# Functors on the category of quasi-fibrations

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**Abstract**

We consider the following questions: when can we extend a continuous endofunctor on *Top* the category of topological spaces to a fibrewise continuous endofunctor on *Top*(2) the category of continuous maps? If this is true, does such fibrewise continuous endofunctor preserve fibrations? In this paper, we define *Fib* the topological category of cell-wise trivial fibre spaces over polyhedra and show that any continuous endofunctor on *Top* induces a fibrewise continuous endofunctor on *Fib* preserving the class of quasi-fibrations.

*Key words:* Continuous functor; fibration; fibrewise
*1991 MSC:* Primary 55R70, Secondary 55Q25, 55P10.

## 1 Introduction

In 1965, Hilton introduced in [6] a category of continuous maps and their commutative diagrams to give a homotopy theory of continuous maps. Following Hilton, James [8,9] extended this idea to study spaces and maps from the fibrewise point of view. A continuous map, whose target is a fixed space $B$, is called a *fibrewise space over* $B$. As an extension of the notion of a pointed space, James introduced the notion of a fibrewise pointed space: In fact in [9], James studied fibrewise continuous endofunctors to develop the homotopy theory on the category of fibrewise (pointed) spaces, e.g, $\Sigma_B$ and $C_B$ on the category of fibrewise spaces and $\Sigma^B_B$, $C^B_B$ and $\Omega^B_B$ on the category of fibrewise pointed spaces. In 2006, May and Sigurdsson established in [12] a homotopy

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theory of fibrewise pointed spaces in the homotopy category. In this paper, we discuss about fibrewise continuous functors not in the homotopy category but in the topological category itself.

Such extensions, in turn, give some information on the homotopy properties of topological spaces, which are studied by Crabb and James [1], Hardie [5], James [10], James and Morris [11], Oda [13], Smith [17], Sakai [15,16] and etc.

Let *Top* be the category of compactly generated Hausdorff spaces and continuous maps and $Top_*$ be the full subcategory of *Top* whose objects are all pointed spaces. Let us recall the definition of the category $Top(2)$ and its comma categories $Top_{\mathrm{B}}$ and $Top^{\mathrm{B}}_{\mathrm{B}}$. Let us begin with the definition of $Top(2)$.

**Definition 1.1** *An object of the category $Top(2)$ is a continuous map $p : E \to B$ of Top, which is often denoted by $(p{:}E{\to}B)$ and is called a projection. A morphism between $(p_1{:}E_1{\to}B_1)$ and $(p_2{:}E_2{\to}B_2)$ of $Top(2)$ is a pair of maps $(f{:}E_1{\to}E_2,\ g{:}B_1{\to}B_2)$ together with a commutative diagram given as follows:*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

We introduce the following category $Top^{\mathrm{A}}_{\mathrm{B}}$ for a pair $(B, A)$ of spaces, while $A$ is assumed to be either $\emptyset$ or $B$ in this paper.

**Definition 1.2** *An object of the category $Top^{\mathrm{A}}_{\mathrm{B}}$ is a pair consisting of morphisms $p : E \to B$ and $s : B \to E$ of Top such that $p{\circ}s$ is the inclusion $i^A_B : A \hookrightarrow B$, which are often called a projection and a section, respectively. A morphism between $(p_1{:}E_1{\to}B, s_1{:}A{\to}E_1)$ and $(p_2{:}E_2 \to B, s_2{:}A{\to}E_2)$ in $Top^{\mathrm{A}}_{\mathrm{B}}$ is given by a map $f : E_1 \to E_2$ together with commutative diagrams given as follows:*

$$\begin{array}{ccccc} E_1 & & \xrightarrow{f} & & E_2, \\ & {\scriptstyle p_1}\searrow & & \swarrow{\scriptstyle p_2} & \\ & & B & & \end{array} \qquad \begin{array}{ccccc} & & A & & \\ & {\scriptstyle s_1}\swarrow & & \searrow{\scriptstyle s_2} & \\ E_1 & & \xrightarrow[f]{} & & E_2. \end{array}$$

Then we clearly obtain that $(i^A_B{:}A{\to}B, 1_A{:}A{\to}A)$ is the initial object of $Top^{\mathrm{A}}_{\mathrm{B}}$ and that $(1_B{:}B{\to}B, i^A_B{:}A{\to}B)$ is the terminal object of $Top^{\mathrm{A}}_{\mathrm{B}}$. In the case when $A = \emptyset$, we abbreviate $Top^{\mathrm{A}}_{\mathrm{B}}$ as $Top_{\mathrm{B}}$.

To extend the homotopy theory on *Top* or $Top_*$ to $Top_{\mathrm{B}}$ or $Top^{\mathrm{B}}_{\mathrm{B}}$, it is often a necessary and sufficient procedure to extend a continuous endofuntor on *Top* or $Top_*$ to a fibrewise continuous endofunctor with appropriate properties on $Top_{\mathrm{B}}$ or $Top^{\mathrm{B}}_{\mathrm{B}}$.

Let $f$ and $g$ be morphisms in $Top^{\mathrm{A}}_{\mathrm{B}}$. Then $f$ and $g$ are called homotopic in $Top^{\mathrm{A}}_{\mathrm{B}}$ if there exists a homotopy in $Top^{\mathrm{A}}_{\mathrm{B}}$.

**Definition 1.3** *An endofunctor $\Phi : Top \to Top$ is said to be continuous if $\Phi : map(X, Y) \to map(\Phi(X), \Phi(Y))$ is a continuous map for any $X$ and $Y$ in Top.*

In the homotopy theory, many basic constructions are performed by using continuous endofunctor on *Top* such as the suspension functor $\Sigma$, the loop functor $\Omega$ and etc. In the effort to study Hopf invariants and Lusternik-Schnirelmann (L-S)theory on $Top^{\mathrm{B}}_{\mathrm{B}}$, we realise the following questions, which is essential to work on L-S theory.

**Question 1** *(1) When can we extend a given continuous endofunctor on Top to a fibrewise continuous endofunctor on $Top(2)$?*

*(2) Assume that the answer to the above is yes. Does the resulting fibrewise continuous endofunctor on $Top(2)$ preserve (quasi) fibrations?*

An answer has been made by James as follows.

**Theorem 1.4 (James [9])** *A given continuous endofunctor $\Phi$ on Top which satisfies $X \subset \Phi(X)$ induces a fibrewise continuous endofunctor on $Top(2)$.*

**Corollary 1.5 (James [9])** *Cone functor $C$, suspension functor $\Sigma$, James' reduced product functor $J$, and localisation functor $R$ (see Iwase [7]) can be extended to $Top(2)$, because these functors satisfy the condition in Theorem 1.4.*

We can observe that Theorem 1.4 does not answer on Question 1 (1) for the loop functor $\Omega$ nor on Question 1 (2). Since it is technically difficult to extend a given continuous endofunctor defined on *Top* to a fibrewise continuous endofunctor on the entire $Top(2)$, we work in a slightly small subcategory of *Top*, where we generalise the above result of James.

## 2 Main Theorem

We define a full subcategory *Fib* of $Top(2)$ and its comma category $Fib^{\mathrm{A}}_{\mathrm{B}}$ to give affirmative answers to the above questions without giving any conditions on the continuous endofunctors themselves.

Firstly, we define the category *Fib*:

**Definition 2.1** *An object ($p$:$E \to B$) of the full-subcategory Fib of $Top(2)$ is a fibre-space over a (possibly infinite) polyhedron with the following two con-*

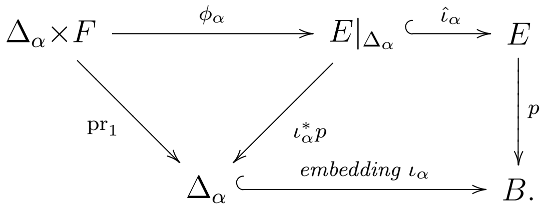
*ditions:*

(1) *(weak topology) The topology of $E$ is the weak topology with respect to subspaces $\{E|_{\Delta_\alpha}; \alpha \in \Lambda\}$, i.e., $A$ is closed in $E$ if and only if $A \cap E|_{\Delta_\alpha}$ is closed in $E|_{\Delta_\alpha}$ for each $\alpha \in \Lambda$.*

(2) *(cell-wise triviality) For each simplex $\Delta_\alpha$ of $B$, there exist a space $F$ (a fibre of $p$ on $\Delta_\alpha$) and a homeomorphism*

$$\phi_\alpha : \Delta_\alpha \times F_\alpha, \to E|_{\Delta_\alpha}$$

*such that the following diagram is commutative.*

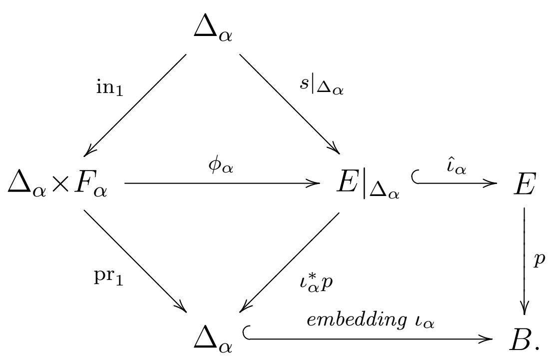



Secondly, we introduce the comma categories $Fib_{\mathrm{B}}$ and $Fib_{\mathrm{B}}^{\mathrm{B}}$ in a slightly general form as follows:

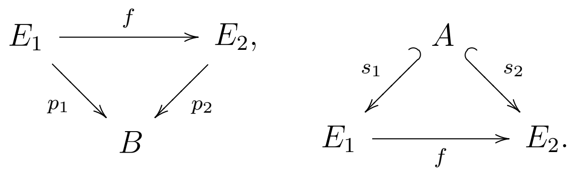
**Definition 2.2** *An object of the category $Fib_{\mathrm{B}}^{\mathrm{A}}$ is a pair consisting of morphisms $p : E \to B$ and $s : A \to E$ of Top such that $p \circ s$ is the inclusion $i_{\mathrm{B}}^{A} : A \hookrightarrow B$, which are often called a projection and a secion, respectively and satisfy the (pointed) cell-wise triviality condition:*
*For each simplex $\Delta_\alpha$ of $B$, there exist a homeomorphism*

$$\phi_\alpha : \Delta_\alpha \times F_\alpha \to E|_{\Delta_\alpha},$$

*which is an identification map, and the commutative diagram*




*A morphism between $(p_1{:}E_1 \to B, s_1{:}A \to E_1)$ and $(p_2{:}E_2 \to B, s_2{:}A \to E_2)$ in $Fib_{\mathrm{B}}^{\mathrm{A}}$ is given by a map $f : E_1 \to E_2$ together with commutative diagrams* *given as follows:*



Then we clearly obtain that $(i_B^A{:}A{\to}B, 1_A{:}A{\to}A)$ is the initial object of $Fib_{\mathrm{B}}^{\mathrm{A}}$ and that $(1_B{:}B{\to}B, i_B^A{:}A{\to}B)$ is the terminal object of $Fib_{\mathrm{B}}^{\mathrm{A}}$. In the case when $A = \emptyset$, we abbreviate $Fib_{\mathrm{B}}^{\mathrm{A}}$ as $Fib_{\mathrm{B}}$.

Let $\alpha$, $\beta$ : $Fib(\mathcal{C}) \to Top$ be functors such that

$$\alpha(E{\to}B) = E,\ \alpha(f,g) = f,$$
$$\beta(E{\to}B) = B,\ \beta(f,g) = g.$$

Our main results are described as follows.

**Theorem 2.3** *For any continuous endofunctor $\Phi$ : Top $\to$ Top, there exists a fibrewise continuous endofunctor $\Phi(2)$ : Fib $\to$ Fib which enjoys the following properties.*

$$\alpha\circ\Phi(2)(p{:}E{\to}B) = \coprod_{b\in B}\Phi(E_b), \quad \alpha\circ\Phi(2)(f,g) = \coprod_{b\in B}\Phi(f_b), \tag{1}$$
$$\beta\circ\Phi(2)(p{:}E{\to}B) = B, \quad \beta\circ\Phi(2)(f,g) = g, \tag{2}$$
$$\begin{cases}\Phi(2)(\mathrm{pr}_1{:}B{\times}F{\to}B) = (\mathrm{pr}_1{:}B{\times}\Phi(F){\to}B),\\ \Phi(2)(g{\times}h, g) = (g{\times}\Phi(h), g) \quad \textit{for} \quad h : F_1 \to F_2.\end{cases} \tag{3}$$

**Theorem 2.4** *For any natural transformation $\theta : \Phi_1 \to \Phi_2$ of fibrewise continuous endofunctors $\Phi_i$ : Top $\to$ Top $(i = 1, 2)$, there exists a natural transformation $\theta(2) : \Phi_1(2) \to \Phi_2(2)$ which enjoys the following properties.*

$$\alpha\circ\theta(2)(p{:}E{\to}B) = \coprod_{b\in B}\theta(E_b), \tag{1}$$
$$\beta\circ\theta(2)(p{:}E{\to}B) = \mathrm{id}_B, \tag{2}$$
$$\theta(2)(\mathrm{pr}_1{:}B{\times}F{\to}B) = \mathrm{id}_B{\times}\theta(F). \tag{3}$$

**Remark 2.5** *If $\theta$ is a natural equivalence, then so is $\theta(2)$.*

Finally, we define classes of quasi-fibrations in *Fib* and $Fib_{\mathrm{B}}^{\mathrm{A}}$.

**Definition 2.6** *(1) An object $(p{:}E{\to}B)$ of Fib is in the class* $\mathrm{q}(Fib)$*, if and only if $p$ is a quasi-fibration with a constant fibre $F$.*

*(2) An object $(p{:}E{\to}B;s{:}A{\to}E)$ of $Fib_{\mathrm{B}}^{\mathrm{A}}$ is in the class* $\mathrm{q}(Fib_{\mathrm{B}}^{\mathrm{A}})$*, if and only if $p$ is a quasi-fibration with a constant fibre $F$.*

Then we obtain the following properties of $\Phi(2)$ for an endofunctor $\Phi$ on *Top*.

**Theorem 2.7** *For a continuous endofunctor $\Phi$ on Top, the following two statements hold.*

*(1)* $\Phi(2) : Fib \to Fib$ *preserves* $\mathrm{q}(Fib)$,
*(2)* $\Phi(2) : Fib^{\mathrm{A}}_{\mathrm{B}} \to Fib^{\mathrm{A}}_{\mathrm{B}}$ *preserves the class* $\mathrm{q}(Fib^{\mathrm{A}}_{\mathrm{B}})$.

By Theorem 2.3, we obtain many fibrewise continuous endofunctors such as the fibrewise cone, the fibrewise suspension, the fibrewise loop space, the fibrewise reduced product space, the fibrewise localisation and etc. Combining this with Theorem 2.7, we have the following.

**Corollary 2.8** *The continuous endofunctors $C$, $\Sigma$, $\Omega$, $J$ and $R$ preserve the class $\mathrm{q}(Fib)$, and hence induce fibrewise continuous endofunctors $C^B_B$, $\Sigma^B_B$, $\Omega^B_B$, $J^B_B$ and $R^B_B$, which preserve the class $\mathrm{q}(Fib^{\mathrm{B}}_{\mathrm{B}})$.*

If $p : E \to B$ is a Hurewicz fibration, then it is known that $j_B : J^B_B E \to \Omega^B_B \Sigma^B_B E$ is a fibrewise pointed homotopy equivalence (see Theorem 2.3 [15]). As an application of Corollary 2.8, we obtain the following result.

**Theorem 2.9** *Let $(p : E \to B)$ be an object of $Fib^{\mathrm{B}}_{\mathrm{B}}$. If the fibre $F$ has the homotopy type of a CW-complex, then $j_B : J^B_B E \to \Omega^B_B \Sigma^B_B E$ is a fibrewise pointed homotopy equivalence.*

To define James Hopf invariants on a fibrewise space, we use the James filtration $J_k(X)$ for a space $X$ which is natural with respect to $X$, by following Sakai [15]. Then the above James filtration $J_k$ gives a continuous functor and the combinatorial extension of a fibrewise shrinking map $J_k : (J_k X, X) \to (\wedge^k X, *)$ gives a natural transformation

$$J_k : JX \longrightarrow J(\wedge^k X).$$

Thus we obtain the following result from Theorems 2.3 and 2.4.

**Corollary 2.10** *The natural transformation $J_k : J \to J(\wedge^k(\ ))$ induces a natural transformation $J_k : J^B_B \to J^B_B(\wedge^k_B(\ ))$.*

Then by Theorem 2.9, we can give a definition of a James Hopf invariant on $Fib^{\mathrm{B}}_{\mathrm{B}}$.

**Definition 2.11** *Let $[\ ,\ ]^B_B$ denote a fibrewise pointed homotopy set. Then we define a James Hopf invariant over $B$*

$$h^B_k : [\Sigma^B_B K, \Sigma^B_B E]^B_B \to [\Sigma^B_B K, \Sigma^B_B(\wedge^k_B E)]^B_B$$

*as the following composite:*

$$\begin{aligned}[\Sigma_B^B K, \Sigma_B^B E]_B^B &\xrightarrow{ad} [K, \Omega_B^B \Sigma_B^B E]_B^B \\ &\xrightarrow{(j_B^{-1})_*} [K, J_B^B E]_B^B \xrightarrow{(J_k)_*} [K, J_B^B(\wedge_B^k E)]_B^B \\ &\xrightarrow{(j_B)_*} [K, \Omega_B^B \Sigma_B^B(\wedge_B^k E)]_B^B \xrightarrow{ad^{-1}} [\Sigma_B^B K, \Sigma_B^B(\wedge_B^k E)]_B^B,\end{aligned}$$

*where ad denotes the adjoint map.*

## 3 Topology of $\coprod_{b\in B} \Phi(E_b)$

Let $\Phi : Top \to Top$ be a continuous functor and $(p{:}E{\to}B)$ an object of $Fib$. Since $\phi_\alpha : \Delta_\alpha \times F \to \iota_\alpha^* E$ is a homeomorphism for each simplex $\Delta_\alpha$ and $\Phi$ a continuous functor, we obtain a natural homeomorphism $\Phi(2)(\phi_\alpha) : \Delta_\alpha \times \Phi(F) \to \coprod_{b\in\Delta_\alpha} \Phi(E_b)$ for each simplex $\Delta_\alpha$ given by the formula

$$\Phi(2)(\phi_\alpha)(b, y) = \Phi(\hat{\phi}_\alpha(b))(y),$$

where $\hat{\phi}_\alpha(b) : F \to E_b$ is defined by $\hat{\phi}_\alpha(b)(x) = \phi_\alpha(b, x)$. Using it, we topologise $(\coprod_{b\in B} \Phi(E_b), \mathcal{O})$ as follows:

**Definition 3.1** *We topologise $\coprod_{b\in\Delta_\alpha} \Phi(E_b)$ as follows. A subset $F$ of $\coprod_{b\in\Delta_\alpha} \Phi(E_b)$ is said to be closed in $(\coprod_{b\in\Delta_\alpha} \Phi(E_b), \mathcal{O}_\alpha)$ if the inverse image $\Phi(2)(\phi_\alpha)^{-1}(F)$ is closed in $(\Delta_\alpha \times \Phi(F), \mathcal{O}'_\alpha)$. For the total space $\coprod_{b\in B} \Phi(E_b)$, we give the weak topology by the filtration $\{\coprod_{b\in\Delta_\alpha} \Phi(E_b), \alpha \in \Lambda\}$: let $A$ be a subset of $\coprod_{b\in B} \Phi(E_b)$. Then $A$ is said to be closed in $(\coprod_{b\in B} \Phi(E_b), \mathcal{O})$ if $A \cap \coprod_{b\in\Delta_\alpha} \Phi(E_b)$ is closed in $(\coprod_{b\in\Delta_\alpha} \Phi(E_b), \mathcal{O}_\alpha)$ for each $\alpha \in \Lambda$.*

Then we have the following Proposition:

**Proposition 3.2** *$\mathcal{O}|_{\coprod_{b\in\Delta_\alpha} \Phi(E_b)} = \mathcal{O}_\alpha$ on $\coprod_{b\in\Delta_\alpha} \Phi(E_b)$ for each $\alpha \in \Lambda$.*

*Proof:* Assume that $A_0$ is closed in $(\coprod_{b\in\Delta_\alpha} \Phi(E_b), \mathcal{O}|_{\coprod_{b\in\Delta_\alpha} \Phi(E_b)})$. Then there exists a closed set $A \subseteq (\coprod_{b\in B} \Phi(E_b), \mathcal{O})$ such that $A \cap \coprod_{b\in\Delta_\alpha} \Phi(E_b) = A_0$. By the definition of the topology of $(\coprod_{b\in B} \Phi(E_b), \mathcal{O})$, $A \cap \coprod_{b\in\Delta_{\alpha'}} \Phi(E_b)$ is closed in $(\coprod_{b\in\Delta_{\alpha'}} \Phi(E_b), \mathcal{O}_{\alpha'})$ for each $\alpha' \in \Lambda$. Therefore $A_0 = A \cap \coprod_{b\in\Delta_\alpha} \Phi(E_b)$ is closed in $(\coprod_{b\in\Delta_\alpha} \Phi(E_b), \mathcal{O}_\alpha)$.

Conversely assume that $A_0$ is closed in $(\coprod_{b\in\Delta_\alpha}\Phi(E_b),\mathcal{O}_\alpha)$. We prove that $A_0\cap\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b)$ is closed in $(\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b),\mathcal{O}_{\alpha'})$ for any $\alpha'\neq\alpha$ by dicussing the following two cases:

(Case when $\Delta_\alpha\cap\Delta_{\alpha'}=\emptyset$) Obviously we have $A_0\cap\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b)=\emptyset$ which is closed in $(\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b),\mathcal{O}_{\alpha'})$.

(Case when $\Delta_\alpha\cap\Delta_{\alpha'}\neq\emptyset$) Set $\Delta_{\alpha_0}=\Delta_\alpha\cap\Delta_{\alpha'}$. Then we obtain the following commutative diagram.

$$\begin{array}{ccccc}
\Delta_{\alpha'}\times F & \hookleftarrow & \Delta_{\alpha_0}\times F & \hookrightarrow & \Delta_\alpha\times F \\
\approx\big\downarrow\phi_{\alpha'} & & \approx\big\downarrow\phi_{\alpha_0} & & \approx\big\downarrow\phi_\alpha \\
\coprod_{b\in\Delta_{\alpha'}} E_b & \hookleftarrow & \coprod_{b\in\Delta_{\alpha_0}} E_b & \hookrightarrow & \coprod_{b\in\Delta_\alpha} E_b
\end{array}$$

Since $\Phi$ is a continuous functor, we obtain the following commutative diagram.

$$\begin{array}{ccccc}
\Delta_{\alpha'}\times\Phi(F) & \hookleftarrow & \Delta_{\alpha_0}\times\Phi(F) & \hookrightarrow & \Delta_\alpha\times\Phi(F) \\
\approx\big\downarrow\phi_{\alpha'} & & \approx\big\downarrow\phi_{\alpha_0} & & \approx\big\downarrow\phi_\alpha \\
\coprod_{b\in\Delta_{\alpha'}} \Phi(E_b) & \hookleftarrow & \coprod_{b\in\Delta_{\alpha_0}} \Phi(E_b) & \hookrightarrow & \coprod_{b\in\Delta_\alpha} \Phi(E_b)
\end{array}$$

Since $\Delta_{\alpha_0}$ is a face of $\Delta_{\alpha'}$, $\Delta_{\alpha_0}\times\Phi(F)$ is closed in $\Delta_{\alpha'}\times\Phi(F)$. Therefore $A_0\cap\coprod_{b\in\Delta_{\alpha_0}}\Phi(E_b)=A_0\cap\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b)$ is closed in $(\coprod_{b\in\Delta_{\alpha'}}\Phi(E_b),\mathcal{O}_{\alpha'})$. □

## 4 Proof of Theorem 2.3 and Theorem 2.7

We define $\Phi(2)(E\to B)$ and $\Phi(2)(f,g)$ by the conditions (1) and (2) of Theorem 2.3. Since $\alpha\circ\Phi(2)(f,g)$ is continuous on each simplex, it is continuous on B by Proposition 3.2. By definition, the conditions (1), (2) and (3) are clearly satisfied.
Next we prove Theorem 2.7. It is sufficient to show that $\coprod_{b\in B}\Phi(E_b)\to B$ is a quasi-fibration. Let $B_n$ be a subspace of $B$ of dimension up to $n$. Then $B$ is the inductive limit of a sequence of subspaces $B_0\subset B_1\subset\cdots\subset B$,

satisfying the first separation axiom (points are closed). Due to the result of Theorem 2.2 of Dold and Thom [3], it is sufficient to that $\coprod_{b\in B}\Phi(E_b)|_{B_n}\to B_n$ is a quasi-fibration for each $n \geq 0$. We show by induction on $n$. If $n = 0$, then $\coprod_{b\in B}\Phi(E_b)\to B_0$ is trivial. Assume that $\coprod_{b\in B}\Phi(E_b)|_{B_n}\to B_n$ is a quasi-fibration. Set $\check{\Delta}_\alpha^{n+1} = \Delta_\alpha^{n+1} - \{b_\alpha^{n+1}\}$, where $b_\alpha^{n+1}$ is a centroid of $\Delta_\alpha^{n+1}$. Since $\check{\Delta}_\alpha^{n+1}$ is a deformation retract of $\partial\Delta_\alpha^{n+1}$, we see that $B_n\cup\check{\Delta}_\alpha^{n+1}\simeq B_n$ and $\coprod_{b\in B}\Phi(E_b)|_{B_n\cup\check{\Delta}_\alpha^{n+1}}\simeq\coprod_{b\in B}\Phi(E_b)|_{B_n}$. By the induction hypothesis and this, $\coprod_{b\in B}\Phi(E_b)|_{B_n\cup\check{\Delta}_\alpha^{n+1}}\to B_n\cup\check{\Delta}_\alpha^{n+1}$ is a quasi-fibration. Similarly we see that a projection $\coprod_{b\in B}\Phi(E_b)|_{B_n\cup\bigcup_{\alpha\in\Lambda}\check{\Delta}_\alpha^{n+1}}$ $\to B_n\cup\bigcup_{\alpha\in\Lambda}\check{\Delta}_\alpha^{n+1}$ is a quasi-fibration. Since $\coprod_{b\in B}\Phi(E_b)|_{\mathrm{Int}\Delta_\alpha^{n+1}}\to\mathrm{Int}\Delta_\alpha^{n+1}$ is a quasi-fibration for each $\alpha\in\Lambda$, $\coprod_{b\in B}\Phi(E_b)|_{\bigcup_{\alpha\in\Lambda}\mathrm{Int}\Delta_\alpha^{n+1}}\to\bigcup_{\alpha\in\Lambda}\mathrm{Int}\Delta_\alpha^{n+1}$ is a quasi-fibration. Making use of the result of Theorem 2.15 of [3], we see that $\coprod_{b\in B}\Phi(E_b)|_{B_n\cup\bigcup_{\alpha\in\Lambda}\Delta_\alpha^{n+1}}$ $\to B_n\cup\bigcup_{\alpha\in\Lambda}\Delta_\alpha^{n+1} = B_{n+1}$ is a quasi-fibration.

## 5 Proof of Theorem 2.4 and Theorem 2.9

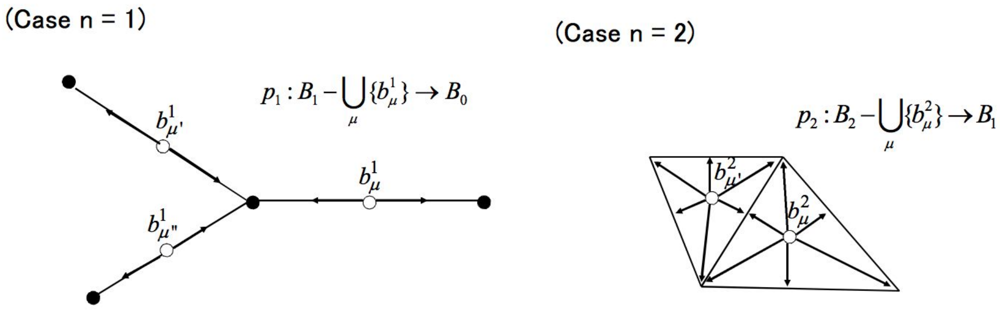
We show Theorem 2.4. We define $\theta(2)(p{:}E\to B)$ by the conditions (1) and (2) of Theorem 2.4. Since $\alpha\circ\theta(2)(f,g)$ is continuous on each simplex, it is continuous on B by Proposition 3.2.

To show Theorem 2.9, we require the following proposition.

**Proposition 5.1** *For each $n \geq 0$, there exists an open covering $\{U_\lambda^n|\ \lambda\in\Lambda_n = \cup_{i=0}^n\Gamma_i\}$ of $B_n$ which satisfies the following two properties.*

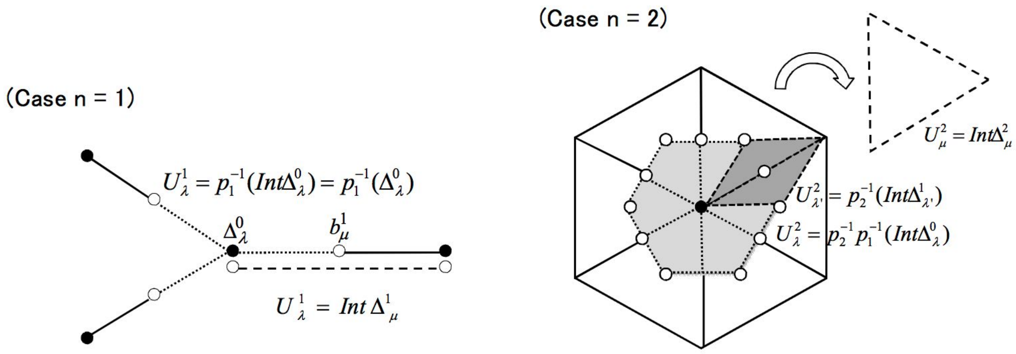
*(1) For each $\lambda\in\Gamma_k\subset\Lambda_n$, $Int\Delta_\lambda^k\subset U_\lambda^n$ $(0\leq k\leq n)$,*
*(2) $Int\Delta_\lambda^k$ is a deformation retract of $U_\lambda^n$.*

*Proof:* (Case when $\dim B < \infty$) Let $p_n : B_n - \cup_\lambda\{b_\lambda^n\}\to B_{n-1}$ be a retraction and $H_n : (B_n - \cup_\lambda\{b_\lambda^n\})\times[0,1]\to B_n$ a homotopy between $id$ and $p_n$.




Then we may define the open set $U^n_\lambda$ for $\lambda \in \Gamma_k \subset \Lambda_n$ as follows:

$$U^n_\lambda := \begin{cases} Int\Delta^k_\lambda & \text{if } k = n, \\ p_n^{-1} p_{n-1}^{-1} \cdots p_{k+1}^{-1}(Int\Delta^k_\lambda) & \text{if } k < n. \end{cases}$$




By the construction, it is obvious that $U^n_\lambda$ satisfies the property (1). To show that $U^n_\lambda$ satisfies the property (2), we define a homotopy $H^n_\lambda : U^n_\lambda \times [0,1] \to U^n_\lambda$ to satisfy

$$\begin{aligned} &\text{(i) } H^n_\lambda(x,t) &&= x \quad \text{for} \quad 0 \le t \le \frac{1}{2^{n-k}} \quad \text{and} \\ &\text{(ii) } H^n_\lambda(x,1) &&\in Int\Delta^k_\lambda \quad \text{if} \quad \lambda \in \Gamma_k. \end{aligned}$$

We define $H^n_\lambda$ by induction on $n$.
($n = 0$)

$$H^0_\lambda(x,t) = x \quad \text{if} \quad 0 \le t \le 1.$$

($n > 0$) By induction hypothesis, we may assume that $H^m_\lambda$ is defined for $m < n$. Then we define

Case $(k = n)$

$$H^n_\lambda(x,t) = x \quad \text{if} \quad 0 \leq t \leq 1$$

Case $(k = 0,\ 1,\ \cdots,\ n-1)$

$$H^n_\lambda(x,t) = \begin{cases} x & \left(0 \leq t \leq \dfrac{1}{2^{n-k}}\right) \\ H_n(x, 2^{n-k}\left(t - \frac{1}{2^{n-k}}\right)) & \left(\dfrac{1}{2^{n-k}} \leq t \leq \dfrac{1}{2^{n-k-1}}\right) \\ H^{n-1}_\lambda(p_n(x), t) & \left(\dfrac{1}{2^{n-k-1}} \leq t \leq 1\right) \end{cases}$$

(well-definedness) When $t = \frac{1}{2^{n-k}}$, we have $H_n(x, 2^{n-k}\left(t - \frac{1}{2^{n-k}}\right)) = H_n(x, 0)$ $= x$. When $t = \frac{1}{2^{n-k-1}}$, we have $H_n(x, 2^{n-k}\left(t - \frac{1}{2^{n-k}}\right)) = H_n(x, 1) = p_n(x)$. By induction hypothesis, we have $H^{n-1}_\lambda(p_n(x), t) = H^{n-1}_\lambda(p_n(x), \frac{1}{2^{(n-1)-k}}) = p_n(x)$. Therefore, $H^n_\lambda$ is well-defined.

From the construction, $H^n_\lambda$ is continuous. Since $H^n_\lambda(x, 0) = x = \mathrm{id}_{U^n_\lambda}(x)$ and $H^n_\lambda(x, 1) = H^{n-1}_\lambda(p_n(x), 1) = p_{k+1} \circ \cdots \circ p_n(x) \in Int\Delta^k_\lambda$ by induction hypothesis, $H^n_\lambda$ is a homotopy between the identity $\mathrm{id}_{U^n_\lambda}$ and the retraction $p_{k+1} \circ \cdots \circ p_n$. Thus, $Int\Delta^k_\lambda$ is a deformation retract of $U^n_\lambda$.
(Case when $\dim B = \infty$) Let $U_\lambda = \cup_{n \geq k} U^n_\lambda$ $(\lambda \in \cup^k_{i=0}\Gamma_i)$. Since $U_\lambda \cap B_n = U^n_\lambda$ is open in $B_n$, $U_\lambda$ is open in $B = \lim\limits_{n \to \infty} B_n$. Then we define a homotopy $H_\lambda : U_\lambda \times [0,1] \to B$ as $H_\lambda|_{U^n_\lambda \times [0,1]} = H^n_\lambda$. Obviously, $H_\lambda$ is well-defined by the construction. Since $H^n_\lambda$ is continuous, $H_\lambda$ is continuous. □

Since $B$ is a CW-complex, $B$ is paracompact. So, $\{U^n_\lambda\}$ is numerable. Moreover $\{U^n_\lambda\}$ is contractible by Proposition 5.1. By the construction of $\{U^n_\lambda\}$, $\coprod_{b \in B} \Phi(E_b) \to B$ is trivial on $U^n_\lambda$ for each $\lambda \in \Lambda_n,\ n \geq 0$. Due to the result of Theorem 6.4 of Dold [2], the projection $p : \coprod_{b \in B} \Phi(E_b) \to B$ has the property WCHP, in other words, for each homotopy $\bar{H} : X \times [0,1] \to B$, $p$ has the ordinary CHP for the following:

$$\hat{H} : X \times [-1,1] \to B, \hat{H}(x \times [-1,0]) = \bar{H}(x,0), \hat{H}|X \times [0,1] = \bar{H}.$$

By the assumption of the fibre, $J_b E$ and $\Omega_b \Sigma_b E$ are homotopy equivalence for each $b \in B$. Therefore $J^B_B E$ and $\Omega^B_B \Sigma^B_B E$ are fibre homotopy equivalence due to the result of Theorem 6.3 of Dold [2].

From the cell-wise triviality condition of $Fib^{\mathrm{B}}_{\mathrm{B}}$, $\coprod_{b \in B} \Phi(E_b) \to B$ is trivial (in the sense of $Fib^{\mathrm{B}}_{\mathrm{B}}$) on $U^n_\lambda$ for each $\lambda \in \Lambda_n,\ n \geq 0$. So $J^B_B E$ and $\Omega^B_B \Sigma^B_B E$ are fibrewise pointed homotopy equivalence due to the result of Theorem 3.9 of [4]. This completes the proof of Theorem 2.9.

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