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Square Rings Associated to Elements in Homotopy Groups of Spheres

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Abstract. In this paper we compute for $\alpha \in \pi_{n-2}(S^{m-1})$ with $n < 3m-3$ the full homotopy category consisting of finite one point unions $\Sigma C_\alpha \vee \ldots \vee \Sigma C_\alpha$ with $\Sigma C_\alpha = S^m \cup_{\Sigma\alpha} e^n$. For this we describe the square ring $\mathrm{End}(\Sigma C_\alpha)$ only in terms of primary homotopy operations on spheres. In low dimensions with $n - m \leq 19$ these homotopy operations are computed in the book of Toda [**T**], so that we get this way many explicit examples of square rings. In particular we shall describe algebraically the square rings $\mathrm{End}(\Sigma\mathbb{C}P_2)$, $\mathrm{End}(\Sigma\mathbb{H}P_2)$ and $\mathrm{End}(\Sigma\mathcal{C}a)$ where $\mathbb{C}P_2$ and $\mathbb{H}P_2$ are the complex and quaternionic projective plane respectively and where $\mathcal{C}a$ is the Cayley plane. The structure of $\mathrm{End}(\Sigma C_\alpha)$ leads to a theory of extensions for square rings.

1. Introduction

For pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of pointed maps $X \to Y$. Hence $[X, Y]$ is the set of morphisms in the homotopy category $\mathrm{Top}^*/_{\simeq}$. In this paper spaces are CW-complexes.

We consider a suspended space ΣX which is $(m-1)$–connected and of dimension $< 3m-3$ with $m \geq 2$ (that is ΣX is *metastable*) and we consider the full subcategory

$$\underline{\underline{\mathrm{Add}}}(\Sigma X) \subset \mathrm{Top}^*/_{\simeq} \tag{1.1}$$

consisting of one point unions $\bigvee^k \Sigma X$ of k-copies of the space ΣX with $k \geq 0$. On the other hand we associate with ΣX the diagram

$$\begin{aligned} \mathrm{End}(\Sigma X) = Q = (Q_e \xrightarrow{\bar{H}} Q_{ee} \xrightarrow{\bar{P}} Q_e),\\ Q_e = [\Sigma X, \Sigma X], \quad Q_{ee} = [\Sigma X, \Sigma X \wedge X] \end{aligned} \tag{1.2}$$

where $\bar{H}$ is the Hopf invariant and $\bar{P}$ is induced by the Whitehead product $[1_{\Sigma X}, 1_{\Sigma X}] : \Sigma X \wedge X \to X$. Here Q_e and Q_{ee} are groups by the co–H–structure of ΣX and Q_e is also a monoid by composition of maps. Moreover since ΣX is metastable the group Q_{ee} is abelian. The next lemma is shown in [**BHP**].

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Lemma 1.1. *The diagram* $Q = \mathrm{End}(\Sigma X)$ *has the structure of a square ring (see Section 3 below) and the algebraic biproduct completion* $\underline{\underline{\underline{\mathrm{Add}}}}(Q)$ *of the square ring* Q *is isomorphic to the category* $\underline{\underline{\underline{\mathrm{Add}}}}(\Sigma X)$.

The lemma shows that the computation of the square ring $Q = \mathrm{End}(\Sigma X)$ yields an algebraic characterization of the category $\underline{\underline{\underline{\mathrm{Add}}}}(\Sigma X)$ by the category $\underline{\underline{\underline{\mathrm{Add}}}}(Q)$ in which the object corresponding to $\bigvee^k \Sigma X$ is denoted by $\coprod^k Q$ and in which the morphisms are certain matrices defined in (2.3) below. If ΣX is *stable* (that is, if ΣX is of dimension $\leq 2m-2$), then $Q_{ee} = 0$ and $Q_e = \mathrm{End}(\Sigma X)$ is a ring, i.e. the endomorphism ring of an object in an additive category. In this case the lemma states the well known fact that $\underline{\underline{\underline{\mathrm{Add}}}}(\Sigma X)$ is isomorphic to the category of free Q_e-modules $\coprod^k Q = \bigoplus^k Q_e$.

The k-th *general linear group* of the square ring Q is the group

$$GL(Q,k) = \mathrm{Aut}(\coprod^k Q) \tag{1.3}$$

of automorphisms of $\coprod^k Q$ in the category $\underline{\underline{\underline{\mathrm{Add}}}}(Q)$. Hence we obtain by (1.1) the following computation of the group of homotopy equivalences $\mathrm{Aut}(\bigvee^k \Sigma X)$ in $\mathrm{Top}^*/\simeq$.

Corollary 1.2. *For* $k \geq 0$ *one has a canonical isomorphism of groups*

$$\mathrm{Aut}(\bigvee^k \Sigma X) \cong GL(Q,k)$$

where the right hand side is algebraically determined by the square ring $Q = \mathrm{End}(\Sigma X)$ *in (1.2).*

The purpose of this paper is the computation of the square ring $\mathrm{End}(\Sigma X)$ if ΣX is a 2-cell complex. Let $\pi_k(S^m)$ be the k-th homotopy group of the m-sphere S^m. We consider an element

$$\alpha \in \pi_{n-2}(S^{m-1}) \quad \text{with} \quad n-2 \geq m-1 \geq 1, n < 3m-3 \tag{1.4}$$

which yields the mapping cone C_α and its suspension ΣC_α which are CW-complexes of the form

$$\begin{cases} C_\alpha = S^{m-1} \cup_\alpha e^{n-1} \\ \Sigma C_\alpha = S^m \cup_{\Sigma\alpha} e^n \end{cases}$$

The assumptions on m, n show that ΣC_α is in the meta-stable range so that the endomorphism square ring $\mathrm{End}(\Sigma C_\alpha)$ is defined. For example, if $\alpha = \eta_2 \in \pi_3(S^2)$ is the Hopf map then $C_\alpha = \mathbb{C}P_2$ is the complex projective plane and the square ring $\mathrm{End}(\Sigma\mathbb{C}P_2)$ was computed in (8.6) of [**BHP**]. In this paper we describe more generally $\mathrm{End}(\Sigma C_\alpha)$ only in terms of primary homotopy operations on spheres. In low dimensions with $n - m \leq 19$ these homotopy operations are computed in the book of Toda [**T**], so that we get this way many explicit examples of square rings. In particular we shall describe algebraically the square rings $\mathrm{End}(\Sigma\mathbb{H}P_2)$ and $\mathrm{End}(\Sigma\mathcal{C}a)$ where $\mathbb{H}P_2$ is the quaternionic projective plane and where $\mathcal{C}a$ is the Cayley plane.

2. The biproduct completion of a square ring

We recall the definition of square group and square ring from [**BHP**].

Definition 2.1. A *square group* $Q = (Q_e \xrightarrow{\bar{H}} Q_{ee} \xrightarrow{\bar{P}} Q_e)$ is given by a group Q_e and an abelian group Q_{ee}. Both groups are written additively. Moreover $\bar{P}$ is a homomorphism and $\bar{H}$ is a quadratic function, that is, the cross effect

$$\langle x|y\rangle_{\bar{H}} = \bar{H}(x+y) - \bar{H}(x) - \bar{H}(y)$$

is linear in $x, y \in Q_e$. In addition the following properties are satisfied for $x, y \in Q_e$ and $u, v \in Q_{ee}$.

(1) $\langle \bar{P}(u)|y\rangle_{\bar{H}} = 0$ and $\langle x|\bar{P}(v)\rangle_{\bar{H}} = 0$
(2) $\bar{P}(\langle x|y\rangle_{\bar{H}}) = x + y - x - y$
(3) $\bar{P}\bar{H}\bar{P}(u) = \bar{P}(u) + \bar{P}(u)$

Definition 2.2. A *square ring* $Q = (Q_e \xrightarrow{\bar{H}} Q_{ee} \xrightarrow{\bar{P}} Q_e)$ is given by a square group $(Q_e \xrightarrow{\bar{H}} Q_{ee} \xrightarrow{\bar{P}} Q_e)$ for which Q_e has the additional structure of a monoid with unit $1 \in Q_e$ and the multiplication is denoted by $x \circ y \in Q_e$. This monoid structure induces a ring structure on the abelian group $\bar{R} = \mathrm{cok}(\bar{P})$ through the canonical projection $Q_e \xrightarrow{\bar{\epsilon}} \bar{R}$. We write $\bar{\epsilon}(a) = \bar{a}$. Moreover the abelian group Q_{ee} is an $\bar{R}\otimes\bar{R}\otimes\bar{R}^{op}$-module with action denoted by $(\bar{t}\otimes\bar{s})\cdot u\cdot\bar{r} \in Q_{ee}$ for $\bar{t}, \bar{s}, \bar{r} \in \bar{R}, u \in Q_{ee}$. In addition the following properties are satisfied where $\bar{H}(2) = \bar{H}(1+1)$.

(1) $\langle x|y\rangle_{\bar{H}} = (\bar{y}\otimes\bar{x})\cdot\bar{H}(2)$
(2) $T = \bar{H}\bar{P} - 1$ is an isomorphism of abelian groups satisfying $T((\bar{t}\otimes\bar{s})\cdot u\cdot\bar{r}) = (\bar{s}\otimes\bar{t})\cdot T(u)\cdot\bar{r}$.
(3) $\bar{P}(u)\circ x = \bar{P}(u\cdot\bar{x})$
(4) $x\circ\bar{P}(u) = \bar{P}((\bar{x}\otimes\bar{x})\cdot u)$
(5) $\bar{H}(x\circ y) = (\bar{x}\otimes\bar{x})\cdot\bar{H}(y) + \bar{H}(x)\cdot\bar{y}$
(6) $(x+y)\circ z = x\circ z + y\circ z + \bar{P}((\bar{x}\otimes\bar{y})\cdot\bar{H}(z))$
(7) $x\circ(y+z) = x\circ y + x\circ z$

By [**BP**], we know that the category of square groups is the same as the category of quadratic functors $\underline{\underline{\mathrm{Gr}}} \to \underline{\underline{\mathrm{Gr}}}$ where $\underline{\underline{\mathrm{Gr}}}$ is the category of groups. With respect to the monoidal structure in this category a square ring is also a monoid in the category of square groups; see [**BP**].

We remark that the definition of a square ring in [**BHP**] or [**BP**] uses also the equation

(8) $\bar{H}\bar{P}\bar{H}(x) + \bar{H}(x+x) - 4\bar{H}(x) = \bar{H}(2)\cdot\bar{x}$.

which is redundant. In fact, by the condition (6) of Definition 2.2, we have

$$2\circ x = (1+1)\circ x = 1\circ x + 1\circ x + \bar{P}((\bar{1}\otimes\bar{1})\cdot\bar{H}(x)) = x + x + \bar{P}\bar{H}(x).$$

Applying $\bar{H}$ using the condition (1) of Definition 2.1 we get

$$\bar{H}(2\circ x) = \bar{H}(x+x) + \bar{H}(\bar{P}\bar{H}(x)) + \langle x+x|\bar{P}\bar{H}(x)\rangle_{\bar{H}} = \bar{H}(x+x) + \bar{H}\bar{P}\bar{H}(x).$$

On the other hand by condition (5) we have

$$\bar{H}(2\circ x) = (\bar{2}\otimes\bar{2})\cdot\bar{H}(x) + \bar{H}(2)\cdot\bar{x} = 4\bar{H}(x) + \bar{H}(2)\cdot\bar{x}.$$

Comparing the equations we obtain (8).

Definition 2.3. Given a square ring Q as above we define the *biproduct completion* $\underline{\underline{\mathrm{Add}}}(Q)$. We obtain the category $\underline{\underline{\mathrm{Add}}}(Q)$ in terms of matrices as follows. Objects in $\underline{\underline{\mathrm{Add}}}(Q)$ are denoted by $\coprod^{x} Q$ will $x \in \{0, 1, 2, \cdots\}$. For $x = 0$ this is the initial object and for $x = 1$ we write $Q = \coprod^{1} Q$. Sets of morphisms are defined by product sets

$$\mathrm{Mor}(Q, \coprod^{x} Q) = (\prod_{i=1}^{x} Q_e) \times (\prod_{1 \le i < j \le x} Q_{ee})$$
$$f \in \mathrm{Mor}(\coprod^{y} Q, \coprod^{x} Q) = \prod_{k=1}^{y} \mathrm{Mor}(Q, \coprod^{x} Q)$$

where we write $f = (f_i^k, f_{ij}^k)$. Now let $g = (g_k^s, g_{k\ell}^s)$ be an element in $\mathrm{Mor}(\coprod^{z} Q, \coprod^{y} Q)$. Then the *composition*

$$fg = ((fg)_i^s, (fg)_{ij}^s)$$

is given by the coordinates

$$(fg)_i^s = f_i^1 \circ g_1^s + f_i^2 \circ g_2^s + \cdots + f_i^y \circ g_y^s + \sum_{k<\ell} \bar{P}((\overline{f_i^k} \otimes \overline{f_i^\ell}) \cdot g_{k\ell}^s)$$
$$(fg)_{ij}^s = \sum_k (f_{ij}^k \cdot \overline{g_k^s}) + \sum_{k<\ell} ((\overline{f_i^k} \otimes \overline{f_j^\ell}) \cdot g_{k\ell}^s + (\overline{f_i^\ell} \otimes \overline{f_j^k}) \cdot T g_{k\ell}^s + \overline{(f_i^\ell \cdot g_\ell^s)} \otimes \overline{(f_j^k \cdot g_k^s)} \cdot \bar{H}(2))$$

3. The main result

We associate with α in (1.4) the following data determined by α.

Definition 3.1. Given $\alpha \in \pi_{n-2}(S^{m-1})$ with $n < 3m-3$. Let $U_\alpha \subset \pi_n(S^m)$ be the subgroup generated by $\eta_m(\Sigma^2\alpha)$ and $(\Sigma\alpha)\eta_{n-1}$ where η_t is the Hopf element, $t \ge 2$. Then the quotient group $\pi_n(S^m)/U_\alpha$ is part of the diagram

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$
$$M_e = \pi_n(S^m)/U_\alpha, \quad M_{ee} = \pi_n(S^{2m-1}) \tag{3.1}$$

where H is given by the Hopf invariant and P is induced by the Whitehead square $[\iota_m, \iota_m] : S^{2m-1} \to S^m$, that is, $P(u) = [\iota_m, \iota_m]_* u$. The element $\Sigma\alpha$ is a torsion element of *order* k in $\pi_{n-1}(S^m)$ so that $\Sigma(k\alpha) = 0$ and hence by the exactness of the EHP-sequence with $n \le 3m-3$ (see [**B1**] A.6.7)

$$\pi_n(S^{2m-1}) \overset{P_0}{\to} \pi_{n-2}(S^{m-1}) \overset{E_0}{\to} \pi_{n-1}(S^m) \overset{H_0}{\to} \pi_{n-1}(S^{2m-1}) \to \cdots,$$

there exists

$$\begin{cases} \mu \in \pi_n(S^{2m-1}) \quad \text{with} \\ P_0(\mu) = [\iota_{m-1}, \iota_{m-1}]_* (\Sigma^2)^{-1}\mu = -k\alpha \end{cases} \tag{3.2}$$

Here we use the inverse $(\Sigma^2)^{-1}$ of the double suspension $\Sigma^2 : \pi_{n-2}(S^{2m-3}) \cong \pi_n(S^{2m-1})$. Moreover let

$$\lambda = \Sigma^2 H(\alpha) \tag{3.3}$$

be given by the Hopf invariant $H : \pi_{n-2}(S^{m-1}) \to \pi_{n-2}(S^{2m-3})$.

Theorem 3.2. *In terms of the data (M, λ, μ, k) associated to α we define below a square ring $Q(M, \lambda, \mu, k)$ together with an isomorphism*

$$\mathrm{End}(\Sigma C_\alpha) \cong Q(M, \lambda, \mu, k)$$

of square rings.

For $k \geq 1$ let $R = \mathbb{Z}\times_k\mathbb{Z}$ be the subring of $\mathbb{Z}\times\mathbb{Z}$ consisting of all pairs $a = (a_0, a_1)$ with $a_0 - a_1 \equiv 0 \mod k$. This is the pull back ring of $\mathbb{Z} \longrightarrow \mathbb{Z}/k \longleftarrow \mathbb{Z}$. Then $1 = \eta(1) = (1, 1) \in R$ is the unit and we have an augmentation $\epsilon : R \to \mathbb{Z}$ with $\epsilon(a) = a_0$ for $a = (a_0, a_1)$. The kernel of ϵ is generated by $\bar{k} = (0, k)$ so that 1 and $\bar{k}$ form a $\mathbb{Z}$-basis of the free abelian group $\mathbb{Z}\times_k\mathbb{Z}$. We have a surjection map

$$\deg : [\Sigma C_\alpha, \Sigma C_\alpha] \twoheadrightarrow \mathbb{Z}\times_k\mathbb{Z} = R \tag{3.4}$$

which carries $u : \Sigma C_\alpha \to \Sigma C_\alpha$ to the pair $\deg(u) = (a_0, a_1)$ where a_0 is the degree of $H_m(u)$ on $H_m(\Sigma C_\alpha) = \mathbb{Z}$ and a_1 is the degree of $H_n(u)$ on $H_n(\Sigma C_\alpha) = \mathbb{Z}$. We shall prove the following crucial lemma.

Lemma 3.3. *For the square ring End(ΣC_α) given by diagram (1.2) and for the data (M, λ, μ, k) in definition 3.1 one gets a commutative diagram*

$$\begin{array}{ccccc}
M_e & \xrightarrow{\quad H \quad} & M_{ee} & \xrightarrow{\quad P \quad} & M_e \\
{\scriptstyle i}\downarrow & & \| & & \downarrow{\scriptstyle i} \\
[\Sigma C_\alpha, \Sigma C_\alpha] & \xrightarrow{\bar{H}} & [\Sigma C_\alpha, \Sigma C_\alpha \wedge C_\alpha] & \xrightarrow{\bar{P}} & [\Sigma C_\alpha, \Sigma C_\alpha] \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle \deg} & \nearrow_{h} & & & \\
R & & & &
\end{array}$$

where the column $M_e \overset{i}{\hookrightarrow} [\Sigma C_\alpha, \Sigma C_\alpha] \overset{\deg}{\to} R$ is a split short exact sequence of abelian groups. For a splitting s of deg *let $h = \bar{H}s$. We can choose s such that $s(1) = \iota_\alpha$ is the identity of ΣC_α and $s(\bar{k}) = \mu_0$ such that $\bar{H}(\mu_0) = h(\bar{k}) = \mu$ and $\bar{H}(1 + 1) = h(1 + 1) = \lambda$.*

Definition 3.4. A quadratic $\mathbb{Z}$-module [**B2**]

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e) \tag{3.5}$$

consists of abelian groups M_e and M_{ee} and homomorphisms H and P satisfying $HPH = 2H$ and $PHP = 2P$. We consider $k \in \mathbb{N}$ and $\lambda, \mu \in M_{ee}$ with relations

$$\begin{cases} P(\lambda) = 0 \\ \\ HP(\mu) = 2\mu + k\lambda. \end{cases} \tag{3.6}$$

One can easily check that (M, λ, μ, k) in (3.1) satisfies these relations. Then the *square ring* $Q(M, \lambda, \mu, k)$ with

$$Q = (R \oplus M_e \xrightarrow{\bar{H}} M_{ee} \xrightarrow{\bar{P}} R \oplus M_e) \tag{3.7}$$

is defined as follows with $R = \mathbb{Z} \times_k \mathbb{Z}$. Let $h : R \to M_{ee}$ be the unique function satisfying

$$\begin{cases} h(1) = 0, \quad h(\bar{k}) = \mu, \\ \\ \langle a, b \rangle_h = h(a + b) - h(a) - h(b) = a_0 b_0 \lambda. \end{cases} \tag{3.8}$$

Then $\bar{H}$ is the function given by

$$\bar{H}(a,x) = h(a) + H(x). \tag{3.9}$$

Moreover $\bar{P}$ is defined by

$$\bar{P}(y) = (0, P(y)) \in R \oplus M_e. \tag{3.10}$$

As a group $R \oplus M_e$ is the direct sum of abelian groups and the monoid structure of $R \oplus M_e$ is given by the product formula

$$(a,x) \circ (b,y) = (a \cdot b, x \cdot b_1 + a_0 * y + \Delta(a,b)) \tag{3.11}$$

where

$$a_0 * y = a_0 y + \frac{a_0(a_0-1)}{2} PH(y) \tag{3.12}$$

and

$$\Delta(a,b) = \frac{a_0(a_0-1)(b_1-b_0)}{2k} P(\mu). \tag{3.13}$$

Now $\bar{R} = \operatorname{cok}(\bar{P}) = R \oplus \operatorname{cok}(P)$ is a ring and the projection $\bar{R} \to R$ is a ring homomorphism. Moreover M_{ee} as $\bar{R} \otimes \bar{R} \otimes \bar{R}^{op}$-module is defined by

$$(\overline{(a,x)} \otimes \overline{(b,y)}) \cdot u \cdot \overline{(c,z)} = (a_0 b_0) \cdot u \cdot c_1 \tag{3.14}$$

for $\overline{(a,x)}, \overline{(b,y)}, \overline{(c,z)} \in \bar{R}$ and $u \in M_{ee}$.

We shall prove that $Q = Q(M, \lambda, \mu, k)$ given by 3.5 through 3.14 above is a well-defined square ring. Using the section s in Lemma 3.3 one obtains the isomorphism

$$R \oplus M_e = [\Sigma C_\alpha, \Sigma C_\alpha] \tag{3.15}$$

as in the proof of Corollary 8.5 carrying (a,x) to $s(a) + i(x)$. This is an isomorphism of abelian groups and of monoids and this isomorphism yields the isomorphism $Q \cong \operatorname{End}(\Sigma C_\alpha)$ of square rings in Theorem 3.2. We point out that in general the group $[\Sigma C_\alpha, W]$ for some space W needs not to be abelian, e.g, for $W = \Sigma C_\alpha \vee \Sigma C_\alpha$ the group $[\Sigma C_\alpha, W]$ is abelian if and only if $H(\alpha) = 0$, in other words, C_α is itself a co-H-space.

4. Examples and applications

We consider the special case of the square ring $Q = Q(M, \lambda, \mu, k)$ for a quadratic $\mathbb{Z}$-module M with $M_e = 0$. In this case we obtain for the ring $R = \mathbb{Z} \times_k \mathbb{Z}$ and for $\lambda, \mu \in M_{ee}$ with $2\mu + \lambda = 0$ the *square ring*

$$Q(M_{ee}, \lambda, \mu, k) = Q = (R \xrightarrow{h} M_{ee} \xrightarrow{0} R) \tag{4.1}$$

with $h(1) = 0, h(\bar{k}) = \mu$ and $\langle a, b \rangle_h = h(a+b) - h(a) - h(b) = a_0 b_0 \lambda$. Moreover M_{ee} is an $R \otimes R \otimes R^{op}$-module by $(a \otimes b) \cdot u \otimes c = (a_0 b_0) \cdot u \cdot c_1$.

For the complex projective plane $\mathbb{C}P_2 = C_{\eta_2}$ where $\alpha = \eta_2$ is the Hopf map we get as a special case of (4.1):

Example 4.1. For $\alpha = \eta_2 \in \pi_3(S^2)$ one has

$$\operatorname{End}(\Sigma \mathbb{C}P_2) \cong Q(M_{ee}, \lambda, \mu, k) \quad \text{with}$$

$$M_{ee} = \mathbb{Z}, \quad \lambda = 1, \quad \mu = -1, \quad k = 2.$$

This example was also computed in (8.6)(2) of [**BHP**]. Moreover the category $\underline{\underline{\mathrm{Add}}}(\Sigma\mathbb{C}P_2)$ was computed by [**U**] and [**Y**]. Using the computation of the square ring $Q = \mathrm{End}(\Sigma\mathbb{C}P_2)$ above we know that $\underline{\underline{\mathrm{Add}}}(\Sigma\mathbb{C}P_2) = \underline{\underline{\mathrm{Add}}}(Q)$ is algebraically determined by Q. These results can be generalised as follows.

We describe the square rings for the Hopf maps ν_4, σ_8 for which the mapping cones

$$C_{\nu_4} = \mathbb{H}P_2 \quad \text{and} \quad C_{\sigma_8} = \mathcal{C}a$$

are the *quaternionic projective space* and the *Cayley plane* respectively. By inspection of Toda's book [**T**] we get the following square rings:

EXAMPLE 4.2. For $\alpha = \nu_4 \in \pi_7(S^4)$ one has

$$\mathrm{End}(\Sigma\mathbb{H}P_2) \cong Q(M_{ee}, \lambda, \mu, k) \quad \text{with}$$

$$M_{ee} = \mathbb{Z}, \quad \lambda = 1, \quad \mu = -12, \quad k = 24.$$

EXAMPLE 4.3. For $\alpha = \sigma_8 \in \pi_{15}(S^8)$ one has

$$\mathrm{End}(\Sigma\mathcal{C}a) \cong Q(M_{ee}, \lambda, \mu, k) \quad \text{with}$$

$$M_{ee} = \mathbb{Z}, \quad \lambda = 1, \quad \mu = -120, \quad k = 240.$$

Hence the examples $\mathrm{End}(\Sigma\mathbb{C}P_2)$, $\mathrm{End}(\Sigma\mathbb{H}P_2)$ and $\mathrm{End}(\Sigma\mathcal{C}a)$ are special cases of the square ring Q in (4.1). The endomorphism square rings of $\Sigma\mathbb{C}P_2$, $\Sigma\mathbb{H}P_2$, $\Sigma\mathcal{C}a$ satisfy $P = 0$. The next examples satisfy $P \neq 0$.

EXAMPLE 4.4. For the double Hopf map $\alpha = \eta_3^2 \in \pi_5(S^3)$ one has

$$\mathrm{End}(\Sigma C_\alpha) \cong Q(M, \lambda, \mu, k) \quad \text{with}$$

$$M = (\mathbb{Z} \oplus \mathbb{Z}/6 \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,1)} \mathbb{Z} \oplus \mathbb{Z}/6), \quad \lambda = 0, \quad \mu = 0, \quad k = 2.$$

EXAMPLE 4.5. For the Whitehead square $\alpha = [\iota_5, \iota_5] \neq 0$ in $\pi_9(S^5) = \mathbb{Z}/2$ one has

$$\mathrm{End}(\Sigma C_\alpha) \cong Q(M, \lambda, \mu, k) \quad \text{with}$$

$$M = (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}), \quad \lambda = 0, \quad \mu = 1, \quad k = 1.$$

EXAMPLE 4.6. For the Whitehead square $\alpha = [\iota_8, \iota_8] \in \pi_{15}(S^8)$ one has

$$\mathrm{End}(\Sigma C_\alpha) \cong Q(M, \lambda, \mu, k) \quad \text{with}$$

$$M = (\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{(1,1,1)} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2), \quad \lambda = 2, \quad \mu = -1, \quad k = 1.$$

In Theorem 3.13 of Oka, Sawashita and Sugawara [**OSS**] extending results of Oka [**O**], the group $\mathrm{Aut}(S^n \cup_f e^m)$ is computed up to an extension problem if $f = \Sigma\alpha$ is a suspension. Also in Theorem A of Yamaguchi [**Y**] and in Section 2 of Unsöld [**U**], the group $\mathrm{Aut}(\bigvee^k \Sigma^r \mathbb{C}P^2)$ is determined for $r \geq 1$. By our result we get:

PROPOSITION 4.7. *Let $\alpha \in \pi_{n-2}(S^{m-1})$ be with $n < 3m-3$. Then the group $\mathrm{Aut}(S^m \cup_f e^m)$ with $f = \Sigma\alpha$ is the group of units in the monoid Q_e determined by the square ring $Q = Q(M, \lambda, \mu, k)$ given by α. In fact, the monoid of self maps of $\Sigma C_\alpha = S^m \cup_f e^n$ coincides with the monoid Q_e. In addition the group*

$$\mathrm{Aut}(\bigvee^k \Sigma C_\alpha) \cong GL(Q, k).$$

is algebraically determined by the square ring $Q = \mathrm{End}(\Sigma C_\alpha)$ for $k \geq 1$.

We now consider for a metastable space ΣX the groups $M = [\Sigma X, W]$ where W is a pointed space. For $Q = \mathrm{End}(\Sigma X)$ in (4.1) we get the *operations*

$$(4.2) \qquad \begin{cases} [\Sigma X, W] \times [\Sigma X, \Sigma X] \longrightarrow [\Sigma X, W] \\ \\ [\Sigma X, W] \times [\Sigma X, W] \times [\Sigma X, \Sigma X \wedge X] \longrightarrow [\Sigma X, W] \end{cases}$$

which carry (m, a) and (m, n, x) to the composites $m \circ a$ and $[m, n] \circ x$ respectively where $[m, n] \in [\Sigma X \wedge X, W]$ is the Whitehead product of $m, n \in [\Sigma X, W]$. These operations give $M = [\Sigma X \wedge X, W]$ the following structure of a Q-module. The structure of $[\Sigma X \wedge X, W]$ as a Q-module determines completely the functor

$$\underline{\underline{\mathrm{Add}}}(\Sigma X)^{op} \longrightarrow \underline{\underline{\mathrm{Set}}}$$

which carries $\bigvee^k \Sigma X$ to the set of homotopy classes $[\bigvee^k \Sigma X, W]$; see [**BHP**].

DEFINITION 4.8. A *Q–module* M is given by a group M which we write additively and by Q-operations which are functions

$$M \times Q_e \longrightarrow M, \quad (m, a) \longmapsto m \cdot a,$$
$$M \times M \times Q_{ee} \longrightarrow M, \quad (m, n, x) \longmapsto [m, n] \cdot x.$$

For $a, b \in Q_e$, $x, y \in Q_{ee}$, $m, n \in M$ the following relations hold where $[M] = \{[m, n] \cdot x\,;\, m, n \in M,\, x \in Q_{ee}\} \subset M$:

$m \cdot 1 = m$, $(m \cdot a) \cdot b = m \cdot (a \cdot b)$, $m \cdot (a + b) = m \cdot a + m \cdot b$,
$(m + n) \cdot a = m \cdot a + n \cdot a + [m, n] \cdot H(a)$,
$m \cdot P(x) = [m, m] \cdot x$,
$[m, n] \cdot T(x) = [n, m] \cdot x$,
$[m \cdot a, n \cdot b] \cdot x = [m, n] \cdot (a \otimes b) \cdot x$ and $([m, n] \cdot x) \cdot a = [m, n] \cdot (x \cdot a)$,
$[m, n] \cdot x$ is linear in m, n and x,
$[m, n] \cdot x = 0$ for $m \in [M]$.

These equations imply that the commutator in M satisfies

$$n + m - n - m = -n - m + n + m = [m, n] \cdot H(2).$$

Hence M is a group of nilpotency degree 2 and $[M]$ is central in M. Morphisms in the category $\underline{\underline{\mathrm{Mod}}}(Q)$ of Q-modules are homomorphisms $M \to M'$ which are compatible with the Q-operations.

Since we computed $\mathrm{End}(\Sigma C_\alpha)$ we then get:

PROPOSITION 4.9. *For $\alpha \in \pi_{n-2}(S^{m-1})$ with $n < 3m-3$ the group $[\Sigma C_\alpha, W]$ is a Q-module where $Q = Q(M, \lambda, \mu, k) \cong \mathrm{End}(\Sigma C_\alpha)$ is given by α as in section 3.*

REMARK 4.10. Let ΣX be a metastable space and let G be a connected topological group. Then the set $[X, B]$ of homotopy classes of maps from X to G has the structure of a Q-module where $Q = \mathrm{End}(\Sigma X)$ is the endomorphism square ring of ΣX. This follows since we have natural isomorphism of groups

$$[X, G] = [X, \Omega BG] = [\Sigma X, BG]$$

where BG is the classifying space of G. In particular the groups

$$[\mathbb{R}P_2, G], \quad [\mathbb{C}P_2, G], \quad [\mathbb{H}P_2, G], \quad [\mathcal{C}a, G]$$

have the structure of a Q-module where Q is the endomorphism square ring for $\Sigma\mathbb{R}P_2$, $\Sigma\mathbb{C}P_2$, $\Sigma\mathbb{H}P_2$, $\Sigma\mathcal{C}a$ respectively; in fact, algebraic descriptions of these square rings are given in (8.2) of [**BHP**], (4.1), (4.2) and (4.3).

5. A quadratic action

The following three sections are purely algebraic. We study first a quadratic action denoted by $*$ which will be used in the next section for the computation of certain square rings. This way we show that the square ring $Q(M, \lambda, \mu, k)$ used in our main result is in fact a well defined square ring satisfying all properties in Definition 2.2.

Let R be an augmented ring with unit, i.e. two ring homomorphisms $\eta : \mathbb{Z} \to R$ and $\epsilon : R \to \mathbb{Z}$ are given to satisfy $\epsilon\eta = 1_{\mathbb{Z}}$. We write $\eta(\ell) = \ell$ for $\ell \in \mathbb{Z}$ and $\epsilon(a) = \tilde{a}$ for $a \in R$. For example the ring $R = \mathbb{Z} \times_k \mathbb{Z}$ is augmented by $\varepsilon(a) = \tilde{a} = a_0 \in \mathbb{Z}$ with $a = (a_0, a_1)$ and the unit is $1 = (1, 1)$.

A quadratic R-module $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$ is given by right R-modules M_e, M_{ee} and R-linear homomorphisms H, P with $HPH = 2H$ and $PHP = 2P$.

For any quadratic R-module $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$, we define a left action of R on M_e by

$$\tilde{a}{*}x = \tilde{a}x + \frac{\tilde{a}(\tilde{a}-1)}{2}PH(x),$$

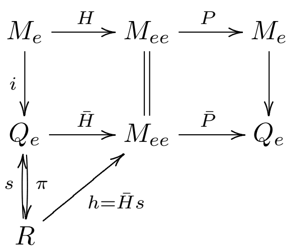
$\tilde{a} \in \mathbb{Z}$ for $a \in R, x \in M_e$. Then the following proposition holds.

PROPOSITION 5.1. *The action $*$ satisfies following formulas.*

(1) $\tilde{a}{*}(\widetilde{b{*}x}) = (\widetilde{ab}){*}x$
(2) $(\widetilde{a+b}){*}x = \tilde{a}{*}x + \tilde{b}{*}x + \tilde{a}\tilde{b}PH(x)$
(3) $\tilde{a}{*}(x+y) = \tilde{a}{*}x + \tilde{a}{*}y$
(4) $\tilde{a}{*}(x{\cdot}b) = (\tilde{a}{*}x){\cdot}b$
(5) $H(\tilde{a}{*}x) = \tilde{a}\tilde{a}H(x)$
(6) $\tilde{a}{*}P(x) = \tilde{a}\tilde{a}P(x)$

6. Square extension

We study square extensions which are motivated by the commutative diagram in Lemma 3.3. Let R be an augmented ring and let $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$ be a quadratic R–module. For the group $Q_e = R \oplus M_e$ (given by the direct sum of the abelian groups R and M_e) we consider the following extension diagram



This is a generalization of the diagram in Lemma 3.3. Here i and s are the inclusions for $Q_e = R \oplus M_e$ and π is the projection. We now consider conditions on (H, P, h) which yield a square group and a square ring respectively.

Proposition 6.1. *Given (H, P, h) as above, the following two conditions are equivalent.*

(1) $Q = (Q_e \xrightarrow{\bar{H}} M_{ee} \xrightarrow{\bar{P}} Q_e)$ *is a square group with*

$$\bar{H}(a, x) = h(a) + H(x) \quad \textit{and} \quad \bar{P}(u) = iP(u) = (0, P(u))$$

for $a \in R$, $x \in M_e$ and $u \in M_{ee}$.

(2) *The data (H,P,h) satisfies the following conditions.*

i) *A cross effect $\langle a|b\rangle_h = h(a + b) - h(a) - h(b)$ is linear in $a, b \in R$.*

ii) *$P(\langle a|b\rangle_h) = 0$, in other words, $Ph(a)$ is linear in a.*

Proof: By the definitions of $\bar{H}$ and cross effects, we have

$$\begin{aligned}\langle (a, x)|(b, y)\rangle_{\bar{H}} &= \bar{H}((a, x) + (b, y)) - \bar{H}(a, x) - \bar{H}(b, y) \\ &= h(a + b) + H(x + y) - h(a) - H(x) - h(b) - H(y) = h(a + b) - h(a) - h(b).\end{aligned}$$

Thus we have

$$\langle (a, x)|(b, y)\rangle_{\bar{H}} = \langle a|b\rangle_h. \tag{6.1}$$

Suppose (1). The condition (1) of Definition 2.1 implies the condition (2i) by (6.1). The condition (2) of Definition 2.1 implies $\bar{P}(\langle a, b\rangle_h) = a + b - a - b = 0$, and hence we have (2ii).

Conversely suppose (2). By (6.1), the condition (2i) implies the condition (2) of Definition 2.1. The condition (1) of Definition 2.1 is a direct consequence of $\operatorname{im} \bar{P} = 0 \oplus \operatorname{im} P \subset 0 \oplus M_e = \ker \pi$ and (6.1). The condition (2) of Definition 2.1 is obtained by (6.1) and the condition (2ii). The condition (3) of Definition 2.1 is automatically satisfied since M is a quadratic R-module. *qed.*

For a given quadratic R-module $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$, let $\bar{R} = R \oplus (M_e/\operatorname{im}(P))$ and let

$$\bar{\epsilon} : R \oplus M_e \to R \oplus (M_e/\operatorname{im}(P)) = \bar{R} \text{ and}$$
$$p : \bar{R} = R \oplus (M_e/\operatorname{im}(P)) \to R$$

be the canonical projections. We write $\bar{\epsilon}(a, x) = \overline{(a, x)}$. There is an action of $R \otimes R \otimes R^{op}$ on M_{ee} given by

$$(t \otimes s) \cdot u \cdot r = \widetilde{ts}(u \cdot r), \tag{6.2}$$

which makes M_{ee} an $R \otimes R \otimes R^{op}$-module. We show the following theorem.

THEOREM 6.2. *Given* (H, P, h) *as above and a fuction* $\Delta : R{\times}R \to M_e$ *with* $\Delta(0, b) = \Delta(1, b) = \Delta(a, 0) = \Delta(a, 1) = 0$ *the following two statements are equivalent.*

(1) $Q = (Q_e \xrightarrow{\bar{H}} M_{ee} \xrightarrow{\bar{P}} Q_e)$ *is a square ring with*

$$\bar{H}(a, x) = h(a) + H(x) \quad \text{and} \quad \bar{P}(u) = (0, P(u))$$

for $a \in R$, $x \in M_e$ *and* $u \in M_{ee}$ *and multiplication* $\circ$ *of the monoid* Q_e *given by*

$$(a, x)\circ(b, y) = (ab, x{\cdot}b + \tilde{a}{*}y + \Delta(a, b))$$

for $a, b \in R$, $x, y \in M_e$, *which yields a ring structure on* $\bar{R} = R\oplus(M_e/\operatorname{im}(P)) = Q_e/\operatorname{im}(\bar{P})$ *as an extension of* R *with the action of* $\bar{R}\otimes\bar{R}\otimes\bar{R}^{op}$ *on* M_{ee} *through* $p\otimes p\otimes p$.

(2) *The data* (H,P,h,Δ) *satisfy the following conditions.*
  i) $Ph(2) = 0$,
  ii) $\langle a|b\rangle_h = \tilde{a}\tilde{b}h(2)$,
  iii) $h(a{\cdot}b) + H(\Delta(a, b)) = \tilde{a}\tilde{a}h(b) + h(a){\cdot}b$,
  iv) $\Delta(a, b){\cdot}c + \Delta(ab, c) = \tilde{a}{*}\Delta(b, c) + \Delta(a, bc)$,
  v) $\Delta(a, b + c) = \Delta(a, b) + \Delta(a, c)$,
  vi) $\Delta(a + b, c) = \Delta(a, c) + \Delta(b, c) + \tilde{a}\tilde{b}Ph(c)$.

We call the data (H, P, h, Δ) with the properties in the theorem a *square extension.*

Proof: Firstly, we observe the properties of the multiplication $\circ$ on $Q_e = R\oplus M_e$.

$$(a, x)\circ(1, 0) = (a, x{\cdot}1 + \Delta(a, 1)) = (a, x),$$
$$(1, 0)\circ(b, y) = (b, \tilde{1}{*}y + \Delta(1, b)) = (b, y).$$

Thus $(1, 0)$ gives the two-sided unit for $\circ$. The following equations illustrates the conditions for $\circ$ to satisfy the associativity law in Q_e.

$$\begin{aligned}
(a, x)\circ((b, y)\circ(c, z)) &= (a, x)\circ(bc, \tilde{b}{*}z + y{\cdot}c + \Delta(b, c))\\
&= (abc, \tilde{a}{*}(\tilde{b}{*}z + y{*}c + \Delta(b, c)) + x{\cdot}(bc) + \Delta(a, bc))\\
&= (abc, (\tilde{a}\tilde{b}){*}z + (\tilde{a}{*}y){\cdot}c + x{\cdot}(bc) + \tilde{a}{*}\Delta(b, c) + \Delta(a, bc)),\\
((a, x)\circ(b, y))\circ(c, z) &= (ab, \tilde{a}{*}y + x{\cdot}b + \Delta(a, b))\circ(c, z)\\
&= (abc, (\tilde{a}\tilde{b}){*}z + (\tilde{a}{*}y){\cdot}c + (x{\cdot}b){\cdot}c + \Delta(a, b){\cdot}c + \Delta(ab, c)).
\end{aligned}$$

Thus $\circ$ gives a monoid structure on Q_e if and only if the condition (2iv) is satisfied. Next we see $Q_e\circ\operatorname{im}(\bar{P}) \subset \operatorname{im}(\bar{P})$ and $\operatorname{im}(\bar{P})\circ Q_e \subset \operatorname{im}(\bar{P})$. For $(a, x), (b, y) \in Q_e$ and $u \in M_{ee}$, we have the following equations by the definition of $\circ$:

$$(a, x)\circ(0, P(u)) = (0, \tilde{a}{*}P(u) + x{\cdot}0 + \Delta(a, 0)) = (0, \tilde{a}\tilde{a}P(u)) = \bar{P}(\tilde{a}\tilde{a}u),$$
$$(0, P(u))\circ(b, y) = (0, \tilde{0}{*}x + P(u){\cdot}b + \Delta(0, b)) = (0, P(u){\cdot}b) = \bar{P}(u{\cdot}b).$$

Thus $\circ$ induces a monoid structure also on $\bar{R}$ such that the canonical projection $\bar{\epsilon} : Q_e \to \bar{R} = Q_e/\operatorname{im}(\bar{P})$ preserves the monoid structures. Moreover the following equations illustrates the conditions for $\circ$ to satisfy the distributive laws in Q_e.

$$\begin{aligned}
&((a,x)+(b,y))\circ(c,z) = (a+b,x+y)\circ(c,z)\\
&\qquad = ((a+b)c, \widetilde{(a+b)}*z + (x+y)\cdot c + \Delta(a+b,c))\\
&\qquad = (ac+bc, (\tilde{a}+\tilde{b})*z + x\cdot c + y\cdot c + \Delta(a+b,c))\\
&\qquad = (ac+bc, \tilde{a}*z + x\cdot c + \tilde{b}*z + y\cdot c + \Delta(a+b,c) + \tilde{a}\tilde{b}PH(z)),\\
&(a,x)\circ(c,z) + (b,y)\circ(c,z) = (ac, \tilde{a}*z + x\cdot c + \Delta(a,c)) + (bc, \tilde{b}*z + y\cdot c + \Delta(b,c))\\
&\qquad = (ac+bc, \tilde{a}*z + x\cdot c + \tilde{b}*z + y\cdot c + \Delta(a,c) + \Delta(b,c)),\\
&(a,x)\circ((b,y)+(c,z)) = (a,x)\circ(b+c,y+z)\\
&\qquad = (a(b+c), \tilde{a}*(y+z) + x\cdot(b+c) + \Delta(a,b+c))\\
&\qquad = (ab+ac, \tilde{a}*y + x\cdot b + \tilde{a}*z + x\cdot c + \Delta(a,b+c))\\
&(a,x)\circ(b,y) + (a,x)\circ(c,z) = (ab, \tilde{a}*y + x\cdot b + \Delta(a,b)) + (ac, \tilde{a}*z + x\cdot c + \Delta(a,c))\\
&\qquad = (ab+ac, \tilde{a}*y + x\cdot b + \tilde{a}*z + x\cdot c + \Delta(a,b) + \Delta(a,c)).
\end{aligned}$$

Thus the multiplication $\circ$ gives a monoid structure in Q_e with conditions (6) and (7) of Definition 2.2 if and only if the conditions (2v) and (2vi) are satisfied. Hence the conditions (2iv), (2v) and (2vi) imply that the multiplication $\circ$ induces a ring structure on $\bar{R}$ such that the canonical projection $p : \bar{R} \to R$ as well as $\bar{\epsilon} : Q_e \to \bar{R}$ preserves the ring structures, which induces an action of $\bar{R}\otimes\bar{R}\otimes\bar{R}^{op}$ on M_{ee} through $p\otimes p\otimes p : \bar{R}\otimes\bar{R}\otimes\bar{R}^{op} \to R\otimes R\otimes R^{op}$: For any $\overline{(a,x)}, \overline{(b,y)} \in \bar{R}$, we have

$$p(\overline{(a,x)}\circ\overline{(b,y)}) = p(\overline{(a,x)\circ(b,y)}) = p(\overline{(ab, x\cdot b + \tilde{a}*y + \Delta(a,b))}) = ab.$$

Also the conditions (2iv), (2v) and (2vi) imply conditions (2), (3) and (4) of Definition 2.2:

$$\begin{aligned}
&TT(u) = HP(T(u)) - T(u)\\
&\qquad = \bar{H}\bar{P}(HP(u)-u) - (HP(u)-u) = HP(u) - HP(u) + u = u,\\
&T(((\overline{(a,x)}\otimes\overline{(b,y)})\cdot u\cdot\overline{(c,z)}) = T(\tilde{a}\tilde{b}u\cdot c) = HP(\tilde{a}\tilde{b}u\cdot c) - \tilde{a}\tilde{b}u\cdot c\\
&\qquad = \tilde{a}\tilde{b}HP(u)\cdot c - \tilde{a}\tilde{b}u\cdot c = \tilde{a}\tilde{b}(HP(u)-u)\cdot c = \tilde{b}\tilde{a}T(u)\cdot c = (\overline{(b,y)}\otimes\overline{(a,x)})\cdot\overline{T}(u)\cdot\overline{(c,z)},\\
&\bar{P}(u\cdot\overline{(a,x)}) = (0, P(u\cdot a)) = (0, P(u)\cdot a) = \bar{P}(u)\cdot\overline{(a,x)},\\
&\bar{P}((\overline{(a,x)}\otimes\overline{(a,x)})\cdot u) = (0, P(\tilde{a}\tilde{a}u)) = (0, \tilde{a}\tilde{a}P(u)) = (0, \tilde{a}*P(u)) = (a,x)\circ\bar{P}(u),
\end{aligned}$$

where $T = \bar{H}\bar{P} - 1 = HP - 1$. Thus the conditions (2), (3), (4), (6) and (7) of Definition 2.2 are satisfied with inducing a ring structure on $\bar{R}$ with an action on M_{ee} via $p\otimes p\otimes p$ if and only if $\circ$ gives a multiplication with the conditions (2iv) (2v) and (2vi).

Secondly, Proposition 6.1 shows that the conditions (2i) and (2ii) are necessary and sufficient conditions for Q_e to be square group satisfying the condition (1) of Definition 2.2, since we have

$$\begin{aligned}
\langle(a,x)|(b,y)\rangle_{\bar{H}} &= \bar{H}((a,x)+(b,y)) - \bar{H}(a,x) - \bar{H}(b,y)\\
&= \bar{H}(a+b,x+y) - \bar{H}(a,x) - \bar{H}(b,y)\\
&= h(a+b) + H(x+y) - H(x) - h(a) - H(y) - h(b)\\
&= \langle a|b\rangle_h = \tilde{a}\tilde{b}h(2) = \tilde{a}\tilde{b}\bar{H}(2),
\end{aligned}$$

if (2ii) is satisfied.

Finally, the condition (2iii) is equivalent to the condition (5) of Definition 2.2, since

$$\begin{aligned}
&\bar{H}((a,x)\circ(b,y)) = \bar{H}(ab, x{\cdot}b + \tilde{a}*y + \Delta(a,b)) \\
&\qquad = h(ab) + H(x){\cdot}b + H(\tilde{a}*y) + H(\Delta(a,b)) \\
&\qquad = h(ab) + H(\Delta(a,b)) + \tilde{a}\tilde{a}H(y) + H(x){\cdot}b \\
&(\overline{(a,x)}\otimes\overline{(a,x)}){\cdot}\bar{H}(b,y) + \bar{H}(a,x){\cdot}\overline{(b,y)} = \tilde{a}\tilde{a}(h(b)+H(y)) + (h(a)+H(x)){\cdot}\overline{(b,y)} \\
&\qquad = \tilde{a}\tilde{a}h(b) + h(a){\cdot}b + \tilde{a}\tilde{a}H(y) + H(x){\cdot}b
\end{aligned}$$

This completes the proof of the theorem. *qed.*

7. The square ring $Q(M, \lambda, \mu, k)$

For $k \geq 1$ let $R = \mathbb{Z}\times_k\mathbb{Z}$ be the subring of $\mathbb{Z}\times\mathbb{Z}$ consisting of all pairs $a = (a_0, a_1)$ with $a_0 - a_1 \equiv 0 \mod k$. This is the pull back ring of $\mathbb{Z} \longrightarrow \mathbb{Z}/k \longleftarrow \mathbb{Z}$. Then $\eta(\ell) = (\ell, \ell) \in \mathbb{Z}\times_k\mathbb{Z} = R$ gives the unit $1 = \eta(1) = (1, 1)$. The augmentation $\epsilon : R = \mathbb{Z}\times_k\mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $\epsilon(a) = a_0$ for $a = (a_0, a_1)$. A free $\mathbb{Z}$-basis of $R = \mathbb{Z}\times_k\mathbb{Z} \cong \mathbb{Z}\oplus\mathbb{Z}$ is given by 1 and $\bar{k}$ where $\bar{k} = (0, k)$ a generator of $\ker(\epsilon)$.

PROPOSITION 7.1. *Let $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$ be a quadratic $\mathbb{Z}$-module. Also let $k \in \mathbb{N}$ and $\lambda, \mu \in M_{ee}$ with relations $P(\lambda) = 0$ and $HP(\mu) = 2\mu + k\lambda$ be given. Then we obtain a square extension (H, P, h, Δ) as follows. Let*

(1) *the right action of R on M_e and M_{ee} be the multiplication given as $x{\cdot}(a_0, a_1) = a_1 x$ so that the homomorphisms H and P are R-linear and*

(2) *$h : \mathbb{Z} \times_k \mathbb{Z} \longrightarrow M_{ee}$ be the unique quadratic function satisfying*

$$\begin{cases} h(1) = 0, \quad h(\bar{k}) = \mu, \\ \langle a|b\rangle_h = a_0 b_0 \lambda \end{cases}$$

for $a, b \in \mathbb{Z} \times_k \mathbb{Z}$. Moreover let

(3) *$\Delta : (\mathbb{Z} \times_k \mathbb{Z}) \times (\mathbb{Z} \times_k \mathbb{Z}) \longrightarrow M_e$ be defined by*

$$\Delta((a_0, a_1), (b_0, b_1)) = \frac{a_0(a_0-1)(b_1-b_0)}{2k} P(\mu). \tag{7.1}$$

Since (H, P, h, Δ) is a square extension we thus obtain by theorem 6.2 the well defined square ring

$$Q(M, \lambda, \mu, k) = ((\mathbb{Z} \times_k \mathbb{Z})\oplus M_e \xrightarrow{\bar{H}} M_{ee} \xrightarrow{\bar{P}} (\mathbb{Z} \times_k \mathbb{Z})\oplus M_e)$$

which coincides with Definition 3.4.

Proof: One can easily check all the necessary conditions as follows. Firstly we give the following relations which makes our computations easy.

$$\begin{aligned}
&h(a_0, a_1) = \frac{a_0(a_0-1)}{2}\lambda + \frac{a_1 - a_0}{k}\mu, \quad \text{and} \\
&h(2) = h(2, 2) = \lambda \quad \text{and} \quad Ph(a_0, a_1) = \frac{a_1 - a_0}{k} P(\mu).
\end{aligned}$$

Then Definition 6.2(2i) and 6.2(2ii) are obtained by the equations $P(h(2)) = P(\lambda) = 0$ and $\langle a|b\rangle_h = a_0 b_0 \lambda = a_0 b_0 h(2)$.

Definition 6.2(2iii) is obtained as follows.

$$\begin{aligned}
&\widetilde{(a_0,a_1)}\widetilde{(a_0,a_1)}h(b_0,b_1)+h(a_0,a_1)\cdot(b_0,b_1)={a_0}^2h(b_0,b_1)+b_1h(a_0,a_1)\\
&\quad={a_0}^2\frac{b_0(b_0-1)}{2}\lambda+{a_0}^2\frac{b_1-b_0}{k}\mu+\frac{a_0(a_0-1)}{2}b_1\lambda+\frac{a_1-a_0}{k}b_1\mu\\
&\quad=\frac{{a_0}^2b_0(b_0-1)}{2}\lambda+\frac{a_0(a_0-1)}{2}b_0\lambda+\frac{a_0(a_0-1)(b_1-b_0)}{2k}k\lambda+\frac{{a_0}^2(b_1-b_0)+(a_1-a_0)b_1}{k}\mu\\
&\quad=\frac{{a_0}^2{b_0}^2-a_0b_0}{2}\lambda+\frac{a_0(a_0-1)(b_1-b_0)}{2k}(HP(\mu)-2\mu)+\frac{{a_0}^2(b_1-b_0)+(a_1-a_0)b_1}{k}\mu\\
&\quad=\frac{(a_0b_0)^2-a_0b_0}{2}\lambda+\frac{a_1b_1-a_0b_0}{k}\mu+H(\frac{a_0(a_0-1)(b_1-b_0)}{2k}P(\mu))\\
&\quad=h((a_0b_0,a_1b_1))+H(\Delta((a_0,a_1),(b_0,b_1)))=h((a_0,a_1)\cdot(b_0,b_1))+H(\Delta((a_0,a_1),(b_0,b_1))).
\end{aligned}$$

Definition 6.2(2iv) is obtained as follows.

$$\begin{aligned}
&\Delta((a_0,a_1),(b_0,b_1))\cdot(c_0,c_1)+\Delta((a_0,a_1)(b_0,b_1),(c_0,c_1))\\
&\quad=c_1\frac{a_0(a_0-1)(b_1-b_0)}{2k}P(\mu)+\frac{a_0b_0(a_0b_0-1)(c_1-c_0)}{2k}P(\mu)\\
&\quad=\frac{a_0(a_0-1)(b_1-b_0)c_1+a_0b_0(a_0b_0-1)(c_1-c_0)}{2k}P(\mu),
\end{aligned}$$

and hence

$$\begin{aligned}
&\widetilde{(a_0,a_1)}*\Delta((b_0,b_1),(c_0,c_1))+\Delta((a_0,a_1),(b_0,b_1)(c_0,c_1))\\
&\quad=a_0*(\frac{b_0(b_0-1)(c_1-c_0)}{2k}P(\mu))+\frac{a_0(a_0-1)(b_1c_1-b_0c_0)}{2k}P(\mu)\\
&\quad=\frac{a_0a_0b_0(b_0-1)(c_1-c_0)+a_0(a_0-1)(b_1c_1-b_0c_0)}{2k}P(\mu)\\
&\quad=\frac{a_0b_0(a_0b_0-1)(c_1-c_0)+a_0(a_0-1)(b_1-b_0)c_1}{2k}P(\mu)\\
&\quad=\Delta((a_0,a_1),(b_0,b_1))\cdot(c_0,c_1)+\Delta((a_0,a_1)(b_0,b_1),(c_0,c_1)).
\end{aligned}$$

Definition 6.2(2v) is obtained as follows.

$$\Delta((a_0,a_1),(b_0,b_1)+(c_0,c_1))=\frac{a_0(a_0-1)(b_1+c_1-b_0-c_0)}{2k}P(\mu),$$

and hence

$$\begin{aligned}
&\Delta((a_0,a_1),(b_0,b_1))+\Delta((a_0,a_1),(c_0,c_1))=\frac{a_0(a_0-1)(b_1-b_0)}{2k}P(\mu)+\frac{a_0(a_0-1)(c_1-c_0)}{2k}P(\mu)\\
&\quad=\frac{a_0(a_0-1)(b_1-b_0)+a_0(a_0-1)(c_1-c_0)}{2k}P(\mu)=\frac{a_0(a_0-1)(b_1+c_1-b_0-c_0)}{2k}P(\mu)\\
&\quad=\Delta((a_0,a_1),(b_0,b_1)+(c_0,c_1)).
\end{aligned}$$

Definition 6.2(2vi) is obtained as follows.

$$\begin{aligned}
&\Delta((a_0,a_1)+(b_0,b_1),(c_0,c_1)) = \Delta((a_0+b_0,a_1+b_1),(c_0,c_1))\\
&\quad= \frac{(a_0+b_0)(a_0+b_0-1)(c_1-c_0)}{2k}P(\mu)\\
&\quad= \frac{a_0(a_0-1)(c_1-c_0)}{2k}P(\mu) + \frac{b_0(b_0-1)(c_1-c_0)}{2k}P(\mu) + \frac{2a_0b_0(c_1-c_0)}{2k}P(\mu)\\
&\quad= \Delta((a_0,a_1),(c_0,c_1)) + \Delta((b_0,b_1),(c_0,c_1)) + a_0b_0\frac{(c_1-c_0)}{k}P(\mu)\\
&\quad= \Delta((a_0,a_1),(c_0,c_1)) + \Delta((b_0,b_1),(c_0,c_1)) + a_0b_0P(h(c_0,c_1))
\end{aligned}$$

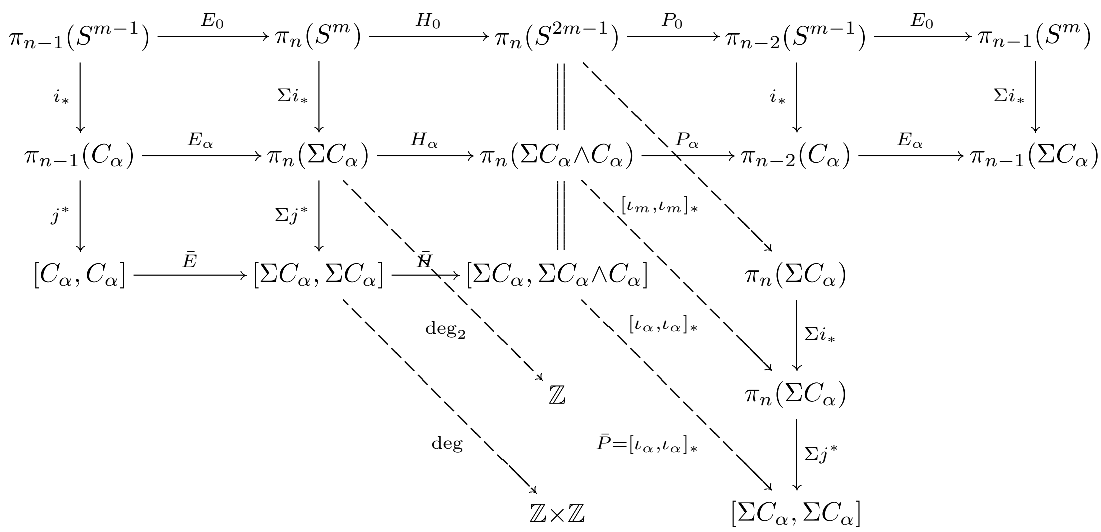
qed.

8. Proof of Lemma 3.3

Now let $\alpha \in \pi_{n-2}(S^{m-1})$ be an element as in (1.4) which induces the following cofibration sequence.

$$S^{n-2} \xrightarrow{\alpha} S^{m-1} \xrightarrow{i} C_\alpha \xrightarrow{j} S^{n-1}.$$

We give here a picture of related maps and Hopf invariants.



where $\deg : [\Sigma C_\alpha, \Sigma C_\alpha] \to \mathrm{End}(H^m(\Sigma C_\alpha))\times \mathrm{End}(H^n(\Sigma C_\alpha)) = \mathbb{Z}\times\mathbb{Z}$ and $\deg_2 : \pi_n(\Sigma C_\alpha) \to \mathrm{Hom}(H^n(S^n), H^n(\Sigma C_\alpha)) = \mathbb{Z}$ is taking the degree of a map. We remark that the EHP sequences as rows are exact, when $n < 3m-3$ by Toda [**T**] with H_0 and H_α the Hilton-Hopf invariants.

Proposition 8.1. *The homomorphism* deg *has its image in the pull back ring $R = \mathbb{Z}\times_k\mathbb{Z}$, where k is the order of the suspension element $\Sigma\alpha$ in the group $\pi_{n-1}(S^m)$.*

Proof: A map $f : \Sigma C_\alpha \to \Sigma C_\alpha$ with $\deg(f) = (d_0, d_1)$ induces the commutative diagram

$$
\text{(8.1)} \qquad
\begin{array}{ccccccc}
S^{n-1} & \xrightarrow{\Sigma\alpha} & S^m & \xrightarrow{\Sigma i} & \Sigma C_\alpha & \xrightarrow{\Sigma j} & S^n \\
\downarrow{\scriptstyle d_1\iota_{n-1}} & & \downarrow{\scriptstyle d_0\iota_m} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle d_1\iota_n} \\
S^{n-1} & \xrightarrow{\Sigma\alpha} & S^m & \xrightarrow{\Sigma i} & \Sigma C_\alpha & \xrightarrow{\Sigma j} & S^n.
\end{array}
$$

Thus we have $d_1\Sigma\alpha = \Sigma\alpha\circ d_1\iota_{n-1} = d_0\iota_m\circ\Sigma\alpha = d_0\Sigma\alpha$, and hence $(d_1 - d_0)\Sigma\alpha = 0$, $d_1 - d_0 \equiv 0 \mod k$. *qed.*

Lemma 8.2. *Let $\alpha \in \pi_{n-2}(S^{m-1})$. If $k\Sigma\alpha = 0$, then for any choice of an element $\mu_1 \in \pi_{n-2}(S^{2m-3})$ with $[\iota_{m-1}, \iota_{m-1}]_*\mu_1 = -k\alpha$, we can find out an element $\mu_0 \in \pi_n(\Sigma C_\alpha)$ with $H_\alpha(\mu_0) = \Sigma^2\mu_1$ and $\deg_2(\mu_0) = k$, i.e. the degree of $\mu_{0*} : H_n(S^n; \mathbb{Z}) \to H_n(\Sigma C_\alpha; \mathbb{Z})$ is k.*

Proof: We use the exact sequences of homotopy groups associated to the pairs $(\Omega S^m, S^{m-1})$ and $(\Omega\Sigma C_\alpha, C_\alpha)$, since there is the following commutative diagram (see [**W**] and [**B1**]).

$$
\begin{array}{ccccccc}
\pi_n(S^m) & \xrightarrow{H_0} & \pi_n(S^{2m-1}) & \xrightarrow{P_0} & \pi_{n-2}(S^{m-1}) & \xrightarrow{E_0} & \pi_{n-1}(S^m) \\
\| & & \uparrow{\scriptstyle J_2(S^{m-1})_*} & & \| & & \| \\
\pi_{n-1}(\Omega S^m) & \xrightarrow{\hat{H}_0} & \pi_{n-1}(\Omega S^m, S^{m-1}) & \xrightarrow{\hat{P}_0} & \pi_{n-2}(S^{m-1}) & \xrightarrow{\hat{E}_0} & \pi_{n-2}(\Omega S^m) \\
\downarrow{\scriptstyle \Omega\Sigma i_*} & & \|{\scriptstyle \Omega\Sigma i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \Omega\Sigma i_*} \\
\pi_{n-1}(\Omega\Sigma C_\alpha) & \xrightarrow{\hat{H}_\alpha} & \pi_{n-1}(\Omega\Sigma C_\alpha, C_\alpha) & \xrightarrow{\hat{P}_\alpha} & \pi_{n-2}(C_\alpha) & \xrightarrow{\hat{E}_\alpha} & \pi_{n-2}(\Omega\Sigma C_\alpha) \\
\| & & \downarrow{\scriptstyle J_2(C_\alpha)_*} & & \| & & \| \\
\pi_n(\Sigma C_\alpha) & \xrightarrow{H_\alpha} & \pi_n(\Sigma C_\alpha \wedge C_\alpha) & \xrightarrow{P_\alpha} & \pi_{n-2}(C_\alpha) & \xrightarrow{E_\alpha} & \pi_{n-1}(\Sigma C_\alpha)
\end{array}
$$

where $J_2(S^{m-1})_*$ and $J_2(C_\alpha)_*$ are surjective. Then we see that it is sufficient to show the existence of elements $\mu_1 \in \pi_{n-2}(S^{2m-3})$ and $\mu_0 \in \pi_n(\Sigma C_\alpha)$ with $[\iota_{m-1}, \iota_{m-1}]_*\mu_1 = -k\alpha$, $H_\alpha(\mu_0) = \Sigma^2\mu_1$ and $\deg_2(\mu_0) = k$. In fact, for any other element $\mu_1' \in \pi_{n-2}(S^{2m-3})$ with $[\iota_{m-1}, \iota_{m-1}]_*\mu_1' = -k\alpha$, there is an element $\mu_2' \in \pi_{n-1}(\Omega S^m, S^{m-1})$ with $J_2(S^{m-1})_*\mu_2' = \Sigma^2\mu_1'$, and hence $\hat{P}_0(\mu_2') = -k\alpha = \hat{P}_0\hat{H}_\alpha(\mu_0)$. Then we can take an element $\gamma \in \pi_n(S^m)$ with $\mu_2' = \hat{H}_\alpha(\mu_0) + H_0(\gamma) = \hat{H}_\alpha(\mu_0 + \gamma)$. By putting $\mu_0' = \mu_0 + \gamma \in \pi_{n-1}(C_\alpha)$, we get $H_\alpha(\mu_0') = \Sigma^2(\mu_1')$ with $\deg_2(\mu_0') = k$.

Since the order of $\Sigma\alpha$ is $k \geq 1$, we have $\hat{E}_0(k\alpha) = 0$, and hence there is an extension $\widehat{k\alpha} : (C(S^{n-2}), S^{n-2}) \to (\Omega S^m, S^{m-1})$ of $k\alpha$. Let $H : (C(S^{n-2}), S^{n-2}) \to (C_\alpha, S^{m-1})$ be a relative homeomorphism giving a null-homotopy of α. By adding k-copies of H, we obtain a null-homotopy $kH : (C(S^{n-2}), S^{n-2}) \to (C_\alpha, S^{m-1})$ of $k\alpha$ which gives the commutative diagram

$$
\begin{array}{ccc}
C(S^{n-2}) & \xrightarrow{kH} & C_\alpha \\
\downarrow & & \downarrow \\
S^{n-1} & \xrightarrow{k\iota_{n-1}} & S^{n-1},
\end{array}
$$

where the two columns are the canonical collapsions. Since the two maps $\widehat{k\alpha}$ and kH coincide on S^{n-2}, by gluing $-\widehat{k\alpha}$ with the direction altered to $kH : 0 \to k\alpha$ we get a new map

$$\mu_0 = kH - \widehat{k\alpha} : S^{n-1} \to \Omega\Sigma S^{m-1} \cup C_\alpha \subset \Omega\Sigma C_\alpha$$

which gives the following diagram commutative up to homotopy.

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\mu_0} & \Omega\Sigma S^{m-1}\cup C_\alpha & \hookrightarrow & \Omega\Sigma C_\alpha \\ \| & & \downarrow & & \downarrow \\ S^{n-1} & \xrightarrow{k\iota_{n-1}} & S^{n-1} & = & S^{n-1}. \end{array}$$

By taking the adjoint we get $\mu_0 \in \pi_{n-1}(\Omega\Sigma C_\alpha)$ as an element in $\pi_n(\Sigma C_\alpha)$. Then by the definition of Hilton-Hopf invariant H_0, $H_0(\mu_0) = \mu_0$. *qed.*

Next we show the following Propositions.

PROPOSITION 8.3. *There is a central extension*

$$0 \longrightarrow \pi_n(S^m)/U_\alpha \longrightarrow [\Sigma C_\alpha, \Sigma C_\alpha] \xrightarrow{\deg} \mathbb{Z} \times_k \mathbb{Z} \longrightarrow 0$$

where the subgroup U_α of $\pi_n(S^m)$ is generated by the elements $\eta_m(\Sigma^2\alpha)$ and $(\Sigma\alpha)\eta_{n-1}$ with $\eta_t = \Sigma^{t-2}\eta \in \pi_{t+1}(S^t)$ the Hopf element, $t \geq 2$.

Proof: Since $\mathbb{Z} \times_k \mathbb{Z} \cong \mathbb{Z}\{(1,1)\}\oplus\mathbb{Z}\{(0,k)\}$ as modules, deg is surjective by Lemma 8.2.

Let $f : \Sigma C_\alpha \to \Sigma C_\alpha$ be an element in $\ker\deg$. Then $f\circ\Sigma i : S^m \to S^m$ must be trivial, since $\deg(f) = (0,0)$. Hence there exists a map $f_1 : S^n \to \Sigma C_\alpha$ such that $f \sim f_1\circ\Sigma j$. We also see that $\Sigma j\circ f : S^n \to S^n$ is trivial. Hence there exists a map $f_0 : S^n \to S^m$ such that $f_1 \sim \Sigma i\circ f_0$ and $f \sim \Sigma i\circ f_0\circ\Sigma j$. Conversely, an element $f_0 : S^n \to S^m$ induces $\Sigma i\circ f_0\circ\Sigma j : \Sigma C_\alpha \to \Sigma C_\alpha$ which is in $\ker\deg$. Thus we have a short exact sequence

$$0 \longrightarrow \mathrm{im}(\Sigma j^*\circ\Sigma i_*) \longrightarrow [\Sigma C_\alpha, \Sigma C_\alpha] \xrightarrow{\deg} \mathbb{Z} \times_k \mathbb{Z} \longrightarrow 0 \tag{8.2}$$

and an isomorphism $\mathrm{im}(\Sigma j^*\circ\Sigma i_*) \cong \pi_n(S^m)/\ker\Sigma j^*\circ\Sigma i_*$. Since $S^n = \Sigma C_\alpha/S^m$ co-acts on ΣC_α, the image $\mathrm{im}\,\Sigma j^*$ is in the center of $[\Sigma C_\alpha, \Sigma C_\alpha]$, and hence so is $\mathrm{im}(\Sigma j^*\circ\Sigma i_*)$. Thus the short exact sequence (8.2) is a central extension.

So we are left to show that $\ker\Sigma j^*\circ\Sigma i_* = U_\alpha$. Let g be an element in $\ker\Sigma j^*\circ\Sigma i_*$, i.e, $\Sigma i\circ g\circ\Sigma j \sim 0$. Let us consider the diagram

$$\begin{array}{ccccccc} & & \Sigma C_\alpha & \xrightarrow{\Sigma j} & S^n & \xrightarrow{\Sigma^2\alpha} & S^{m+1} \\ & & & & & \searrow^{g} & \\ & & & & S^{n-1} & \xrightarrow{\Sigma\alpha} S^m & \xrightarrow{\Sigma i} \Sigma C_\alpha \end{array}$$

with cofibration rows. Let $F_{\Sigma i} \xrightarrow{\widehat{\Sigma i}} S^m$ denote the homotopy fibre of Σi with CW decomposition

$$F_{\Sigma i} \simeq S^{n-1} \cup (\text{cells in dimension} \geq n + m - 2 > n),$$

where $\widehat{\Sigma i}|_{S^{n-1}} = \Sigma\alpha$. Since $\Sigma i\circ g\circ\Sigma j \sim 0$, there is a map $g_0 : \Sigma C_\alpha \to F_{\Sigma i}$ whose image is in $S^{n-1} \subset F_{\Sigma i}$ such that $g\circ\Sigma j \sim \widehat{\Sigma i}\circ g_0$. For the dimensional reasons, we can take an element $x\eta_{n-1} \in \pi_n(S^{n-1})$ such that $g_0 \sim x\eta_{n-1}\circ\Sigma j$, and hence $g\circ\Sigma j \sim x\Sigma\alpha\circ\eta_{n-1}\circ\Sigma j$. Thus we get that $(g - x\Sigma\alpha\circ\eta_{n-1})\circ\Sigma j \sim 0$. Then there exists an element $y\eta_m \in \pi_{m+1}(S^m)$ such that $g - x\eta_{n-1} \sim y\eta_m$, and hence g is in U_α. This implies that $\ker \Sigma j^*\circ\Sigma i_* \subseteq U_\alpha$. The converse is clear. *qed.*

Proposition 8.4. *The epimorphism* deg *has a splitting homomorphism* $s : \mathbb{Z} \times_k \mathbb{Z} \to [\Sigma C_\alpha, \Sigma C_\alpha]$ *given by the following formula.*

$$s(a_0, a_1) = a_0\iota_\alpha + \frac{(a_1 - a_0)}{k}\mu_0\circ\Sigma j$$

Proof: Since $s(1,1) = \iota_\alpha$ and $s(0,k) = \mu_0\circ\Sigma j$, it is sufficient to show that ι_α and $\mu_0\circ\Sigma j$ commutes up to homotopy. To see this, we take adjoint maps of them. The adjoint of the identity ι_α is the canonical inclusion $\widehat{\iota_\alpha} : C_\alpha \hookrightarrow \Omega\Sigma C_\alpha$ and the adjoint of $\mu_0\circ\Sigma j$ is described as a composition $\widehat{\mu_0}\circ j$ where $\widehat{\mu_0} : S^{n-1} \to \Omega\Sigma C_\alpha$ is the adjoint of μ_0. Also the adjoint of the commutator of ι_α and $\mu_0\circ\Sigma j$ is given, up to sign, by the composition

$$\langle\widehat{\iota_\alpha}, \widehat{\mu_0}\circ j\rangle : C_\alpha \xrightarrow{\bar{\Delta}} C_\alpha\wedge C_\alpha \xrightarrow{1\wedge j} C_\alpha\wedge S^{n-1} \xrightarrow{\widehat{\iota_\alpha}\wedge\widehat{\mu_0}} \Omega\Sigma C_\alpha\wedge\Omega\Sigma C_\alpha \xrightarrow{c} \Omega\Sigma C_\alpha,$$

where $\bar{\Delta} : C_\alpha \to C_\alpha\wedge C_\alpha$ is the reduced diagonal map and c denotes the commutator of the first and second projections $\Omega\Sigma C_\alpha\times\Omega\Sigma C_\alpha \to \Omega\Sigma C_\alpha$. Since C_α is of dimension $n - 1 < n + m - 2$, we can compress $\bar{\Delta}$ into a subspace $S^{m-1}\wedge S^{m-1}$ which is collapsed in $C_\alpha\wedge S^{n-1}$. Thus we have $\langle\widehat{\iota_\alpha}, \widehat{\mu_0}\circ j\rangle \sim 0$, and hence ι_α and $\mu_0\circ\Sigma j$ are commutative up to homotopy. *qed.*

Corollary 8.5. *The group* $[\Sigma C_\alpha, \Sigma C_\alpha]$ *is an abelian group isomorphic with* $(\mathbb{Z} \times_k \mathbb{Z}) \times (\pi_n(S^m)/U_\alpha)$.

Proof: Let $\phi : (\mathbb{Z} \times_k \mathbb{Z}) \times (\pi_n(S^m)/U_\alpha) \to [\Sigma C_\alpha, \Sigma C_\alpha]$ be the homomorphism given by

$$\phi(a, x) = s(a) + \Sigma j^*\circ\Sigma i_*(x).$$

Then by Propositions 8.3 and 8.4, one can easily see that ϕ is an isomorphism of groups. Thus the group $[\Sigma C_\alpha, \Sigma C_\alpha]$ is an abelian group isomorphic with $(\mathbb{Z} \times_k \mathbb{Z}) \times (\pi_n(S^m)/U_\alpha)$. *qed.*

Thus for any choice of the data λ, μ in (3.1), the homomorphism s given in Proposition 8.4 gives a splitting of the homomorphism $\deg : [\Sigma C_\alpha, \Sigma C_\alpha] \to R$ of abelian groups. We summarise the results obtained in this section as follows.

Lemma 3.3 *For the square ring* $End(\Sigma C_\alpha)$ *given by (1.2) and for the data* (M, λ, μ, k) *in (3.1) one gets a commutative diagram*

$$\begin{array}{ccccc}
M_e & \xrightarrow{H} & M_{ee} & \xrightarrow{P} & M_e \\
{\scriptstyle i}\downarrow & & \| & & \downarrow{\scriptstyle i} \\
[\Sigma C_\alpha, \Sigma C_\alpha] & \xrightarrow{\bar{H}} & [\Sigma C_\alpha, \Sigma C_\alpha\wedge C_\alpha] & \xrightarrow{\bar{P}} & [\Sigma C_\alpha, \Sigma C_\alpha] \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle \deg} & \nearrow{\scriptstyle h} & & & \\
R & & & &
\end{array}$$

where the column $M_e \overset{i}{\hookrightarrow} [\Sigma C_\alpha, \Sigma C_\alpha] \overset{\deg}{\to} R$ *is a split short exact sequence of abelian groups. For a splitting* s *of* deg, *let* $h = \bar{H}s$. *We can choose* s *such that* $s(1) = \iota_\alpha$

is the identity of ΣC_α *and* $s(\bar{k}) = \mu_0$ *such that* $\bar{H}(\mu_0) = h(\bar{k}) = \mu$ *and* $\bar{H}(1+1) = h(1+1) = \lambda$.

9. Proof of Theorem 3.2

Using Lemma 3.3 with notations in (3.1), we show Theorem 3.2

PROPOSITION 9.1. *We have the following formulae for any* $\ell \in \mathbb{Z}$, $f, f' \in [\Sigma C_\alpha, \Sigma C_\alpha]$ *and* $g, g' \in \pi_n(\Sigma C_\alpha)$.

(1) $(f + (g\circ\Sigma j))\circ f' = f\circ f' + (g\circ\Sigma j)\circ f'$.
(2) $f\circ s(\ell,\ell) = \ell f$.
(3) $s(\ell,\ell)\circ(g'\circ\Sigma j) = \ell g'\circ\Sigma j + \frac{\ell(\ell-1)}{2}\bar{P}\bar{H}(g'\circ\Sigma j)$.
(4) $(g\circ\Sigma j)\circ(g'\circ\Sigma j) = (\deg_2 g')g\circ\Sigma j$.

Proof: Firstly we show the formula (1). By a Hilton-Milnor theorem with $n \leq 3m-3$ (see [**W**]), we have

$$(f + (g\circ\Sigma j))\circ f' = f\circ f' + (g\circ\Sigma j)\circ f' + [f, g\circ\Sigma j]\circ H_\alpha(f'),$$
$$[f, g\circ\Sigma j]\circ H_\alpha(f') = [f, g]\circ(\Sigma\iota_\alpha\wedge j)\circ(\Sigma i\wedge i)\circ H_\alpha(f') = 0.$$

Secondly we show the formula (2). For a map $f : \Sigma C_\alpha \to \Sigma C_\alpha$, we have $f\circ s(\ell,\ell) = f\circ(\ell\iota_\alpha) = f\circ(\iota_\alpha + \cdots \iota_\alpha) = f + \cdots + f = \ell f$.

Thirdly we show the formula (3). For $\ell = 2$, a Hilton-Milnor theorem with $n \leq 3m-3$ (see [**W**]) implies

$$2\iota_\alpha\circ(g'\circ\Sigma j) = 2g'\circ\Sigma j + [\iota_m, \iota_m]\circ H_\alpha(g'\circ\Sigma j) = 2g\circ\Sigma j + \bar{P}\bar{H}(g'\circ\Sigma j),$$

By the induction on $\ell \geq 2$, we get the desired formula (3).

We show the last formula (4). We have $(g\circ\Sigma j)\circ(g'\circ\Sigma j) = g\circ(\Sigma j\circ g'\circ\Sigma j) = g\circ((\deg_2 g')\iota_n) = (\deg_2 g')f$. *qed.*

COROLLARY 9.2. *The splitting* $s : \mathbb{Z}\times_k\mathbb{Z} \to [\Sigma C_\alpha, \Sigma C_\alpha]$ *satisfies the following formulae.*

(1) $(g\circ\Sigma j)\circ s(a_0, a_1) = a_1(g\circ\Sigma j)$, *where* g *is in* $\pi_n(\Sigma C_\alpha)$ *or* $\pi_n(\Sigma C_\alpha\wedge C_\alpha)$.
(2) $s(a_0, a_1)\circ s(b_0, b_1) = s(a_0b_0, a_1b_1) + \frac{a_0(a_0-1)(b_1-b_0)}{2k}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j)$.

Proof: The formula (1) is clear by the proof of Proposition 9.1 (4). So we show the formula (2) using Proposition 9.1 (1) through (4).

$$\begin{aligned}
s(a_0,a_1)\circ s(b_0,b_1) &= s(a_0,a_1)\circ(s(b_0,b_0) + s(0,b_1-b_0)) \\
&= s(a_0,a_1)\circ s(b_0,b_0) + s(a_0,a_1)\circ s(0,b_1-b_0) \\
&= b_0s(a_0,a_1) + (s(a_0,a_0) + s(0,a_1-a_0))\circ s(0,b_1-b_0) \\
&= b_0s(a_0,a_1) + s(a_0,a_0)\circ s(0,b_1-b_0) + s(0,a_1-a_0)\circ s(0,b_1-b_0) \\
&= s(a_0b_0,a_1b_0) + a_0s(0,b_1-b_0) + \frac{b_1-b_0}{k}s(a_0,a_0)\circ s(0,k) + (b_1-b_0)s(0,a_1-a_0) \\
&= s(a_0b_0,a_1b_1) + \frac{(b_1-b_0)}{k}(\frac{a_0(a_0-1)}{2}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j)) \\
&= s(a_0b_0,a_1b_1) + \frac{a_0(a_0-1)(b_1-b_0)}{2k}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j)
\end{aligned}$$

qed.

For the two elements $\bar{H}(2\iota_\alpha), H_\alpha(\mu_0)\circ\Sigma j \in [\Sigma C_\alpha, \Sigma C_\alpha\wedge C_\alpha]$, the following holds.

Proposition 9.3.

(1) $\bar{H}(2\iota_\alpha) = \Sigma^2 H(\alpha)$ *where* $\iota_\alpha : C_\alpha \to C_\alpha$ *denotes the identity.*
(2) $\bar{P}\bar{H}(2\iota_\alpha) = 0$.
(3) $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = 2H_\alpha(\mu_0)\circ\Sigma j + k\bar{H}(2\iota_\alpha)$.
 i) $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = 0$ *and* $2H_\alpha(\mu_0)\circ\Sigma j + k\bar{H}(2\iota_\alpha) = 0$ *when* m *is odd.*
 ii) $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = 2\bar{H}(\mu_0)\circ\Sigma j$ *and* $k\bar{H}(2\iota_\alpha) = 0$ *when* m *is even.*

Proof: Firstly we show (1). By a Hilton-Milnor theorem with $n \leq 3m-3$ (see [**W**]), we have

$$(\iota_1 + \iota_2)\circ(2\iota_\alpha) = \iota_1\circ(2\iota_\alpha) + \iota_2\circ(2\iota_\alpha) + [\iota_1, \iota_2]\circ\bar{H}(2\iota_\alpha),$$

where $\iota_t : \Sigma C_\alpha \to \Sigma C_\alpha \vee \Sigma C_\alpha$ is the inclusion to the t-th factor. Since $[\iota_1, \iota_2]\circ\bar{H}(2\iota_\alpha)$ is in the center of the group $[\Sigma C_\alpha, \Sigma C_\alpha \vee \Sigma C_\alpha]$ for dimensional reasons, we have the relation

$$\iota_1 + \iota_2 - \iota_1 - \iota_2 = [\iota_2, \iota_1]\circ\bar{H}(2\iota_\alpha).$$

The adjoint of $[\iota_2, \iota_1]$ is given by a Samelson product (commutator) $c_\alpha : C_\alpha \wedge C_\alpha \to \Omega\Sigma C_\alpha$ of the adjoints of ι_1 and ι_2. Also the adjoint of $\iota_1 + \iota_2 - \iota_1 - \iota_2$ is given by the composition of the commutator c_α with reduced diagonal map $\hat{\Delta}_2 : C_\alpha \to C_\alpha \wedge C_\alpha$ which is given by the suspension of the Hilton-Hopf invariant $H_0(\alpha)$ (see Theorem 5.14 of Boardmann and Steer [**BS**]). Thus we have $[\iota_2, \iota_1]\circ\Sigma^2 H_0(\alpha) = [\iota_2, \iota_1]\circ\bar{H}(2\iota_\alpha)$, and hence $\bar{H}(2\iota_\alpha) = \Sigma^2 H_0(\alpha)$.

Secondly we show (2). By a Hilton-Milnor theorem with $n \leq 3m-3$ (see [**W**]), we have

$$(2\iota_\alpha)f = 2f + [\iota_\alpha, \iota_\alpha]\circ\bar{H}(f).$$

For $f = \ell\iota_\alpha$ with $\ell \in \mathbb{Z}$, we then have $2\ell\iota_\alpha = 2\ell\iota_\alpha + [\iota_\alpha, \iota_\alpha]\circ\bar{H}(\ell\iota_\alpha)$, and hence we have $\bar{P}\bar{H}(\ell\iota_\alpha) = [\iota_\alpha, \iota_\alpha]\circ\bar{H}(\ell\iota_\alpha) = 0$.

So we are left to show (3). For the dimensional reasons, Hilton-Hopf invariant satisfies the following derivation formula.

$$\bar{H}(\ell\iota_\alpha\circ(\mu_0\circ\Sigma j)) = \bar{H}(\ell\iota_\alpha)\circ(\mu_0\circ\Sigma j) + (\Sigma(\ell\iota_m)\wedge(\ell\iota_m))\circ(H_\alpha(\mu_0)\circ\Sigma j)$$

Since the image of $\bar{H}(\ell\iota_\alpha)$ is in S^{2m-1}, it factors through $\Sigma j : \Sigma C_\alpha \to S^n$, and $\Sigma j\circ\mu_0\circ\Sigma j = (k\iota_n)\circ\Sigma j = k\Sigma j$. It then follows that $\bar{H}(\ell\iota_\alpha)\circ(\mu_0\circ\Sigma j) = k\bar{H}(\ell\iota_\alpha)$, and hence $\bar{H}((\ell\iota_\alpha)\circ(\mu_0\circ\Sigma j)) = k\bar{H}(\ell\iota_\alpha) + \ell^2 H_\alpha(\mu_0)\circ\Sigma j$. By putting $\ell = 2$, we get

$$\bar{H}((2\iota_\alpha)\circ(\mu_0\circ\Sigma j)) = k\bar{H}(2\iota_\alpha) + 4H_\alpha(\mu_0)\circ\Sigma j.$$

On the other hand, by a Hilton-Milnor theorem with $n \leq 3m-3$ (see [**W**]), we have

$$(2\iota_\alpha)\circ\mu_0 = 2\mu_0 + [\iota_\alpha, \iota_\alpha]\circ H_\alpha(\mu_0) = 2\mu_0 + [\iota_m, \iota_m]\circ H_\alpha(\mu_0)$$

and hence

$$\begin{aligned}\bar{H}((2\iota_\alpha)\circ(\mu_0\circ\Sigma j)) &= H_\alpha((2\iota_\alpha)\circ\mu_0)\circ\Sigma j \\ &= 2H_\alpha(\mu_0)\circ\Sigma j + \Sigma i\circ H_0([\iota_m, \iota_m]\circ H_\alpha(\mu_0))\circ\Sigma j = 2H_\alpha(\mu_0)\circ\Sigma j + \bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j)\end{aligned}$$

where $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = \Sigma i\circ H_0([\iota_m, \iota_m])\circ H_\alpha(\mu_0)\circ\Sigma j$. Thus we get the relation $2H_\alpha(\mu_0)\circ\Sigma j + \bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = k\bar{H}(2\iota_\alpha) + 4H_\alpha(\mu_0)\circ\Sigma j$, and hence $\bar{H}\bar{P}(H_\alpha(\mu_0)) = k\bar{H}(2\iota_\alpha) + 2H_\alpha(\mu_0)\circ\Sigma j$.

In case when m is odd, the Whitehead square $[\iota_m, \iota_m]$ has order 2, an¡d hence its Hilton-Hopf invariant is trivial. Thus $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = 0$ and $k\bar{H}(2\iota_\alpha) + 2H_\alpha(\mu_0)\circ\Sigma j = 0$.

In case when m is even, the Whitehead square $[\iota_m, \iota_m]$ has order ∞ and its Hilton-Hopf invariant is 2. Thus $\bar{H}\bar{P}(H_\alpha(\mu_0)\circ\Sigma j) = 2H_\alpha(\mu_0)\circ\Sigma j$ and $k\bar{H}(2\iota_\alpha) = 0$. *qed.*

Then we define the quadratic $\mathbb{Z}$-module

$$(9.1)\qquad \begin{gathered} M = (\pi_n(S^m)/U_\alpha \xrightarrow{H} \pi_n(S^{2m-1}) \xrightarrow{P} \pi_n(S^m)/U_\alpha) \quad \text{with elements} \\ \lambda \in \pi_n(S^{2m-1}), \quad \mu \in \pi_n(S^{2m-1}), \quad k \in \mathbb{N} \end{gathered}$$

as in (3.1) above. With the notations in (1.4), Proposition 7.1 and (9.1) we get: *In terms of the data (M, λ, μ, k) associated to α in (9.1) we obtain by 7.1 the square ring $Q(M, \lambda, \mu, k)$ together with an isomorphism*

$$(9.2)\qquad \mathrm{End}(\Sigma C_\alpha) \cong Q(M, \lambda, \mu, k)$$

of square rings.

Proof: We show that (M, λ, μ, k) satisfies the hypothesis in Proposition 7.1. Since a higher homotopy groups are abelian, M_e and M_{ee} are abelian groups. Also from the fact that H_0 and $[\iota_m, \iota_m]_*$ is a homomorphism, it follows that H and P are homomorphisms. For $x \in M_{ee} = \pi_n(S^{2m-1})$, since x is in stable range, $PHP(x) = P(H([\iota_m, \iota_m])\circ x)$. If m is even, $H([\iota_m, \iota_m]) = 2$ and $PHP(x) = 2P(x)$. But if m is odd, $H([\iota_m, \iota_m]) = 0$ and $PHP(x) = 0$ while the order of $P(x) = [\iota_m, \iota_m]\circ x$ is 2, and hence $PHP(x) = 0 = 2P(x)$. For $x \in M_e$, $H(x)$ is in $M_{ee} = \pi_n(S^{2m-1})$, and hence $H(x)$ is in stable range, $HPH(x) = H([\iota_m, \iota_m])\circ H(x)$. If m is even, $H([\iota_m, \iota_m]) = 2$ and $HPH(x) = 2H(x)$. But if m is odd, $H([\iota_m, \iota_m]) = 0$ and $HPH(x) = 0$. For dimensional reasons, there is an element $x_0 \in \pi_{n-2}(S^{2m-3})$ with $\Sigma^2 x_0 = H(x) = H_0(x)$ and $[\iota_{m-1}, \iota_{m-1}]_* x_0 = P_0(H_0(x)) = 0$. Taking its Hilton-Hopf invariant, we get $2x_0 = 0$ and $2H_0(x) = \Sigma^2(2x_0) = 0$, and hence $HPH(x) = 0 = 2H(x)$. By Proposition 9.3, it follows that (M, λ, μ, k) satisfy the required conditions to define a square extension.

So we are left to show that the isomorphism $\phi : (\mathbb{Z} \times_k \mathbb{Z}) \times (\pi_n(S^m)/U_\alpha) \to [\Sigma C_\alpha, \Sigma C_\alpha]$ in the proof of Corollary 8.5 carries the product given in Theorem 6.2 to the composition. For $a = (a_0, a_1), b = (b_0, b_1) \in \mathbb{Z} \times_k \mathbb{Z}$ and $[x], [y] \in \pi_n(S^m)/U_\alpha$, we obtain by using Proposition 9.1

$$
\begin{aligned}
\phi(a,[x])\circ\phi(b,[y]) &= (s(a)+\Sigma j^*(\Sigma i_*(x)))\circ(s(b)+\Sigma j^*(\Sigma i_*(y)))\\
&= (s(a)+\Sigma i\circ x\circ\Sigma j)\circ s(b)+(s(a_0,a_1)+\Sigma i\circ x\circ\Sigma j)\circ(\Sigma i\circ y\circ\Sigma j)\\
&= s(a)\circ s(b)+(\Sigma i\circ x\circ\Sigma j)\circ s(b)+(a_0\iota_\alpha+(\frac{a_1-a_0}{k}\mu_0+\Sigma i\circ x)\circ\Sigma j)\circ(\Sigma i\circ y\circ\Sigma j)\\
&= s(ab)+b_1\Sigma i\circ x\circ\Sigma j+\frac{a_0(a_0-1)(b_1-b_0)}{2k}\Sigma i\circ[\iota_m,\iota_m]\circ H_\alpha(\mu_0)\circ\Sigma j\\
&\qquad +(a_0\iota_\alpha)\circ\Sigma i\circ y\circ\Sigma j+(\frac{a_1-a_0}{k}\mu_0+\Sigma i\circ x)\circ\Sigma j\circ\Sigma i\circ y\circ\Sigma j\\
&= s(ab)++b_1\Sigma i\circ x\circ\Sigma j+a_0\Sigma i\circ y\circ\Sigma j+\frac{a_0(a_0-1)}{2}\Sigma i\circ[\iota_m,\iota_m]\circ H_\alpha(y)\circ\Sigma j\\
&\qquad +\frac{a_0(a_0-1)(b_1-b_0)}{2k}\Sigma i\circ([\iota_m,\iota_m]\circ H_\alpha(\mu_0))\circ\Sigma j\\
&= s(ab)+\Sigma i\circ(b_1x+a_0y+\frac{a_0(a_0-1)}{2}P_\alpha H_\alpha(y)+\frac{a_0(a_0-1)(b_1-b_0)}{2k}P_\alpha H_\alpha(\mu_0))\circ\Sigma j\\
&= \phi(ab,b_1[x]+a_0[y]+\frac{a_0(a_0-1)}{2}P_\alpha H_\alpha(y)+\Delta(a,b))=\phi((a,[x])\circ(b,[y]))
\end{aligned}
$$

qed.

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