcal{TC}(B)$.

Conversely assume that $\mathrm{cat}^{*}_{\mathrm{B}}(d(B)) = m$, $m \geq 0$ and there is an open cover $\bigcup_{i=0}^{m} U_i = d(B)$ of $d(B) = B\times B$ where U_i is fibrewise compressible into $\Delta(B) \subset B\times B = d(B)$: the compression homotopy is described as $H_i(a,b;t) = (\sigma_i(a,b;t), b)$ for $(a,b) \in U_i$ and $t \in [0,1]$, such that $\sigma_i(a,b;0) = a$ and $\sigma_i(a,b;1) = b$. Hence we can define a section $s_i : U_i \to \mathcal{P}(B)$ by the formula

$$s_i(a,b)(t) = \sigma_i(a,b;t) \quad t \in [0,1].$$

Since $\bigcup_{i=0} U_i = B\times B$, we obtain $\mathcal{TC}(B) \leq m+1$ and hence we have $\mathcal{TC}(B) \leq \mathrm{cat}^{*}_{\mathrm{B}}(d(B)) + 1$. Thus we have $\mathcal{TC}(B) = \mathrm{cat}^{*}_{\mathrm{B}}(d(B)) + 1$. □

3. Proof of Theorem 1.11

Assume that $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R) = m$, where $u \in H^*(B{\times}B, \Delta(B))$ and $f : Y \to d(B) = B{\times}B$ a map of $\mathrm{secat}(f^*\pi) < m$. Then there is an open cover $\bigcup_{i=1}^{m} U_i = Y$ and a series of maps $\{\sigma_i : U_i \to \mathcal{P}(B)\,;\, 1 \le i \le m\}$ satisfying $\pi{\circ}\sigma_i = f|_{U_i}$. Let $\hat{Y} = Y \amalg B$ with projection $p_{\hat{Y}}$ and section $s_{\hat{Y}}$ given by

$$p_{\hat{Y}}|_Y = p_Y, \quad p_{\hat{Y}}|_B = \mathrm{id}_B \quad \text{and} \quad s_{\hat{Y}} : B \hookrightarrow Y \amalg B = \hat{Y}.$$

Then we can extend f to a map $\hat{f} : \hat{Y} \to d(B)$ by the formula

$$\hat{f}|_Y = f, \quad \hat{f}|_B = s_{d(B)} = \Delta.$$

By putting $\hat{U}_i = U_i \amalg B$ which is open in $\hat{Y}$, we obtain an open cover $\bigcup_{i=1}^{m} \hat{U}_i = \hat{Y}$ and a series of maps $\hat{\sigma}_i : \hat{U}_i \to \mathcal{P}(B)$ satisfying $\pi{\circ}\hat{\sigma}_i = \hat{f}|_{\hat{U}_i}$:

$$\hat{\sigma}_i|_{U_i} = \sigma_i, \quad \hat{\sigma}_i|_B = s_{\mathcal{P}(B)}.$$

Hence there is a fibrewise homotopy $\Phi_i : \hat{U}_i{\times}[0,1] \to d(B)$ such that $\Phi_i(y,0) = \hat{f}(y)$ and $\Phi_i(y,1) \in \Delta(B)$ given by the following formula.

$$\Phi_i(y,t) = (\hat{\sigma}_i(y)(t), \hat{\sigma}_i(y)(1)), \quad (y,t) \in \hat{U}_i{\times}[0,1],$$

so that we have $\Phi_i(y,0) = (\hat{\sigma}_i(y)(0), \hat{\sigma}_i(y)(1)) = \pi{\circ}\hat{\sigma}_i(y) = \hat{f}(y)$ and $\Phi_i(y,1) = (\hat{\sigma}_i(y)(1), \hat{\sigma}_i(y)(1)) \in \Delta(B)$. Moreover, for any $(b,t) \in B{\times}[0,1]$, we have $\Phi_i(b,t) = (\hat{\sigma}_i(b)(t), \hat{\sigma}_i(b)(1)) = (s_{\mathcal{P}(B)}(t), s_{\mathcal{P}(B)}(1)) = (b,b)$. Thus Φ_i gives a fibrewise pointed compression homotopy of $\hat{f}|_{\hat{U}_i}$ into $\Delta(B)$. Then it follows that $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(\hat{f}) < m$ and hence we obtain $f^*(u) = 0$ and $\mathrm{wgt}_\pi(u;R) \ge m$. Thus we obtain $\mathrm{wgt}_\pi(u;R) \ge m = \mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R)$.

Conversely assume that $\mathrm{wgt}_\pi(u;R) = m$, where $u \in H^*(B{\times}B, \Delta(B))$ and $f : Y \to B{\times}B$ such that $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f) < m$. Then there exists an open covering $\bigcup_{i=1}^{m} U_i = Y$ with $U_i \supset s_Y(B)$ and a sequence of fibrewise homotopies $\{\phi_i : U_i{\times}[0,1] \to B{\times}B\}$ such that $\phi_i(y,0) = f|_{U_i}(y)$, $\phi_i(y,1) \in \Delta(B)$ and $\mathrm{pr}_2{\circ}\phi_i(y,t) = \mathrm{pr}_2{\circ}f(y)$ for $(y,t) \in U_i{\times}[0,1]$. Hence there is a sequence of maps $\{\sigma_i : U_i \to \mathcal{P}(B)\}$ given by

$$\sigma_i(y)(t) = \mathrm{pr}_1{\circ}\phi_i(y,t), \quad y \in U_i,\ t \in [0,1]$$

such that $\pi{\circ}\sigma_i(y) = (\mathrm{pr}_1{\circ}\phi_i(y,0), \mathrm{pr}_1{\circ}\phi_i(y,1)) = f(y)$ since $\mathrm{pr}_2{\circ}\phi_i(y,t) = \mathrm{pr}_2{\circ}f(y)$ for $(y,t) \in U_i{\times}[0,1]$. Thus we obtain $\mathrm{secat}(f^*\pi) < m$, and hence $f^*(u) = 0$. This implies $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R) \ge m = \mathrm{wgt}_\pi(u;R)$ and hence $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;R) = \mathrm{wgt}_\pi(u;R)$. □

4. Proof of Theorem 1.12

The proof of Lemma 2 in §2 of Milnor [16] implies the following:

Lemma 4.1. *The pair $(B{\times}B, \Delta(B))$ is an NDR-pair.*

Proof: For each vertex β of B, let V_β be the star neighbourhood in B and $V = \bigcup_\beta V_\beta{\times}V_\beta \subset B{\times}B$. Then the closure $\bar{V} = \bigcup_\beta \bar{V}_\beta{\times}\bar{V}_\beta$ is a subcomplex of $B{\times}B$. For the barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ of x and y, resp, we see that $(x,y) \in V$ if and only if $\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta) > 0$ and that $\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta) = 1$ if and only if the barycentric coordinates of x and y are the same, or equivalently, $(x,y) \in \Delta(B)$. Hence we can define a continuous map $v : B{\times}B \to [-1,1]$ by the following formula.

$$v(x,y) = \begin{cases} 2\sum_\beta \mathrm{Min}(\xi_\beta, \eta_\beta) - 1, & \text{if } (x,y) \in \bar{V}, \\ -1, & \text{if } (x,y) \notin V. \end{cases}$$

Then we have that $v^{-1}(1)=\Delta(B)$. Let $U=v^{-1}((0,1])$ an open neighbourhood of $\Delta(B)$. Using Milnor's map s, we obtain a pair of maps (u,h) as follows:

$$u(x,y)=\mathrm{Min}\{1,1-v(x,y)\}\quad\text{and}$$
$$h(x,y,t)=(s(x,y)(\mathrm{Min}\{t,w(x,y)\}),y),$$

where $w(x,y)=u(x,y)+v(x,y)=\mathrm{Min}\{1,1+v(x,y)\}$. Note that $w(x,y)=1$ if $(x,y)\in U$ and that $w(x,y)=0$ if $(x,y)\notin V$. Then $u^{-1}(0)=\Delta(B)$, $u^{-1}([0,1))=U$ and $h(x,y,1)=(y,y)\in\Delta(B)$ if $(x,y)\in U$. Moreover, $\mathrm{pr}_2\circ h(x,y,t)=y$ and $h(x,x,t)=(s(x,x)(t),x)=(x,x)$ for any $x,y\in B$ and $t\in[0,1]$. Thus the data (u,h) gives the fibrewise Strøm structure on $(B\times B,\Delta(B))$. □

5. Proof of Theorem 1.13

Let X be a fibrewise well-pointed space over B and $\hat{X}$ the fiberwise pointed space obtained from X by giving a fibrewise whisker. More precisely, we define $\hat{X}$ be the mapping cylinder of s_X,

$$\hat{X}=X\cup_{s_X}B\times[0,1],\quad X\ni s_X(b)\sim(b,0)\in B\times[0,1]\text{ for any }b\in B,$$

with projection $p_{\hat{X}}$ and section $s_{\hat{X}}$ given by the formulas

$$p_{\hat{X}}|_X=p_X,\quad p_{\hat{X}}|_{B\times[0,1]}(b,t)=b,\quad\text{for }(b,t)\in B\times[0,1],$$
$$s_{\hat{X}}(b)=(b,1)\in B\times[0,1]\subset\hat{X}.$$

Then by the definition of Strøm structure, X is fibrewise pointed homotopy equivalent to $\hat{X}$ the fibrewise whiskered space over B. So we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)=\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(\hat{X})$ and $\mathrm{cat}^*_{\mathrm{B}}(X)=\mathrm{cat}^*_{\mathrm{B}}(\hat{X})$.

Assume that $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)=m\geq 0$. Then it is clear by definition that $\mathrm{cat}^*_{\mathrm{B}}(X)\leq m=\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$.

Conversely assume that $\mathrm{cat}^*_{\mathrm{B}}(X)=m\geq 0$. Then there is an open cover $\bigcup_{i=0}^m U_i=X$ such that U_i is compressible into $s_X(B)\subset X$. Hence there is a fibrewise homotopy $\Phi_i:U_i\times[0,1]\to X$ such that $\Phi_i(x,0)=x$, $\Phi_i(x,1)=s_X(p_X(x))$ and $p_X\circ\Phi_i(x,t)=p_X(x)$. We define $\hat{U}_i$ as follows:

$$\hat{U}_i=U_i\cup_{s_X}(s_X)^{-1}(U_i)\times[0,1]\cup B\times(\frac{2}{3},1].$$

We also define a fibrewise pointed homotopy $\hat{\Phi}_i:\hat{U}_i\times[0,1]\to\hat{X}$ as follows:

$$\hat{\Phi}_i(\hat{x},t)=\begin{cases}\Phi_i(x,t), & \hat{x}=x\in X,\\ \Phi_i(s_X(b),t-3s), & \hat{x}=(b,s)\in(s_X)^{-1}(U_i)\times(0,\frac{t}{3}),\\ s_X(b), & \hat{x}=(b,\frac{t}{3}),b\in(s_X)^{-1}(U_i),\\ (b,\frac{6s-2t}{6-3t}), & \hat{x}=(b,s)\in(s_X)^{-1}(U_i)\times(\frac{t}{3},\frac{2}{3}),\\ (b,\frac{2}{3}), & \hat{x}=(b,\frac{2}{3}),b\in(s_X)^{-1}(U_i),\\ (b,s), & \hat{x}=(b,s)\in B\times(\frac{2}{3},1].\end{cases}$$

It is then easy to see that $\hat{U}_i$'s cover the entire X, and hence we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(\hat{X})\leq m=\mathrm{cat}^*_{\mathrm{B}}(X)$. Thus $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)\leq\mathrm{cat}^*_{\mathrm{B}}(X)$ and hence $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)=\mathrm{cat}^*_{\mathrm{B}}(X)$. In particular, we have $\mathcal{TC}(B)=\mathcal{TC}^{\mathrm{M}}(B)$ for a locally finite simplicial complex B. □

6. Fibrewise A_∞ structures

From now on, we work in the category $\underline{\underline{\mathcal{T}}}_B^B$. For any X a fibrewise pointed space over B, we denote by $p_X : X \to B$ its projection and by $s_X : B \to X$ its section.

We say that a pair (X, A) of fibrewise pointed spaces over B is a fibrewise NDR-pair or that A is a fibrewise NDR subset of X, if the inclusion map $A \hookrightarrow X$ is a fibrewise cofibration, in other words, the inclusion has the fibrewise (strong) Strøm structure (see Crabb-James [1]). Since B is the zero object in $\underline{\underline{\mathcal{T}}}_B^B$, for any given fibrewise pointed space X over B, we always have a pair (X, B) in $\underline{\underline{\mathcal{T}}}_B^B$, where we regard $s_X(B) = B$. When the pair (X, B) is fibrewise NDR, the space X is called fibrewise well-pointed.

Proposition 6.1 (Crabb-James [1]). (1) *If (X, A) and (X', A') are fibrewise NDR-pairs, then so is $(X, A)\times_B(X', A') = (X\times_B X', X\times_B A'\cup A\times_B X')$.*

(2) *If (X, A) is a fibrewise NDR-pair, then so is $(\overset{m}{\Pi}_B X, \overset{m}{\mathrm{T}}_B(X, A))$, which is defined by induction for all $m \geq 1$:*

$$(\overset{1}{\Pi}_B X, \overset{1}{\mathrm{T}}_B(X, A)) = (X, A),$$
$$(\overset{m+1}{\Pi}_B X, \overset{m+1}{\mathrm{T}}_B(X, A)) = (\overset{m}{\Pi}_B X, \overset{m}{\mathrm{T}}_B(X, A))\times_B(X, A).$$

If X is a fibrewise pointed space over B, then by taking $A = B$, we obtain a fibrewise subspace $\overset{m+1}{\mathrm{T}}_B(X, B)$ of $\overset{m+1}{\mathrm{T}}_B X$, which is called an $(m+1)$-fold fibrewise fat-wedge of X, and is often denoted by $\overset{m+1}{\mathrm{T}}_B X$. In addition, the pair $(\overset{m+1}{\Pi}_B X, \overset{m+1}{\mathrm{T}}_B X)$ is a fibrewise NDR-pair for all $m \geq 0$, if X is fibrewise well-pointed.

Examples 6.2. (1) *Let X be a fibrewise pointed space over B with $p_X = pr_2 : X = F\times B \to B$ the canonical projection to the second factor and $s_X = in_2 : B \hookrightarrow F\times B = E$ the canonical inclusion to the second factor. Then X is a fibrewise pointed space over B.*

(2) *Let $X = B\times B$, $p_X = pr_2 : B\times B \to B$ the canonical projection to the second factor and $s_X = \Delta_B : B \hookrightarrow B\times B$ the diagonal. Then X is a fibrewise pointed space over B.*

(3) *Let G be a topological group, EG the infinite join of G with right G action and $BG = EG/G$ the classifying space of G. By considering G as a left G space by the adjoint action, we obtain a fibrewise pointed space $X = EG\times_G G$ with $p_X : EG\times_G G \to BG$ with section $s_X : BG \hookrightarrow EG\times_G\{e\} \subseteq EG\times_G G$.*

(4) *Let B be a space, $X = \mathcal{L}(B)$ the space of free loops on B. Then $p_X : \mathcal{L}(B) \to B$ the evaluation map at $1 \in S^1 \subset \mathbb{C}$ is a fibration with section $s_X : B \to \mathcal{L}(B)$ given by the inclusion of constant loops. In view of Milnor's arguments, this example is homotopically equivalent to the example (3).*

Definition 6.3. *Let $\mathcal{P}_{\mathrm{B}}(X) = \left\{\ell : [0,1] \to X \,\middle|\, \exists_{b\in B} \text{ s.t. } \forall_{t\in[0,1]}\ p_X(\ell(t)){=}b\right\}$ the fibrewise free path space, $\mathcal{L}_{\mathrm{B}}(X) = \{\ell \in \mathcal{P}_{\mathrm{B}}(X) | \ell(1){=}\ell(0)\}$ the fibrewise free loop space and $\mathcal{L}_B^B(X) = \{\ell \in \mathcal{P}_{\mathrm{B}}(X) | \ell(1){=}\ell(0){=}s_X{\circ}p_X(\ell(0))\}$ the fibrewise pointed loop space. For any $m \geq 0$, we define an A_∞ structure of $\mathcal{L}_B^B(X)$ as follows.*

(1) *$E_B^{m+1}(\mathcal{L}_B^B(X))$ as the homotopy pull-back in $\underline{\underline{\mathcal{T}}}_B^B$ of $B \hookrightarrow \overset{m+1}{\Pi}_B X \hookleftarrow \overset{m+1}{\mathrm{T}}_B X$,*

(2) *$P^m_B(\mathcal{L}^B_B(X))$ as the homotopy pull-back in $\underline{\underline{\mathcal{T}}}^B_B$ of $X \xrightarrow{\Delta^{m+1}_B} \overset{m+1}{\Pi_B} X \hookleftarrow \overset{m+1}{\mathrm{T}_B} X$,*

(3) *$e^X_m : P^m_B(\mathcal{L}^B_B(X)) \to X$ as the induced map from the inclusion $\overset{m+1}{\mathrm{T}_B} X \hookrightarrow \overset{m+1}{\Pi_B} X$ by the diagonal $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi_B} X$ and*

(4) *$p^{\mathcal{L}^B_B(X)}_B : E^{m+1}_B(\mathcal{L}^B_B(X)) \to P^m_B(\mathcal{L}^B_B(X))$ as a map of fibrewise pointed spaces induced from the section $s_X : B \to X$, since the section $B \hookrightarrow \overset{m+1}{\Pi_B} X$ is nothing but the composition $\Delta^{m+1}_B \circ s_X : B \xrightarrow{s} X \xrightarrow{\Delta^{m+1}_B} \overset{m+1}{\Pi_B} X$.*

We further investigate to understand an A_∞ stucture in a fiberwise view point, using fibrewise constructions. Clearly, these constructions are *not* exactly the Ganea-type fibre-cofibre constructions but the following.

Proposition 6.4 (Sakai)**.** *Let X be a fibrewise pointed space over B and $m \geq 0$. Then $P^{m+1}_B(\mathcal{L}^B_B(X))$ has the homotopy type of a push-out of $p^{\mathcal{L}^B_B(X)}_B : E^{m+1}_B(\mathcal{L}^B_B(X)) \to P^m_B(\mathcal{L}^B_B(X))$ and the projection $E^{m+1}_B(\mathcal{L}^B_B(X)) \to B$.*

This is a direct consequence of the following lemma.

Lemma 6.5. *Let (X, A) and (X', A') be fibrewise NDR-pairs of fibrewise pointed spaces over B and Z a fibrewise pointed space over B with fibrewise maps $f : Z \to X$ and $g : Z \to X'$. Then the homotopy pull-back $\Omega_{(f,g),k}$ of maps $(f, g) : Z \to X \times_B X'$ and $k : X \times_B A' \cup A \times_B X' \hookrightarrow X \times_B X'$ has naturally the homotopy type of the reduced homotopy push-out $W = \Omega_{g,j} \cup_{p_2} \left\{\Omega_{(f,g),i\times j} \wedge_B (B \times J^+)\right\} \cup_{p_1} \Omega_{f,i}$ of $p_1 : \Omega_{(f,g),i\times j} \to \Omega_{f,i}$ and $p_2 : \Omega_{(f,g),i\times j} \to \Omega_{g,j}$, where $J = [-1, 1]$ and*

$$\Omega_{(f,g),k} = \left\{(z, \ell, \ell') \in Z \times_B \mathcal{P}_{\mathrm{B}}(X) \times_B \mathcal{P}_{\mathrm{B}}(X') \middle| \substack{f(z)=\ell(0),\ g(z)=\ell'(0),\\ (\ell(1),\ell'(1))\in A\times_B X' \cup X\times_B A'}\right\},$$

$$\Omega_{(f,g),i\times j} = \left\{(z, \ell, \ell') \in \Omega_{(f,g),k} \middle| (\ell(1), \ell'(1)) \in A \times_B A'\right\},$$

$$\Omega_{f,i} = \{(z, \ell) \in Z \times_B \mathcal{P}_{\mathrm{B}}(X) | f(z)=\ell(0),\ \ell(1)\in A\},$$

$$\Omega_{g,j} = \{(z, \ell') \in Z \times_B \mathcal{P}_{\mathrm{B}}(X') | g(z)=\ell'(0),\ \ell'(1)\in A'\},$$

$p_1(z, \ell, \ell') = (z, \ell)$ and $p_2(z, \ell, \ell') = (z, \ell')$.

Proof of Outline of the proof. The proof of Lemma 6.5 is quite similar to that of Theorem 1.1 in Sakai [20] (which is based on Iwase [7]) by replacing (Y, B) in [20] by (X', A'), defining and using the following spaces.

$$\widehat{W} = \Omega_{(f,g),i\times \mathrm{id}_{X'}} \times \{-1\} \cup \left\{\Omega_{(f,g),i\times j} \times J\right\} \cup \Omega_{(f,g),id_X\times j} \times \{1\} \subset \Omega_{(f,g),k} \times J,$$

$$\Omega_{(f,g),\mathrm{id}_X\times j} = \left\{(z, \ell, \ell') \in \Omega_{(f,g),k} \middle| (\ell(1), \ell'(1)) \in X \times_B A'\right\},$$

$$\Omega_{(f,g),i\times id_{X'}} = \left\{(z, \ell, \ell') \in \Omega_{(f,g),k} \middle| (\ell(1), \ell'(1)) \in A \times_B X'\right\}.$$

The precise construction of homotopy equivalences and homotopies is identical to that in [20] and is left to the readers. □

Theorem 6.6. *Let X be a fibrewise well-pointed space over B. Then the sequence $\{p^{\mathcal{L}^B_B(X)}_B : E^{m+1}_B(\mathcal{L}^B_B(X)) \to P^m_B(\mathcal{L}^B_B(X))\}$ gives a fibrewise pointed version of A_∞-structure on the fibrewise pointed loop space $\mathcal{L}^B_B(X)$.*

Thus in the case when X is a fibrewise well-pointed space over B, we assume that $P^m_B(\mathcal{L}^B_B(X))$ is an increasing sequence given by homotopy push-outs with a fibrewise fibration $e^X_m : P^m_B(\mathcal{L}^B_B(X)) \to X$ such that $e^X_1 : \mathcal{S}^B_B(\mathcal{L}^B_B(X)) \to X$ is a fibrewise evaluation.

Examples 6.7. (1) *Let X be a fibrewise pointed space over B with $p_X = pr_2 : F{\times}B \to B$ the canonical projection and $s_X = in_2 : B \hookrightarrow F{\times}B$ the canonical inclusion. Then $\mathcal{L}^B_B(X) = \mathcal{L}(F){\times}B$ is given by $p_{\mathcal{L}^B_B(X)} = pr_2 : \mathcal{L}(F){\times}B \to B$ and $s_{\mathcal{L}^B_B(X)} = in_2 : B \hookrightarrow \mathcal{L}(F){\times}B$.*

(2) *Let $X = B{\times}B$ be a fibrewise pointed space over B with $p_X = pr_2 : B{\times}B \to B$ and $s_X = \Delta_B : B \hookrightarrow B{\times}B$ the diagonal. Then $\mathcal{L}^B_B(X) = \mathcal{L}(B)$ the free loop space on B, $p_{\mathcal{L}^B_B(X)} : \mathcal{L}(B) \to B$ the evalation map at $1 \in S^1 \subset \mathbb{C}$ and $s_{\mathcal{L}^B_B(X)} : B \hookrightarrow \mathcal{L}(B)$ the inclusion of constant loops.*

Remark 6.8. *When E is a cell-wise trivial fibration on a polyhedron B (see* [12]*), we can see that the canonical map $e^E_\infty : P^\infty_B(\mathcal{L}^B_B(E)) \to E$ is a homotopy equivalence by a similar arguments given in the proof of Theorem 2.9 of* [12].

7. Fibrewise L-S categories of fibrewise pointed spaces

The fibrewise *pointed* L-S category of an fibrewise pointed space is first defined by James and Morris [13] as the least number (minus one) of open subsets which cover the given space and are contractible by a homotopy fixing the base point in each fibre (see also James [14] and Crabb-James [1]) and is redefined by Sakai in [19] as follows: let X be a fibrewise pointed space over B. For given $k \geq 0$, we denote by $\overset{k+1}{\Pi}_B X$ the $(k{+}1)$-fold fibrewise product and by $\overset{k+1}{\mathrm{T}}_B X$ the $(k{+}1)$-fold fibrewise fat wedge. Then $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq m$ if the $(m{+}1)$-fold fibrewise diagonal map $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi}_B X$ is compressible into the fibrewise fat wedge $\overset{m+1}{\mathrm{T}}_B X$ in $\underline{\underline{\mathcal{T}}}^B_B$. If there is no such m, we say $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \infty$. Let us consider the case when $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) < \infty$. The definition of a fibrewise A_∞ structure yields the following criterion.

Theorem 7.1. *Let X be a fibrewise pointed space over B and $m \geq 0$. Then $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq m$ if and only if $\mathrm{id}_X : X \to X$ has a lift to $P^m_B(\mathcal{L}^B_B(X)) \overset{e^X_m}{\to} X$ in $\underline{\underline{\mathcal{T}}}^B_B$.*

Proof: If $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq m$, then the fibrewise diagonal $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi}_B X$ is compressible into the fibrewise fat wedge $\overset{m+1}{\mathrm{T}}_B X \subset \overset{m+1}{\Pi}_B X$ in $\underline{\underline{\mathcal{T}}}^B_B$. Hence there is a map $\sigma : X \to P^m_B(\mathcal{L}^B_B(X))$ in $\underline{\underline{\mathcal{T}}}^B_B$ such that $e^X_m{\circ}\sigma \sim_B 1_X$ in $\underline{\underline{\mathcal{T}}}^B_B$. The converse is clear by the definition of $P^m_B(\mathcal{L}^B_B(X))$. □

In the rest of this section, we work within the category $\underline{\underline{\mathcal{T}}}_B$ of fibrewise *unpointed* spaces and maps between them. But we concentrate ourselves to consider its full subcategory $\underline{\underline{\mathcal{T}}}^*_B$ of all fibrewise pointed spaces, so in $\underline{\underline{\mathcal{T}}}^*_B$, we have more maps than in $\underline{\underline{\mathcal{T}}}^B_B$ while we have just the same objects as in $\underline{\underline{\mathcal{T}}}^B_B$.

Let X be a fibrewise pointed space over B. For given $k \geq 0$, we denote by $\overset{k+1}{\Pi}_B X$ the $(k{+}1)$-fold fibrewise product and by $\overset{k+1}{\mathrm{T}}_B X$ the $(k{+}1)$-fold fibrewise fat wedge. Then $\mathrm{cat}^*_{\mathrm{B}}(X) \leq m$ if the $(m{+}1)$-fold fibrewise diagonal map $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi}_B X$

is compressible into the fibrewise fat wedge $\overset{m+1}{\mathrm{T}}_B X$ in $\underline{\underline{\mathcal{T}}}^*_B$. If there is no such m, we say $\mathrm{cat}^*_{\mathrm{B}}(X) = \infty$. Let us consider the case when $\mathrm{cat}^*_{\mathrm{B}}(X) < \infty$. The definition of a fibrewise A_∞ structure yields the following.

Theorem 7.2. *Let X be a fibrewise pointed space over B and $m \geq 0$. Then $\mathrm{cat}^*_{\mathrm{B}}(X) \leq m$ if and only if $\mathrm{id}_X : X \to X$ has a lift to $P^m_B(\mathcal{L}^B_B(X)) \overset{e^X_m}{\to} X$ in the category $\underline{\underline{\mathcal{T}}}^*_B$.*

Proof: If $\mathrm{cat}^*_{\mathrm{B}}(X) \leq m$, then the fibrewise diagonal $\Delta^{m+1}_B : X \to \overset{m+1}{\Pi}_B X$ is compressible into the fibrewise fat wedge $\overset{m+1}{\mathrm{T}}_B X \subset \overset{m+1}{\Pi}_B X$ in $\underline{\underline{\mathcal{T}}}^*_B$. Hence there is a map $\sigma : X \to P^m_B(\mathcal{L}^B_B(X))$ in $\underline{\underline{\mathcal{T}}}^*_B$ such that $e^X_m \circ \sigma \sim_B 1_X$ in $\underline{\underline{\mathcal{T}}}^*_B$. The converse is clear by the definition of $P^m_B(\mathcal{L}^B_B(X))$. □

8. Upper and lower estimates

For X a fibrewise pointed space over B, we define a fibrewise version of Ganea's strong L-S category (see Ganea [6]) of X as $\mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ and also a fibrewise version of Fox's categorical length (see Fox [5] and Iwase [10]) of X as $\mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}(X)$.

Definition 8.1. *Let X be a fibrewise pointed space over B.*

(1) $\mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ *is the least number $m \geq 0$ such that there exists a sequence $\{(X_i, h_i) | h_i : A_i \to X_{i-1},\ 0 \leq i \leq m\}$ of pairs of space and map satisfying $X_0 = B$ and $X_m \simeq_B X$ in $\underline{\underline{\mathcal{T}}}^B_B$ with the following homotopy push-out diagrams:*

$$\begin{array}{ccc} A_i & \xrightarrow{p_{A_i}} & B \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle s_{X_i}} \\ X_{i-1} & \longrightarrow & X_i \end{array}$$

(2) $\mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}(X)$ *is the least number $m \geq 0$ such that there exists a sequence $\{X_i | h_i : A_i \to X_{i-1},\ 0 \leq i \leq m\}$ of spaces satisfying $X_0 = B$ and $X_m \simeq_B X$ in $\underline{\underline{\mathcal{T}}}^B_B$ and that $\Delta_B : X_i \to X_i \times_B X_i$ is compressible into $X_i \times_B X_{i-1} \cup B \times_B X_i$ in $X_m \times_B X_m$.*

A lower bound for the fibrewise L-S category of a fibrewise pointed space X over B can be described by a variant of cup length: since X is a fibrewise pointed space over B, there is a projection $p_X : X \to B$ with its section $s_X : B \to X$. Hence we can easily observe for any multiplicative cohomology theory h that

$$h^*(X) \cong h^*(B) \oplus h^*(X, B),$$

where we may identify $h^*(X, B)$ with the ideal $\ker s^*_X : h^*(X) \to h^*(B)$.

Definition 8.2. *For a fibrewise pointed space X over B and any multiplicative cohomology theory h, we define*

$$\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X; h) = \mathrm{Max}\,\{m \geq 0 | \exists \{u_1, \cdots, u_m \in h^*(X, B)\} \text{ s.t. } u_1 \cdots u_m \neq 0\},$$

$$\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{Max}\,\left\{\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X; h) \middle| h \text{ is a multiplicative cohomology theory}\right\}.$$

We often denote $\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(\ ;h)$ by $\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(\ ;R)$ when $h^*(\ ) = H^*(\ ;R)$, where R is a ring with unit.

Let us recall that the relationship between an A_∞-structure and a Lusternik-Schnirelmann category gives the key observation in [7, 8, 9].

On the other hand, Rudyak [17] and Strom [23] introduced a homotopy theoretical version of Fadell-Husseini's category weight, which can be translated into our setting as follows: for any fibrewise pointed space X over B, let $\{p^{\mathcal{L}^B_B(X)}_k : E^k_B(\mathcal{L}^B_B(X)) \to P^{k-1}_B(\mathcal{L}^B_B(X))\,;\, k{\geq}1\}$ be the fibrewise A_∞-structure of $\mathcal{L}^B_B(X)$ in the sense of Stasheff [22] (see also [11] for some more properties). Let h be a generalisd cohomology theory.

Definition 8.3. *For any $u \in h^*(X,B)$, we define*

$$\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;h) = \mathrm{Min}\left\{m{\geq}0 \,\middle|\, (e^X_m)^*(u) \neq 0\right\},$$

where e^X_m is the composition of fibrewise maps $P^m_B(\mathcal{L}^B_B(X)) \hookrightarrow P^\infty_B(\mathcal{L}^B_B(X)) \xrightarrow[\simeq_B]{e^X_\infty} X$.

Using this, we introduce some more invariants as follows.

Definition 8.4. *For any fibrewise pointed space X over B, we define*

$$\begin{aligned}
&\mathrm{wgt}_\pi(X;h) = \mathrm{Max}\left\{\mathrm{wgt}_\pi(u;h) \,|\, u \in h^*(X,B)\right\},\\
&\mathrm{wgt}_\pi(X) = \mathrm{Max}\left\{\mathrm{wgt}_\pi(X;h) | h \textit{ is a generalised cohomology theory}\right\},\\
&\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(X;h) = \mathrm{Max}\left\{\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u;h) \,|\, u \in h^*(X,B)\right\},\\
&\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{Max}\left\{\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(X;h) \,\middle|\, h \textit{ is a generalised cohomology theory}\right\}.
\end{aligned}$$

We often denote $\mathrm{wgt}_\pi(\ ;h)$ and $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(\ ;h)$ by $\mathrm{wgt}_\pi(\ ;R)$ and $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(\ ;R)$ respectively when $h^*(\ ) = H^*(\ ;R)$, where R is a ring with unit. We define versions of module weight for a fibrewise pointed space over B.

Definition 8.5. *For a fibrewise pointed space X over B, we define*

(1) $\mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(X;h) = \mathrm{Min}\left\{m{\geq}0 \,\middle|\, \begin{array}{l}(e^X_m)^* \textit{ is a split mono of (unstable) } h^*h\text{-}\\ \textit{modules}\end{array}\right\}$ *for a generalisd cohomology theory h.*

(2) $\mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{Max}\left\{\mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(X;h) \,\middle|\, h \textit{ is a generalised cohomology theory}\right\}$.

Then we immediately obtain the following result.

Theorem 8.6. *For any fibrewise pointed space X over B, we have*

$$\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}(X) \leq \mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}(X).$$

By Lemma 4.1, we have the following as a corollary of Theorem 1.13.

Corollary 8.7. *For any space B having the homotopy type of a locally finite simplicial complex, we obtain*

$$\mathcal{Z}_\pi(B) \leq \mathrm{wgt}_\pi(B) \leq \mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(d(B)) \leq \mathcal{TC}(B){-}1 \leq \mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}(d(B)) \leq \mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}(d(B)).$$

9. Higher Hopf invariants

For any fibrewise pointed map $f : \mathcal{S}^B_B(V) \to X$ in $\underline{\underline{\mathcal{T}}}^B_B$, we have its adjoint $\mathrm{ad}\, f : V \to \mathcal{L}^B_B(X)$ such that

$$e^X_1 \circ \mathcal{S}^B_B(\mathrm{ad}\, f) = f : \mathcal{S}^B_B(V) \to X.$$

If $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \le m$, then there is a fibrewise pointed map $\sigma : X \to P^m_B\, \mathcal{L}^B_B(X)$ in $\underline{\underline{\mathcal{T}}}^B_B$ such that

$$e^X_1 \circ \sigma \simeq^B_B \mathrm{id}_X : X \to X.$$

Hence both the fibrewise maps $e^X_1 \circ (\sigma \circ f)$ and $e^X_1 \circ \mathcal{S}^B_B(\mathrm{ad}\, f)$ are fibrewise pointed homotopic to f in $\underline{\underline{\mathcal{T}}}^B_B$. Then we have

$$e^X_1 \circ \{\mathcal{S}^B_B(\mathrm{ad}\, f) - (\sigma \circ f)\} \simeq^B_B *_B,$$

where $\simeq^B_B$ denotes the fibrewise pointed homotopy and $*_B$ denotes the fibrewise trivial map in $\underline{\underline{\mathcal{T}}}^B_B$. Thus there is a fibrewise pointed map $H^\sigma_m(f) : \mathcal{S}^B_B(V) \to E^{m+1}_B\, \mathcal{L}^B_B(X)$ such that

$$p^{\mathcal{L}^B_B(X)}_m \circ H^\sigma_m(f) \simeq^B_B \mathcal{S}^B_B(\mathrm{ad}\, f) - (\sigma \circ f).$$

Definition 9.1. *Let X be of $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \le m$, $m \ge 0$. For $f : \mathcal{S}^B_B(V) \to X$, we define*

(1) $H^B_m(f) = \left\{ H^\sigma_m(f) \middle| e^X_1 \circ \sigma \simeq^B_B \mathrm{id}_X \right\} \subset [\mathcal{S}^B_B(V), X]$,

(2) $\mathcal{H}^B_m(f) = \left\{ (\mathcal{S}^B_B)^\infty_* H^\sigma_m(f) \middle| e^X_1 \circ \sigma \simeq^B_B \mathrm{id}_X \right\} \subset \left\{ \mathcal{S}^B_B(V), X \right\}^B_B$,

where, for two fibrewise spaces V and W, we denote by $\{V, W\}^B_B$ the homotopy set of fibrewise stable maps from V to W.

Appendix A. Fibrewise homotopy pull-backs and push-outs

In this paper, we are using A_∞ structures which is constructed using tools in $\underline{\underline{\mathcal{T}}}_B$ and $\underline{\underline{\mathcal{T}}}^B_B$ — especially, finite homotopy limits and colimits, in other words, fibrewise homotopy pull-backs and push-outs in $\underline{\underline{\mathcal{T}}}_B$ and $\underline{\underline{\mathcal{T}}}^B_B$. We show in this section that such constructions are possible even when a fibrewise space has some singular fibres.

First we consider the fibrewise homotopy pull-backs in $\underline{\underline{\mathcal{T}}}^B_B$: let X, Y, Z and E be fibrewise spaces over B and $p : E \to Z$ be a fibrewise fibration in $\underline{\underline{\mathcal{T}}}_B$. For any fibrewise map $f : X \to Z$ in $\underline{\underline{\mathcal{T}}}_B$, there exists a pull-back $X \xleftarrow{f^*p} f^*E \xrightarrow{\hat{f}} E$ of $X \xrightarrow{f} Z \xleftarrow{p} E$ as

$$f^*E = \{(x, e) \in X \times_B E | f(x) = p(e)\}$$

a subspace of $X \times_B E$ together with fibrewise maps $f^*p : f^*E \to X$ and $\hat{f} : f^*E \to E$ given by restricting canonical projections:

$$(f^*p)(x, e) = x, \quad \hat{f}(x, e) = e.$$

Theorem A.1 (Crabb-James [1])**.** *Let $p : E \to Z$ be a fibrewise fibration. For any fibrewise map $f : W \to Z$ in $\underline{\underline{\mathcal{T}}}_B$, $f^*p : f^*E \to W$ is also a fibrewise fibration.*

Let $\pi_t : \mathcal{P}_{\mathrm{B}}(Z) \to Z$ be fibrewise fibrations given by $\pi_t(\ell) = \ell(t)$, $t = 0, 1$ (see also [1]). Then π_0 and π_1 induce a map $\pi : \mathcal{P}_{\mathrm{B}}(Z) \to Z \times_B Z$ to the fibre product of two copies of $p_Z : Z \to B$.

Proposition A.2. *$\pi : \mathcal{P}_{\mathrm{B}}(Z) \to Z{\times}_B Z$ is a fibrewise fibration.*

Proof: For any fibrewise map $\phi : W \to \mathcal{P}_{\mathrm{B}}(Z)$ and a fibrewise homotopy $H : W{\times}[0,1] = W{\times}_B(I_B) \to Z{\times}_B Z$ such that $H(w,0) = \pi{\circ}\phi(w)$ for $w \in W$, we define a fibrewise homotopy $\hat{H} : W{\times}[0,1] = W{\times}_B(I_B) \to \mathcal{P}_{\mathrm{B}}(Z)(\subset \mathcal{P}(Z))$ by

$$\hat{H}(w,s)(t) = \begin{cases} \mathrm{pr}_0{\circ}H(w,s), & \text{if } t = 0, \\ \mathrm{pr}_0{\circ}H(w,s{-}3t), & \text{if } 0 < t < \frac{s}{3}, \\ \pi_0{\circ}\phi(w), & \text{if } t = \frac{s}{3}, \\ \phi(w)(\frac{3t-s}{3-2s}), & \text{if } \frac{s}{3} < t < \frac{3-s}{3}, \\ \pi_1{\circ}\phi(w), & \text{if } t = \frac{3-s}{3}, \\ \mathrm{pr}_1{\circ}H(w,3t{-}3{+}s), & \text{if } \frac{3-s}{3} < t < 1 \\ \mathrm{pr}_1{\circ}H(w,s), & \text{if } t = 0, \end{cases}$$

for $(w,s) \in W{\times}_B I_B$ and $t \in [0,1]$, where $\mathrm{pr}_k : Z{\times}_B Z \subset Z{\times}Z \to Z$ denotes the canonical projection given by $\mathrm{pr}_k(z_0,z_1) = z_k$, $k = 0,1$ for any $(z_0,z_1) \in Z{\times}_B Z$. Then for any $(w,s) \in W{\times}_B I_B$, we clearly have

$$\hat{H}(w,0)(t) = \phi(w)(t), \quad t \in [0,1],$$
$$(\hat{H}(w,s)(0), \hat{H}(w,s)(1)) = (\mathrm{pr}_0{\circ}H(w,s), \mathrm{pr}_1{\circ}H(w,s)) = H(w,s),$$

and hence we have $\hat{H}(w,0) = \phi(w)$ for any $w \in W$ and also $\pi{\circ}\hat{H} = H$. This implies that $\hat{H}$ is a fibrewise homotopy of ϕ covering H. Thus π is a fibrewise fibration. □

This yields the following corollary.

Corollary A.3. *For any fibrewise maps $f : X \to Z$ and $g : Y \to Z$ in $\underline{\underline{\mathcal{T}}}_B$, the induced map $(f{\times}_B g)^*\pi : (f{\times}_B g)^*\,\mathcal{P}_{\mathrm{B}}(Z) \to X{\times}_B Y$ is a fibrewise fibration in $\underline{\underline{\mathcal{T}}}_B$.*

We often call the fibrewise space $(f{\times}_B g)^*\,\mathcal{P}_{\mathrm{B}}(Z)$ together with the projections $\mathrm{pr}_X{\circ}(f{\times}_B g)^*\pi : (f{\times}_B g)^*\,\mathcal{P}_{\mathrm{B}}(Z) \to X$ and $\mathrm{pr}_Y{\circ}(f{\times}_B g)^*\pi : (f{\times}_B g)^*\,\mathcal{P}_{\mathrm{B}}(Z) \to Y$ the homotopy pull-back in $\underline{\underline{\mathcal{T}}}_B$ of $X \xrightarrow{f} Z \xleftarrow{g} Y$. We remark that the above construction can be performed within $\underline{\underline{\mathcal{T}}}_B^B$ if X, Y, Z, f and g are all in $\underline{\underline{\mathcal{T}}}_B^B$, so that we have a pointed version of a fibrewise homotopy pull-back:

Corollary A.4. *For any fibrewise maps $f : X \to Z$ and $g : Y \to Z$ in $\underline{\underline{\mathcal{T}}}_B^B$, the induced map $(f{\times}_B g)^*\pi : (f{\times}_B g)^*\,\mathcal{P}_{\mathrm{B}}(Z) \to X{\times}_B Y$ is a fibrewise fibration in $\underline{\underline{\mathcal{T}}}_B^B$.*

Second we consider the fibrewise homotopy push-outs in $\underline{\underline{\mathcal{T}}}_B^B$: let X, Y, Z and W be fibrewise pointed spaces over B and $i : Z \to W$ be a fibrewise cofibration in $\underline{\underline{\mathcal{T}}}_B^B$. For any fibrewise map $f : Z \to X$ over B, there exists a push-out $X \xrightarrow{f_*i} f_*W \xleftarrow{\check{f}} W$ of $X \xleftarrow{f} Z \xrightarrow{i} W$ as a quotient space of $X{\amalg}_B W$ by gluing $f(z)$ with $i(z)$ together with fibrewise maps f_*i and $\check{f}$ induced from the canonical inclusions.

Theorem A.5 (Crabb-James [1])**.** *Let $i : Z \to W$ be a fibrewise cofibration in $\underline{\underline{\mathcal{T}}}_B$ (or $\underline{\underline{\mathcal{T}}}_B^B$). For any fibrewise map $f : Z \to X$ in $\underline{\underline{\mathcal{T}}}_B$ (or $\underline{\underline{\mathcal{T}}}_B^B$, resp.), $f_*i : X \to f_*W$ is also a fibrewise cofibration in $\underline{\underline{\mathcal{T}}}_B$ (or $\underline{\underline{\mathcal{T}}}_B^B$, resp.).*

Let us recall that $\mathcal{I}_B^B(Z)$ is obtained from $\mathcal{I}_B(Z) = Z{\times}_B(B{\times}[0,1]) = Z{\times}[0,1]$ by identifying the subspace $s_Z(B){\times}[0,1] \subset Z{\times}[0,1]$ with $s_Z(B)$ by the canonical projection to the first factor : $s_Z(B)\times[0,1] \to s_Z(B)$. Let $\iota_t : Z \to \mathcal{I}^B_B(Z)$ be fibrewise cofibration in $\underline{\underline{\mathcal{T}}}^B_B$ given by $\iota_t(z) = q(z,t)$, $0 \le t \le 1$, where $q : Z\times[0,1] \to \mathcal{I}^B_B(Z)$ denotes the identification map. Then ι_0 and ι_1 induce a map $\iota : Z\vee_B Z \to \mathcal{I}^B_B(Z)$ from $Z\vee_B Z$ the push-out of two copies of $s_Z : B \to Z$.

Proposition A.6. *$\iota : Z\vee_B Z \to \mathcal{I}^B_B(Z)$ is a fibrewise cofibration.*

Proof: For any fibrewise map $\phi : \mathcal{I}^B_B(Z) \to W$ and a fibrewise homotopy $H : (Z\vee_B Z)\times[0,1] = (Z\vee_B Z)\times_B I_B \to W$ such that $H(z,0) = \phi\circ\iota(z)$ for $z \in Z\vee_B Z$, we define a fibrewise homotopy $\check{H} : \mathcal{I}_B(Z)\times[0,1] = \mathcal{I}_B(Z)\times_B(I_B) \to W$ by

$$\check{H}(q(z,t),s) = \begin{cases} H(\mathrm{in}_0(z), s{-}3t), & \text{if } 0 \le t \le \frac{s}{3}, \\ \phi(q(z, \frac{3t-s}{3-2s})), & \text{if } \frac{s}{3} \le t \le \frac{3-s}{3}, \\ H(\mathrm{in}_1(z), 3t{-}3{+}s), & \text{if } \frac{3-s}{3} \le t \le 1 \end{cases}$$

for $(q(z,t),s) \in \mathcal{I}^B_B(Z)\times_B I_B$, where $\mathrm{in}_k : Z \hookrightarrow Z\vee_B Z$, $k = 0,1$ denote the canonical inclusion given by $\mathrm{in}_0(z) = (z,*_b)$ and $\mathrm{in}_1(z) = (*_b,z)$, $b = p_Z(z)$ for any $z \in Z$. Then for any $(q(z,t),s) \in \mathcal{I}^B_B(Z)\times_B I_B$, we clearly have

$$\check{H}(q(z,t))(0) = \phi(q(z,t)),$$
$$\check{H}(q(z,0))(s) = H(\mathrm{in}_0(z),s), \quad \check{H}(q(z,1))(s) = H(\mathrm{in}_1(z),s),$$

and hence we have $\check{H}(q(z,t))(0) = \phi(q(z,t))$ for any $q(z,t) \in \mathcal{I}^B_B(Z)$ and also $\check{H}\circ(\iota\times_B 1_{I_B}) = H$. This implies that $\check{H}$ is a fibrewise homotopy of ϕ extending H. Thus ι is a fibrewise cofibration. □

This yields the following corollary.

Corollary A.7. *For any fibrewise maps $f : Z \to X$ and $g : Z \to Y$ in $\underline{\underline{\mathcal{T}}}^B_B$, the induced map $(f\vee_B g)_*\iota : X\vee_B Y \to (f\vee_B g)^*\,\mathcal{I}^B_B(Z)$ is a fibrewise cofibration in $\underline{\underline{\mathcal{T}}}^B_B$.*

We often call the fibrewise space $(f\vee_B g)^*\,\mathcal{I}^B_B(Z)$ together with the inclusions $(f\vee_B g)_*\iota\circ\mathrm{in}_X : X \to (f\vee_B g)_*\,\mathcal{I}^B_B(Z)$ and $(f\vee_B g)_*\iota\circ\mathrm{in}_Y : Y \to (f\vee_B g)_*\,\mathcal{I}^B_B(Z)$ as homotopy push-out in $\underline{\underline{\mathcal{T}}}^B_B$ of $X \xleftarrow{f} Z \xrightarrow{g} Y$.

Quite similarly for a fibrewise space Z in $\underline{\underline{\mathcal{T}}}_B$, we obtain a fibrewise cofibration $\hat{\iota} : Z \amalg Z = Z\times\{0\} \cup Z\times\{1\} \hookrightarrow Z\times[0,1] = \mathcal{I}_B(Z)$. Thus we have the following.

Corollary A.8. *For any fibrewise maps $f : Z \to X$ and $g : Z \to Y$ in $\underline{\underline{\mathcal{T}}}_B$, the induced map $(f\amalg g)_*\hat{\iota} : X\amalg Y \to (f\amalg g)^*\,\mathcal{I}_B(Z)$ is a fibrewise cofibration in $\underline{\underline{\mathcal{T}}}_B$.*

Thus we also have an unpointed version of a fibrewise homotopy push-out.

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E-mail address: `iwase@math.kyushu-u.ac.jp`

E-mail address, Sakai: `sakai@gifu-nct.ac.jp`

(Iwase) Faculty of Mathematics, Kyushu University, Fukuoka 810-8560, Japan

(Sakai) Gifu National College of Technology, Gifu 501-0495, Japan.