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Lusternik-Schnirelmann category of stunted quasi-projective spaces

By

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Abstract

We determine the L-S category of stunted quasi-projective space $Q_{n,m} = Q_n/Q_m$ for $n \leq 4m+3$. As a special case of our main result, the L-S category of Q_3 is determined to satisfy $\operatorname{cat} Q_3 = 3$, which is in a sharp contrast with the result $\operatorname{cat}_{\mathrm{Sp}(3)} Q_3 = 2$ by Fernández-Suárez, Gómez-Tato and Tanré [4].

1. Introduction

In this paper, each space is assumed to have the homotopy type of a CW-complex. The (normalized) Lusternik-Schnirelmann category of X, denoted $\operatorname{cat} X$ is the least number m such that there is a covering of X by $m+1$ open subsets each of which is contractible in X.

Let $\mathbb{H}$ be the quaternion and $S(\mathbb{H}^n)$ be the unit sphere in $\mathbb{H}^n$. In [8], James has defined the (quaternionic) quasi-projective space Q_n is as follows:

$$Q_n = S(\mathbb{H}^n) \times S(\mathbb{H}) / \sim,$$

where $\sim$ is an equivalence relation given by

$$(u, q) \sim (uz, z^{-1}qz) \text{ for } z \in S(\mathbb{H}) \text{ and } (u, 1) \sim (v, 1).$$

Then Q_n is a CW-complex having one cell e^{4r-1} of dimension $4r-1$ for $r = 1, \ldots, n$ [8]. Hence, for $m < n$, Q_m is a subcomplex of Q_n and we denote by $Q_{n,m} = Q_n/Q_m$ the stunted quasi-projective space. There is the following result for $\operatorname{cat} Q_{n,m}$.

Theorem 1.1 (Kishimoto and Kono [9])**.**

$$\operatorname{cat} Q_{n,m} = \begin{cases} 1 & m+1 \leq n \leq 2m+1 \\ 2 & 2m+2 \leq n \leq 3m+2 \end{cases}$$

and

$$\operatorname{cat} Q_{n,m} \geq 3 \textit{ for } n \geq 4m+4.$$

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The main result of this paper is as follows:

Theorem 1.2. $\operatorname{cat} Q_{n,m} = 3$ *for* $3m+3 \leq n \leq 4m+3$.

Corollary 1.1. *When* $n = 3$ *and* $m = 0$, $\operatorname{cat} Q_3 = 3$.

This result answers the question in Fernández-Suárez, Gómez-Tato and Tanré [4], in which they have proved $\operatorname{cat}_{\mathrm{Sp}(3)} Q_3 = 2$ with $\operatorname{cat} Q_3$ itself left unknown, where $\operatorname{cat}_X(A)$ denotes the L-S category of A in X in the sense of Berstein and Ganea (see [1] for its precise definition).

In this paper, we follow the notations in Iwase [5]: let h^* be a multiplicative generalized cohomology. The cup-length of X with the cohomology theory h^* is the least number m such that all $(m+1)$-fold cup products vanish in the reduced cohomology $\tilde{h}^*(X)$. We denote this number by $\operatorname{cup}(X;h)$. To obtain Theorem 1.2, we use the following fact due to Ganea [3] (see Iwase, Mimura and Nishimoto [7] for details).

Fact 1.1. *Let X be an $(n-1)$-connected CW-complex and h^* be a multiplicative generalized cohomology. Then*

$$\operatorname{cup}(X;h) \leq \operatorname{cat} X \leq \frac{\dim X}{n}.$$

2. Proof of Theorem 1.2

To obtain the lower bounds for $\operatorname{cat} Q_{n,m}$, we use the cohomology theory introduced by Iwase and Mimura [6]: Let (X, A) be a pair of space. The cohomology theory h^* is defined by

$$h^*(X, A) = \{X/A, \mathcal{S}[0,2]\},$$

where $\mathcal{S}[0,2]$ is the spectrum obtained from the sphere spectrum $\mathcal{S}$ by killing all homotopy groups of dimensions > 2. Then h^* is an additive and multiplicative cohomology theory with the coefficient ring

$$h^* = h^*(pt) \cong \mathbb{Z}[\varepsilon]/(\varepsilon^3, 2\epsilon), \deg \varepsilon = -1,$$

where $\varepsilon \in h^{-1} = \pi_0^S(\Sigma^{-1}\mathcal{S}) \cong \pi_1^S(\mathcal{S})$ corresponds to the Hopf element η.

Since all the cells in $Q_{n,m}$ are concentrated in dimensions 3 modulo 4, we have

$$h^*(Q_{n,m}) \cong h^*\{1, x_{4m+3}, x_{4m+7}, \cdots, x_{4n-1}\},$$

where $\deg x_{4i-1} = 4i-1$ for $m+1 \leq i \leq n$. We need to show the following

Proposition 2.1. $x_{4\ell+3}^2 = \varepsilon \cdot x_{8\ell+7} \in h^{8\ell+6}(Q_{2\ell+2,\ell})$ *for any* $\ell \geq 0$.

Then we obtain $x_{8m+7}^2 = \varepsilon \cdot x_{16m+15} \in h^{16m+14}(Q_{4m+4,2m+1})$ and

$$\begin{aligned} x_{4m+3}^2 x_{8m+7} &= (\varepsilon \cdot x_{8m+7}) x_{8m+7} = \varepsilon \cdot x_{8m+7}^2 \\ &= \varepsilon^2 \cdot x_{16m+15} \in h^{16m+13}(Q_{4m+4,m}). \end{aligned}$$

Hence we have

$$\begin{aligned} 0 \neq & x_{4m+3}x_{8m+7} = \varepsilon \cdot x_{12m+11} \\ \in & h^{12m+10}(Q_{4m+4,m}) \cong h^{12m+10}(Q_{3m+3,m}) \cong \mathbb{Z}/2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} x_{4m+3}^3 = & (\varepsilon \cdot x_{8m+7})x_{4m+3} = \varepsilon \cdot x_{4m+3}x_{8m+7} \\ = & \varepsilon^2 x_{12m+11} \in h^{12m+9}(Q_{3m+3,m}) \end{aligned}$$

and $3 \leq \mathrm{cup}(Q_{3m+3,m}; h)$. By Fact 1.1, we have

$$3 \leq \mathrm{cup}(Q_{3m+3,m}; h) \leq \mathrm{cup}(Q_{n,m}; h) \leq \mathrm{cat}\, Q_{n,m} \text{ for } 3m+3 \leq n,$$

and

$$\mathrm{cat}\, Q_{n,m} \leq \mathrm{cat}\, Q_{4m+3,m} \leq \frac{16m+11}{4m+3} < 4 \text{ for } n \leq 4m+3.$$

So, we obtain Theorem 1.2.

3. Proof of Proposition 2.1

Let $\gamma : S^{8\ell+6} \to Q_{2\ell+1,\ell}$ be the attaching map of the $(8\ell+7)$-cell of $Q_{2\ell+2,\ell}$. There exist a CW-complex $Q'_{2\ell+1,\ell}$ such that $\Sigma Q'_{2\ell+1,\ell} = Q_{2\ell+1,\ell}$, since $Q_{2\ell+1,\ell}$ is $(4\ell+2)$-connected and $\dim Q_{2\ell+1,\ell} = 8\ell+3$. We need the relation between the attaching map γ and the cup product in the cohomology theory h^*. By the parallel argument to Lemma 3.6 of [6], we obtain the following.

Lemma 3.1. *Let h^* be any multiplicative generalized cohomology theory and let $K = \Sigma Q \cup_f e^q$ for a given map f from S^{q-1} to a suspension of a space Q. Let x and y be the elements of $h^*(K)$ such that y corresponds to the generator of $h^*(S^r)$. Then*

$$x^2 = \pm \bar{H}_1^h(f) \cdot y \text{ in } h^*(K),$$

where $\pm \bar{H}_1^h$ is the composition $\rho^h \circ \lambda_2$ of the Boardman-Steer Hopf invariant $\lambda_2 : \pi_{q-1}(\Sigma Q) \to \pi_q(\Sigma^2 Q \wedge Q)$ (Boardman and Steer [2]) with the Hurewicz homomorphism $\rho^h : \pi_q(\Sigma^2 Q \wedge Q) \to h^{2r}(S^q) \cong h^{2r-q}$ given by $\rho^h(g) = \Sigma_*^{-q} g^*(i^*(x) \otimes i^*(x))$ ($i : \Sigma Q \to K$ is the inclusion).

Since $Q'_{2\ell+1,\ell}$ is $(4\ell+1)$-connected and $\dim Q'_{2\ell+1,\ell} = 8\ell+2$, $Q'_{2\ell+1,\ell}$ has the homotopy type of the suspension. Hence, by [2], we have the equation

$$\lambda_2(\gamma) = \Sigma h_2^J(\gamma),$$

where h_2^J is the 2nd James Hopf invariant.

We consider the adjoint map $\mathrm{ad}(\gamma) : S^{8\ell+5} \to \Omega Q_{2\ell+1,\ell}$ in

$$\Omega Q_{2\ell+1,\ell} \subset \Omega Q_{2\ell+2,\ell} \subset \Omega V_{2\ell+2,\ell},$$

where ΩX is a loop space of a space X and $V_{n,m} = \mathrm{Sp}(n)/\mathrm{Sp}(m)$. We recall from [10] that the homology group of $\Omega\mathrm{Sp}(n)$ with coefficients in $\mathbb{Z}/2$ has the ring structure: $H_*(\Omega\mathrm{Sp}(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[u_2, u_6, \cdots, u_{4n-2}]$, where $\deg u_{4i-2} = 4i-2$ for $1 \le i \le n$, and these generators satisfy the relations $u_{4j+2}Sq_2 = u_{2j}^2$ for $1 \le j \le n-1$, where Sq_* is the dual operation of the Steenrod operation. So, we have the homology of $\Omega V_{2\ell+2,\ell}$:

$$H_*(\Omega V_{2\ell+2,\ell};\mathbb{Z}_2) \cong \mathbb{Z}_2[u_{4\ell+2}, u_{4\ell+6}, \cdots, u_{8\ell+2}, u_{8\ell+6}]$$

with the relation

$$u_{8\ell+6}Sq_2 = u_{4\ell+2}^2. \tag{3.1}$$

Hence $(8\ell+6)$-skeleton of $\Omega(V_{2\ell+2,\ell})$ has a cell decomposition:

$$\begin{aligned}(\Omega V_{2\ell+2,\ell})^{(8\ell+6)} \simeq &(S^{4\ell+2} \cup e^{4\ell+6} \cup \cdots \cup e^{8\ell+2})\\ &\cup_{S^{4\ell+2}} (S^{4\ell+2} \cup_{[\iota_{4\ell+2},\iota_{4\ell+2}]} e^{8\ell+4}) \cup e^{8\ell+6},\end{aligned} \tag{3.2}$$

where $[\iota_{4\ell+2},\iota_{4\ell+2}] : S^{8\ell+3} \to S^{4\ell+2}$ is the Whitehead product of two copies of identity map $\iota_{4\ell+2} : S^{4\ell+2} \to S^{4\ell+2}$. By the relation (3.1), we have

$$(\Omega V_{2\ell+2,\ell})^{(8\ell+6)}/(\Omega V_{2\ell+2,\ell})^{(8\ell+2)} = S^{8\ell+4} \cup_{\eta_{8\ell+4}} e^{8\ell+6}, \tag{3.3}$$

where η_k is a $(k-2)$-fold suspension of the Hopf map $\eta_2 : S^3 \to S^2$ for $k \ge 2$. By the cell decomposition:

$$\begin{aligned}\Omega Q_{2\ell+1,\ell} \simeq &(S^{4\ell+2} \cup e^{4\ell+6} \cup \cdots e^{8\ell+2})\\ &\cup_{S^{4\ell+2}} (S^{4\ell+2} \cup_{[\iota_{4\ell+2},\iota_{4\ell+2}]} e^{8\ell+4})\\ &\cup (\text{cells in dimensions} \ge 8\ell+8)\end{aligned}$$

and

$$(\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} = (\Omega Q_{2\ell+2,\ell})^{(8\ell+4)} = (\Omega V_{2\ell+2,\ell})^{(8\ell+4)},$$

we identify $\mathrm{ad}(\gamma)$ with a map:

$$\mathrm{ad}(\gamma) : S^{8\ell+5} \to (\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} = (\Omega V_{2\ell+2,\ell})^{(8\ell+4)}.$$

In consideration of $(8\ell+6)$-skeleton:

$$\begin{aligned}(\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} \cup_{\mathrm{ad}(\gamma)} e^{8\ell+6} &= (\Omega Q_{2\ell+2,\ell})^{(8\ell+6)}\\ &= (\Omega V_{2\ell+2,\ell})^{(8\ell+6)},\end{aligned}$$

the attaching map of $(8\ell+6)$-cell of (3.2) is equal to $\mathrm{ad}(\gamma)$. So, we have the following commutative diagram:

$$
\begin{array}{ccc}
S^{8\ell+6} & \xrightarrow{=} & S^{8\ell+6} \\
\downarrow{\scriptstyle \Sigma\,\mathrm{ad}(\gamma)} & & \downarrow{\scriptstyle \Sigma\,\mathrm{ad}(\gamma)} \\
(\Sigma\Omega\Sigma Q'_{2\ell+1,\ell})^{(8\ell+5)} & \xrightarrow{=} & (\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+5)} \\
\downarrow{\scriptstyle =} & & \\
(\Sigma J^2)^{(8\ell+5)} & & \downarrow{\scriptstyle proj} \\
\downarrow{\scriptstyle proj} & & \\
(\Sigma J^2)^{(8\ell+5)}/(\Sigma J^1)^{(8\ell+4)} & \xrightarrow{=} & (\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+5)}/(\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+4)} \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\Sigma(Q'_{2\ell+1,\ell} \wedge Q'_{2\ell+1,\ell})^{(8\ell+5)} & \xrightarrow{=} & S^{8\ell+5} \\
\downarrow{\scriptstyle incl} & & \\
\Sigma(Q'_{2\ell+1,\ell} \wedge Q'_{2\ell+1,\ell}), & &
\end{array}
$$

where J^k is the k-stage James reduced product of $Q'_{2\ell+1,\ell}$ and *proj* and *incl* are the projection and the inclusion, respectively. The left column is the definition of the 2nd James Hopf invariant and the right column is equal to $\eta_{8\ell+5} : S^{8\ell+6} \to S^{8\ell+5}$, by (3.3). Thus, we have

$$h_2^J(\gamma) = (incl) \circ \eta_{8\ell+5}.$$

And using Lemma 3.1, we obtain the relation $x_{4\ell+3}^2 = \varepsilon \cdot x_{8\ell+7} \in h^{8\ell+6}(Q_{2\ell+2,\ell})$. This completes the proof.

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