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CERTAIN MISSING TERMS IN AN UNSTABLE ADAMS SPECTRAL SEQUENCE

By

Norio IWASE

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§ 0. Introduction

The purpose of this paper is to give some missing terms in the associated spectral sequence with a continuous completion. As an application, we get a partial answer of the Miller conjecture.

For two topological groups G and H, we denote their classifying spaces by BG and BH. It is well known that any continuous homomorphism from G to H induces a continuous mapping from BG to BH. This turns into a natural mapping from $\mathrm{Hom}(G, H)$ to $\mathrm{Map}_*(BG, BH)$, where $\mathrm{Hom}(G, H)$ is the space of all continuous homomorphisms and $\mathrm{Map}_*(BG, BH)$ is the space of all base point preserving mappings. When does the above mapping induce a bijection between based homotopy sets? We can say that this is valid in a case where G is an elementary Abelian p-group and H is a certain Lie group. To obtain this, we need the following

THEOREM 0.1. *Let Y be a space of finite type whose mod p cohomology is a free commutative algebra. Then the evaluation by mod p cohomology H^**

$$\pi_0(\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, Y_p^\wedge), *) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(H^*(Y), H^*(B(\mathbf{Z}/p\mathbf{Z}))^n)$$

is a bijection of sets with base points for any $n \geqq 0$, where $\mathbf{A}$ denotes the category of commutative associative algebras over $\mathbf{A}(p)$ by its left action.

REMARK 0.2. *In the context of Theorem 0.1, if we further assume that Y is nilpotent, then we can replace $Y_p^\wedge$ with Y. This fact is obtained from the arithmetic square theorem [4] with the fact that Y is $\mathbf{Z}$-local and $\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, X_{\mathbf{Q}})$ is weakly contractible for any $\mathbf{Q}$-local space $X_{\mathbf{Q}}$, quite similarly to the proof of Theorem 1.5 in [9].*

The hypothesis of Theorem 0.1 holds for the classifying space of any finitely generated Abelian group if p is odd, but does not hold for some finite non abelian groups. In fact, the conclusion does not hold for a group with a non-central element of order p. But the mod p cohomology ring has non-trivial relations provided that the group is nilpotent.

EXAMPLE 0.3. *There is a trivial counter example for Theorem* 0.1 *with* $p=2$: *Let* H *be the dihedral group* D_n, $n=2^r\geqq 4$. *Then* H *is finite nilpotent. To show that the mod* 2 *cohomology of* D_n *is not a free commutative algebra, we use the cell-decomposition of* BD_n *(due to M. Kamata and H. Minami* [5]) *together with the Serre spectral sequence and obtain that the Bockstein operation is injective on* $H^1(BD_n)$ *and* $H^*(BD_n)=\mathbf{Z}/2\mathbf{Z}[w]\otimes\mathbf{Z}/2\mathbf{Z}[u, v]/(u^2+uv)$, $\dim w=2$, $\dim u=\dim v=1$. *And so* $\pi_0(\mathrm{Map}_*(B\mathbf{Z}/2\mathbf{Z}, BH), *)=\mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, H)\to \mathrm{Hom}_{\mathbf{A}}(H^*(BH), H^*(B\mathbf{Z}/2\mathbf{Z}))$ *is not bijective.*

EXAMPLE 0.4. *There is also a trivial counter example for Theorem* 0.1 *with p odd: Let* H *be the dihedral group* D_p. *Then* H *is a non-nilpotent group with free commutative cohomology algebra and* $\pi_0(\mathrm{Map}_*(B\mathbf{Z}/p\mathbf{Z}, BH), *)=\mathrm{Hom}(\mathbf{Z}/p\mathbf{Z}, H)\to\mathrm{Hom}_{\mathbf{A}}(H^*(BH), H^*(B\mathbf{Z}/p\mathbf{Z}))$ *is not bijective, either. In addition,* $H^*(BH)=\mathbf{Z}/p\mathbf{Z}[w]\otimes\mathbf{Z}/p\mathbf{Z}[v]/(v^2)$, $\dim v=3$, $\dim w=4$.

COROLLARY 0.5. *Let* p *be any prime and* G *a connected associative mod* p *H-space with a p-torsion free cohomology ring. Then there is a bijection*

$$\pi_0(\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, BG), *)\longrightarrow \mathrm{Hom}_{\mathbf{A}}(H^*(BG), H^*(B(\mathbf{Z}/p\mathbf{Z})^n)).$$

Our examples for Corollary 0.5 is as follows:

EXAMPLE 4.7. *Let* p *be any prime. Then there are bijections*:
1) $\pi_0(\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, BSp(1)), *)\cong\pi_0(\mathrm{Hom}((\mathbf{Z}/p\mathbf{Z})^n, Sp(1)), *)$,
2) $\pi_0(\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z}), BU(n)), *)\cong\pi_0(\mathrm{Hom}((\mathbf{Z}/p\mathbf{Z}), U(n)), *)$.

In the case where the groups G and H are discrete, it is well-known that the connected components of the above two sets are in one to one correspondence with each other, and moreover the above mapping is a weak homotopy equivalence. However, if G is not discrete, then we can not expect it, since there exist phantom mappings in general. In the other case where $G=(\mathbf{Z}/2\mathbf{Z})^n$, $n\geqq 0$ and $H=S^3=Sp(1)$, H. Miller proved the same result as the discrete case in [8] by making use of the following

THEOREM (H. Miller [8, 9]). *Let* G *be* $(\mathbf{Z}/p\mathbf{Z})^n$, $n>0$, p *a prime and* Y *a* *very nice space, i.e. the cohomology ring admits a p-simple system of generators whose vector space M span is closed under the action of modulo p Steenrod algebra* $\mathbf{A}(p)$. *Then the evaluation mapping of mod p cohomology induces a bijection*

$$\pi_0(\mathrm{Map}_*(BG, Y), *) \longrightarrow \mathrm{Hom}_{\mathbf{A}(p)}(M, \bar{H}^*(BG; \mathbf{Z}/p\mathbf{Z})).$$

This theorem is obtained by making use of the Massey-Peterson tower. He has conjectured in [8] that this is still true if 0 is any simply connected space whose mod p cohomology $H^*(Y)$ is of finite type. But it is very delicate if we do not use the Massey-Peterson towers, because of the fact that we have to treat based sets. Theorem 0.1 is thus an answer to this conjecture.

To prove Theorem 0.1, we need a somewhat new convergence theorem of an unstable Adams spectral sequence which was introduced by A. K. Bousfield and D. M. Kan [2]. But unfortunately, the convergence lemma of this spectral sequence does not include the one for the homotopy sets, because of the facts that the homotopy sets form only a set with base point and that the E_r-terms $E_r^{s,t}$ are considered only for $t \geqq s$.

Bousfield and Kan introduced their unstable Adams spectral sequence in terms of simplicial objects. But we construct it by using a symmetric product [3].

Let X and Y be any CW-complexes, R a prime field of characteristic p, H_* ordinary homology theory with a coefficient ring R and $\bar{R}$ the functor taking the mod p infinite symmetric product [3]. Then one can construct an unstable Adams resolution for any CW-complex Y as follows;

$$\begin{array}{ccccccccccc}
\bar{R}Y = & Y_0 & \longleftarrow & Y_1 & \longleftarrow \cdots \longleftarrow & Y_n & \longleftarrow & Y_{n+1} & \longleftarrow & \cdots \longleftarrow & Y_p^{\wedge} \\
\uparrow & \| & & \uparrow & & \uparrow & & \uparrow & & & \\
Y & F_0 & & F_1 & & F_n & & F_{n+1} & & &
\end{array}$$

where $F_n \to Y_n \to Y_{n-1}$ is a fibration with fibre F_n having a weak homotopy type of an Eilenberg-MacLane complex whose homotopy groups are R-modules and $Y\hat{p}$ is the Bousfield-Kan R-completion [2] of Y if Y is a good space such as $H_*(Y; R) = H_*(Y_p^{\wedge})$. Then there is an unstable Adams spectral sequence $\{E_r^{s,t}, s \geqq 0, t-s \geqq 0\}$ converging to $\pi_0(\mathrm{Map}_*(\Sigma^{t-s} X, Y_p^{\wedge}), *)$ under some suitable conditions, $t-s>0$, and its E_2-terms are described as follows,

$$E_2^{s,t} = \mathrm{Ext}_{CA}^s(H_*(\Sigma^t X), H_*(Y))$$

for $s \geqq 0$, $t-s \geqq 0$ where **CA** denotes the category of cocomutative coassociative coalgebra *with unit* over the modulo p Steenrod algebra $A(p)$ by its right action, and Ext^s_{CA} is the derived functor of Hom_{CA} by Bousfield, Kan, Dror, Dwyer and Miller (see Miller [9]).

REMARK. *As Miller remarked in* [9], *the right hand side of the above equality is definable even for negative total dimension,* $t-s<0$, *while the left hand side is defined only for* $t-s\geqq 0$.

In this paper, we define certain terms $\bar{E}_2^{s,s-1}$ for $2\leqq s$ and extend the homotopy exact sequence for the E_2-terms over these groups. The definable range of E_r-terms for negative total dimension, however, is getting smaller and smller, and will eventually vanishing entirely for $r=\infty$. These vanishing terms $\bar{E}_r^{s,t}$ for $s>t$ of the spectral sequence are called the ghost terms, when they exist. Actually at E_r-stage, we can not define naturally ghost terms $\bar{E}_r^{s,t}$ for $t<r-1$, since we need the E_r-terms $\bar{E}_r^{s-r,t-r+1}$ to define $\bar{E}_r^{s,t}$. The following figure represents the E_r-terms.

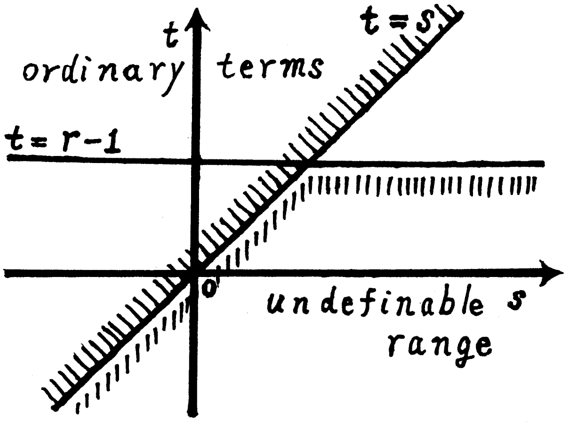

Figure 1

Under the situation, we can describe our convergence theorem which holds even for the based homotopy sets, namely,

THEOREM 3.6. *Let X be a CW-complex and Y a space with a non-degenerate base point. Then the following three statements hold for the evaluation of mod p homology*

$$H\colon \pi_i(\mathrm{Map}_*(X, Y_p^\wedge), *) \longrightarrow \mathrm{Hom}_{CA}(H_*(\Sigma^i X), H_*(Y)).$$

a) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H_*(\Sigma^t X), H_*(Y))=0$ *for* $t-i=s>0$, *then* H *is injective.*
b) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H_*(\Sigma^t X), H_*(Y))=0$ *for* $t-i+1=s>0$, *then* H *is surjective.*
c) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H^*(\Sigma^t X), H_*(Y))=0$ *for* $t-i=s>0$ *and for* $t-i+1=s>0$, *then* H *is bijective.*

Note that both the source and target sets of the bijection H are not naturally groups, if $i=0$.

This paper is organized as follows. In the sections 1 and 2, we recall the notions of cosimplicial spaces and algebra functors and state the key lemma, Lemma 2.4, of the naturality theorem, Theorem 3.3, for the ghost terms. We prove Theorem 3.6 in the section 3 by making use of Theorem 3.3 and the description of the ghost terms to E_2-terms which is given by Proposition 3.5. In the section 4, we introduce a certain algebra functor over a category and prove Theorem 0.1.

Throughout this paper, we work in the category of compactly generated space with a non-degenerate base point. In the remainder of this paper, we often abbreviate $H_*(\ ;\mathbf{Z}/p\mathbf{Z})$ and $H^*(\ ;\mathbf{Z}/p\mathbf{Z})$ by $H_*(\ )$ and $H^*(\ )$, respectively.

The author expresses his gratitude to late Professor S. Oka for his valuable suggestions.

On January 22-nd, 1985, I gave a talk on the result of this paper under the title "Maps between classifying spaces" at the "Symposium on infinite dimensional CW-complexes" which was held in Osaka, Japan.

Then I met J. Lannes on July 27-th, 1986 in Arcata, California who has shown me the preprint [6], whom I thank for his kindness. This result [6] is described as follows: For any given nilpotent space Y of finite type, the free homotopy set $\pi_0(\mathrm{Map}(B(\mathbf{Z}/p\mathbf{Z})^n, Y), *)$ is bijective naturally with the set $\mathrm{Hom}_{\mathbf{A}}(H^*(Y), H^*(B(\mathbf{Z}/p\mathbf{Z})^n))$. His proof for the free homotopy case is entirely different from the one for the based homotopy case in this paper.

The original forms of Miller's theorem and conjecture are those for based mapping spaces. In this sense, our result might be better answer.

§ 1. Preliminaries

Let us recall the notion of a cosimplicial space and prepare the necessary notions.

DEFINITION 1.1. *The triple* $X^\cdot=(\{X^n, n\geqq 0\}, \{d^j: X^{n+1}\to X^n, 0\leqq i\leqq n\}, \{s^j: X^{n+1}\to X^n, 0\leqq j\leqq n\})$ *is a cosimplicial space if* X^n *is a space and the* *following three identities hold:*

1) $d^j d^i = d^i d^{j-1},\ i<j,$
2) $s^j s^i = s^{i-1} s^j,\ i>j,$
3) $s^j d^i = \begin{cases} d^i s^{-1j},\ i<j, \\ \text{identity},\ i=j \text{ or } i=j+1, \\ d^{i-1} s^j,\ i>j+1. \end{cases}$

These identities are called cosimplicial identities.

DEFINITION 1.2. *The standard cosimplicial space* $\Delta^{\cdot}$ *is defined by*

$$\Delta^n = \{(t_0, \dots, t_n) \in [0, 1]^n \mid \textstyle\sum_{j=1}^n t_j = 1\},$$

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}),$$

$$s^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}),$$

and we denote by $\Delta^{[s]\cdot}$ *the s-skeleton of* $\Delta^{\cdot}$.

Let $X^{\cdot}$ be a cosimplicial space. We can construct its total spaces $tot(X^{\cdot})$ and $tot_s(X.)$, $s \geqq 0$:

(1.3) $$tot(X^{\cdot}) = \{\{f_n\colon \Delta^n \to X^n\} \mid d^i f^n = f^{n+1} d^i,\ s^i f^n = f^{n-1} s^i\} \subseteq \textstyle\prod_{n>-1} \mathrm{Map}(\Delta^n, X^n),$$

(1.4) $$tot_s(X^{\cdot}) = \{\{f^n\colon \Delta^{[s]n} \to X^n\} \mid d^i f^n = f^{n+1} d^i,\ s^i f^n = f^{n-1} s^i\} \subseteq \textstyle\prod_{n>-1} \mathrm{Map}(\Delta^{[s]n}, X^n),$$

with the sequences of the constant mappings as the base points.

Let us recall in this section two more notions introduced by Bousfield and Kan [2] which will be used in later sections.

DEFINITION 1.5. *The n-th matching space* $M^n X^{\cdot}$ *of a cosimplicial space* $X^{\cdot}$ *is the following subspace of* $n+1$*-fold product* $(X^n)^{n+1}$ *of* X^n;

$$\{(a_0, \dots, a_n) \in (X^n)^{n+1} \mid s^j a_i = s^{i-1} a_j,\ i>j\}.$$

DEFINITION 1.6. *The normal subspace* $NX^{\cdot}$ *of a cosimplicial space* $X^{\cdot}$ *is the following cosimplicial closed subspace*

$$NX^n = S^{-1}(*, \dots, *)$$

where $S\colon X^n \to M^{n-1} X^{\cdot}$ *is a mapping defined by* $S(a) = (s^0(a), s^1(a), \dots, s^{n-1}(a))$.

§ 2. Algebra functors and group-like cosimplicial spaces

The terminology "algebra functor" is used by J. F. Adams [1] which is equivalent to "triple" or "monad" used by categorists. In this paper, we adopt Adams' terminology.

Let $\underline{C}$ be a category. For an algebra functor $T: \underline{C} \to \underline{C}$. Bousfield and Kan [2] construct a new functor $T^{\cdot}: \underline{C} \to s^0\underline{C}$ where $s^0\underline{C}$ denotes the category of all cosimplicial objects in $\underline{C}$.

DEFINITION 2.1. *For any object X and any morphism f of $\underline{C}$, $(T^{\cdot}X)^n$ and $(T^{\cdot}f)^n$ denote the object $T^{n+1}X$ and the morphism $T^{n+1}f$ in $\underline{C}$ respectively where the functor T^{n+1} is the $n+1$-times composition of the functor T.*

The following is a well-known example due to [2] and [3].

EXAMPLE 2.2. *The functor $\bar{R} = AG(\ ; p)$ is an algebra functor;*

$$\bar{R}(X) = AG(X; p) = SP(X)/p \cdot SP(X),$$

where $SP(X)$ is the infinite symmetric product and $p \cdot SP(X)$ is the p-th power set of $SP(X)$. There is an associated multiplication μ: $\overline{RR}(X) \to \bar{R}(X)$ defined by

$$\mu(\textstyle\sum_i n_i(\sum_j m_{ij}x_{ij})) = \sum_{i,j} (n_i m_{ij})x_{ij},$$

and a unit η: $X \to \bar{R}(X)$ defined by

$$\eta(x) = 1x.$$

For any based space X, $\bar{R}X$ is a topological Abelian group, μ is a continuous homomorphism onto $\bar{R}X$ and η is a homeomorphism onto a closed subspace of $\bar{R}X$. Note that η is *not* a homomorphism even if X is a topological Abelian group Using this functor, we can construct topological p-completion $Y\hat{p}$ of the space Y with base point as in [2]. Inclusions induce the tower of fibrations $\{tot_s(\bar{R}^{\cdot}Y) \to tot_{s-1}(\bar{R}^{\cdot}Y)\}$. By the definition, $tot_s(\bar{R}^{\cdot}Y)$ is R-nilpotent and the tower is equivalent to the realization of the Bousfield and Kan tower $\{R_s(Y)\}$ in [2]. Hence $tot(\bar{R}^{\cdot}Y)$ is equivalent to the p-completion $Y\hat{_p}$.

DEFINITION 2.3. *A cosimplicial space A. is called (Abelian) group-like if it satisfies the following two conditions:*

1) *A^n is a topological (Abelian) group for all $n \geq 0$,*
2) *all the degeneracies s^j and faces d^i for $i > 0$ are continuous homomorphisms*

and d^0 preserves units,

Then the fibration $S: A^{n+1}\to M^nA^\cdot$ has a natural cross section (see J. P. May [7], page 69). We denote the section by C. Let $p_{s+1}: tot_{s+1}(A^\cdot)\to tot_s(A^\cdot)$ be the restriction and observe that $(p_{s+1})^{-1}(*)\cong\{h: \Delta^t\to A^{s+1}\,|\,h(\partial\Delta^t)=\{0\},\ s^jh=0,\ j\geqq 0\}$. We prove the following

LEMMA 2.4. *Let $A^\cdot$ be an Abelian group-like cosimplicial space. Then the natural projection*

$$p_{s+1}: tot_{s+1}(A^\cdot)\longrightarrow tot_s(A^\cdot)$$

is a principal fibration with the fibre $\Omega^{s+1}NA^{s+1}$ and is induced from a mapping

$$\phi_{s+1}: tot_s(A^\cdot)\longrightarrow \Omega^sNA^{s+1}\ for\ s\geqq 0,$$

where $\Omega^tNA^{s+1}=\{f: \Delta^t\to A^{s+1}\,|\,f(\partial\Delta^t)=\{0\},\ s^jf=0,\ j\geqq 0\}$. Moreover, we have

$$\phi_{s+1}j_s\simeq\textstyle\sum_{j=1}^s(-1)^jd^j_\#=\Omega^s\sum_{j+1}^s(-1)^jd^j,$$

$$\phi_{s+1}\mu_{s+1}=m_s(\phi_{s+1}j_s\times\phi_{s+1}),$$

where j_s is the canonical inclusion and m_s is the group addition of Ω^tNA^{s+1} and '$\simeq$' means 'is homotopic to'.

PROOF. Since it is known that the fibration is principal (see Bousfield and Kan [2, Chap. 2 Lemma 2.6]), we are left to prove the latter part of the lemma. Let $\phi_{s+1}(f)=\phi'_{s+1}(f^{s+1})$ and $\phi'_{s+1}(f^{s+1})=f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0))-C_\#S_\#(f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0)))$, for $f=\{f^n\}_{n\geqq 0}$.

This is well-defined since the fibration S and the cross section C are homomorphisms. By the definition of ϕ'_{s+1}, $\phi_{s+1}\mu_s(h,f)=m_s(\phi_{s+1}j_s(h),\phi_{s+1}(f))$. On the other hand, f can be lifted to $tot_{s+1}(A^\cdot)$ if and only if $\phi_{s+1}(f)$ is null-homotopic in Ω^sNA^{s+1}. Actually, assume that f has a lift $\bar f$, then $\phi_{s+1}(f)=\phi'_{s+1}(f^{s+1})$ and $\bar f^{s+1}$ can be regarded as a null-homotopy of f^{s+1}. Conversely assume that $\phi_{s+1}(f)$ is null-homotopic in Ω^sNA^{s+1}, then we can take an extension g_{s+1} of $\phi'_{s+1}(f^{s+1})$ over Δ^{s+1}, and put $\bar g^{s+1}=g_{s+1}+f^{s+1}(e_0)+C(\{f^ss^j-\mathrm{const}\,(s^jf^{s+1}(e_0))\}_j)$. Then $\bar f^{s+1}$ is an extension of f^{s+1} over Δ^{s+1} and a lifting of f^ss^j upon A^{s+1}. Therefore, we can construct a lifting $\bar f$ of f such that $\bar f^t=f^t$, $t\leqq s$; $\bar g^{s+1}$, $t=s+1$ by the cosimplicial identities. To complete the proof of the lemma, we consider more about the mapping $\phi_{s+1}j_s$: $\Omega^sNA^s\to\Omega^sNA^{s+1}$ which is given by the following formula:

$$\phi_{s+1} j_s(g) = \delta(g) \colon d^i x \longrightarrow d^i g(x).$$

Let F be the double adjoint of $\delta = \phi_{s+1} j_s$. Then the mappings F and Fd^i are given as

$$F \colon \partial \Delta^{s+1} \longrightarrow \mathrm{Map}_* (\Omega^s N A^s, A^{s+1}),$$

$$Fd^i \colon \Delta^s \longrightarrow \mathrm{Map}_* (\Omega^s N A^s, A^{s+1}).$$

Since Fd^i satisfies $Fd^i(d^j(x))(f) = d^i f d^j(x) = 0$, F is homotopic to $\sum_{j=0}^{s+1} F_j$ where $F_j \colon \partial \Delta^{s+1} \to A^{s+1}$ is given as follows

$$F_j(d^i(x)) = \begin{cases} F(d^j(x)), & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, the degree of d^j is $(-1)^j$ through the homeomorphism $k \colon \Delta^s / \partial \Delta^s \to \partial \Delta^{s+1}$. Hence we obtain a homotopy

$$\begin{aligned} k^{\#} F_j &\simeq (-1)^j ad(ad(\delta)) d^j_{\#} \\ &= (-1)^j ad(ad(d^j_{\#})) \\ &\simeq ad(ad((-1)^j d^j_{\#})). \end{aligned}$$

Therefore, again by taking double adjoint, we get the following homotopy

$$G_s \colon [0, 1] \longrightarrow \mathrm{Map}_* (\Omega^s N A^s, \mathrm{Map}_* (\partial \Delta^{s+1}, A^{s+1}))$$

such that $G_s(0) = \delta$, $k^{\#} G_s(1) = \sum_{j=0}^{s} (-1)^j d^j_{\#}$. This completes the proof of Lemma 2.4.

Making use of this lemma, we will extend the definition of the E_1-terms $E_1^{t,s}$ and E_2-terms $E_2^{t,s}$ of the unstable Adams spectral sequence to the range with negative total dimension $t - s < 0$, in section 3.

§ 3. A cosimplicial resolution

Let X and Y be CW-complexes and $A^{\cdot} = \bar{R}^{\cdot} Y$ a group-like cosimplicial space. Then $\mathrm{Map}_* (X, A^{\cdot})$ is also a cosimplicial space and we get naturally $tot_s(\mathrm{Map}_* (X, A^{\cdot})) \cong \mathrm{Map}_* (X, tot_s A^{\cdot})$ and hence $\pi_i(tot_s(\mathrm{Map}_* (X, A^{\cdot})), *) \cong \pi_i(\mathrm{Map}_* (X, tot_s A^{\cdot}), *)$. Bousfield and Kan introduced in [2] the homotopy spectral sequence $\{E_r, d_r\}$ with the following extended exact sequences converging to $\pi_i(\mathrm{Map}_* (X, Y_p^{\hat{}}), *)$ for $i \geqq 1$:

$$(3.1)\qquad D_r^{s-r,t-r+1} \longrightarrow E_r^{s,t} \longrightarrow D_r^{s,t} \longrightarrow D_r^{s-1,t-1},$$

for $t \geqq s \geqq 1$ and bijections

$$(3.2)\qquad E_r^{0,t} \longrightarrow D_r^{0,t},\ \text{for } t \geqq 0,$$

where we define inductively as $E_1^{s,t} = \pi_t \operatorname{Map}_* (X, NA^s) = N\pi_t \operatorname{Map}_* (X, A^s) = \pi_t \operatorname{Map}_* (X, A^s) \cap \operatorname{Ker} s^0 \cap \operatorname{Ker} s^1 \cap \cdots \cap \operatorname{Ker} s^{s-1}$ and $D_1^{s,t} = \pi_{t-s}(\operatorname{Map}_* (X, tot_s(A^{\cdot})))$, and as $E_{r+1}^{s,t} = \operatorname{Im}\{D_r^{s+r-1,t+r-1} \to D_r^{s,t}\}/\operatorname{Im}\{d_r\colon E_r^{s-r,t-r+1} \to E_r^{s,t}\}(=\operatorname{Ker}\{E_r^{s,t} \to E_r^{s+r,t+r-1}\}/\operatorname{Im}\{d_r\colon E_r^{s-r,t-r+1} \to E_r^{s,t}\}$ for $t > s$), $d_r\colon E_r^{s,t} \to D_r^{s,t} \to E_r^{s+r,t+r-1}$; $D_{r+1}^{s,t} = \operatorname{Im}\{D_r^{s+1,t+1} \to D_r^{s,t}\}$.

Note that the extended exactness at the set $D_r^{s,s}$ means that $D_r^{s,s}/\{\text{action of } E_r^{s,s}\}$ is mapped injectively onto the set of the elements annihilated by the mapping $D_r^{s,s} \to D_r^{s-1,s-1}$. Also note that the extended exactness at the Abelian group $E_r^{s,s}$ means that $\operatorname{Im}\{D_r^{s-r,t-r+1} \to E_r^{s,s}\}$ is mapped injectively onto the set of the elements annihilated by the mapping $E_r^{s,s} \to D_r^{s,s}$.

Moreover we can define the extra terms $\bar{E}_1^{s,t}$ and $\bar{E}_2^{s,t}$ for $t < s$ as follows:

$$\bar{E}_1^{s,t} = \pi_t \operatorname{Map}_* (X, A^s) \cap \operatorname{Ker} s^0 \cap \operatorname{Ker} s^1 \cap \cdots \cap \operatorname{Ker} s^{s-1}, \quad t \geqq 0,$$

$$\bar{E}_2^{s,t} = \operatorname{Ker}\{d_1\colon \bar{E}_1^{s,t} \to \bar{E}_1^{s+1,t}\}/\operatorname{Im}\{d_1\colon {}'\bar{E}_1^{s-1,t} \to \bar{E}_1^{s,t}\}, \quad t \geqq 1,$$

where ${}'\bar{E}_1^{s-1,t}$ means $\bar{E}_1^{s-t,1}$ when $t < s-1$ and $E_1^{t,t}$ when $t = s-1$, and $d_1 = \sum_{j=0}^{s+1} (-1)^j d_\#^j$.

We obtain the following basic fact on these terms.

Theorem 3.3. *The extra terms fit to the above extended exact sequences with the mappings for* $s \geqq 1$:

$$q_1\colon D_1^{s-1,s-1} \longrightarrow \bar{E}_1^{s,s-1},$$

$$q_2\colon D_2^{s-1,s-1} \longrightarrow \bar{E}_2^{s+1,s},$$

whose annihilating subsets are just the images of $D_1^{s,s} \to D_1^{s-1,s-1}$ *and* $D_2^{s,s} \to D_2^{s-1,s-1}$ *respectively.*

Proof. We need to show the existence of q_i and the exactness at $D_i^{s-1,s-1}$ for $i = 1$ and 2. Let $q_1 = \phi_{s\#}$. Then the exactness at $D_1^{s-1,s-1}$ is easily obtained by Lemma 2.4. Moreover, Lemma 2.4 gives the following commutative diagram

$$\begin{array}{ccccc} E_1^{s,s} & \xrightarrow{j_{s\#}} & D_1^{s,s} & \xrightarrow{p_{s\#}} & D_1^{s-1,s-1} \\ \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle \phi_{s+1\#}} & & \\ \bar{E}_1^{s+1,s} & = & \bar{E}_1^{s+1,s} & \xrightarrow{d_1} & \bar{E}_1^{s+2,s} \end{array}$$

We define q_2 by the formula $q_2(p_{s\#}(x))=\phi_{s+1\#}(x)$ which is well-defined by the commutativity of the above diagram and Lemma 2.4 together with the fact that $d_1\phi_{s+1\#}=0$ which will be proved in the next lemma. Then the exactness at $D_2^{s-1,s-1}$ is obtained by chasing the diagram, making use of the definition of the E_2-terms, the definition of the ghost terms and Lemma 2.4.

LEMMA 3.4. *$d_1\phi_{s+1\#}$ is trivial for $s\geqq 1$.*

PROOF. Before proving this, we remark, by Lemma 2.4, that

$$d_1\phi_{s+1\#}\langle f\rangle=\langle \textstyle\sum_{j=0}^{s+2}(-1)^j d^j(f^{s+1}-\mathrm{const}\,(f^{s+2}(e_0)))\rangle$$

in $\pi_s\,\mathrm{Map}_*(X, A^{s+2})$ for the homotopy class $\langle f\rangle$ of any element f of $\mathrm{Map}_*(X, tot_s(A^\cdot))$. To prove the lemma, it suffices to show that the element

$$\textstyle\sum_{j=0}^{s+2}(-1)^j d^j(f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0)))$$

is homotopic to the constant mapping. The part $\sum_{j=1}^{s+2}(-1)^j d^j(f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0)))$ of the above summand is deformable as follows. For $j\geqq 1$, $d^j(f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0)))=f^{s+2}d^j-\mathrm{const}\,(f^{s+2}(e_0))$ and the summation with $j\geqq 1$ is homotopic to

$$\begin{aligned} &-(f^{s+1}w)^*(f^{s+2}d^0)+\mathrm{const}\,(f^{s+2}(e_0)) \\ &=-(f^{s+1}w)^*(d^0f^{s+1})+\mathrm{const}\,(f^{s+2}(e_0)) \\ &\simeq -d^0f^{s+1}+\mathrm{const}\,(f^{s+2}(e_1)) \\ &\simeq -d^0(f^{s+1}-\mathrm{const}\,(f^{s+1}(e_0))), \end{aligned}$$

where w is a path from e_0 to e_1, $(f^{s+1}w)^*(d^0f^{s+1})$ is the action of path on the mapping, and the above homotopy is given by

$$t\longmapsto -(f^{s+1}w_t)^*(d^0f^{s+1})+\mathrm{const}\,(f^{s+2}(w(t)))$$

and

$$t\longmapsto -d^0(f^{s+1}-\mathrm{const}\,(\ell(t)))+\mathrm{const}\,(d^0(f^{s+1}(e_0)-\ell(t)),$$

where $w_t(u)=w(t+(1-t)u)$, and $\ell(t)$ is a path from 0 to const$(f^{s+1}(e_0))$. This implies the lemma.

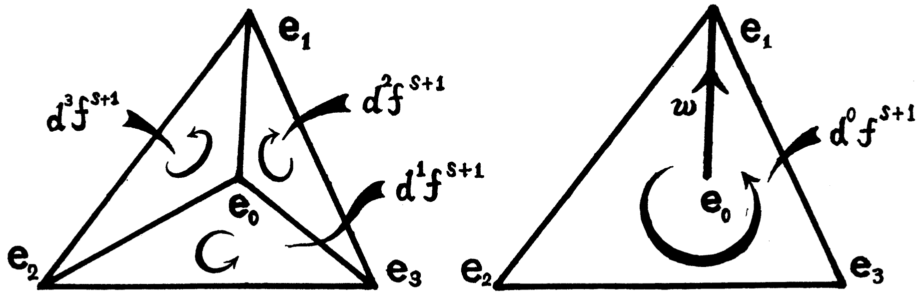

Figure 2

We give here the description of the ghost terms which is obtained by the definition and is similar to the ordinary terms.

PROPOSITION 3.5. *Let X and Y be CW-complexes. Then the ghost terms in E_1-terms and E_2-terms of the Bousfield and Kan unstable Adams spectral sequence are described as*

$$\bar{E}_1^{s,t} \cong \mathrm{Hom}_{CA}(H_*(\Sigma^t X), H_*(\bar{R}^s Y)) \quad \text{for} \quad 0 \leqq t < s,$$

$$\bar{E}_2^{s,t} \cong \mathrm{Ext}_{CA}^s(H_*(\Sigma^t X), H_*(Y)) \quad \text{for} \quad 1 \leqq t < s.$$

PROOF. By the definition, we have

$$\begin{aligned} \bar{E}_1^{s,t} &= \pi_t \,\mathrm{Map}_*(X, N\bar{R}^{s+1}Y) \\ &= \pi_t \,\mathrm{Map}_*(X, \bar{R}^{s+1}Y) \cap \mathrm{Ker}\, s^0 \cap \mathrm{Ker}\, s^1 \cap \cdots \cap \mathrm{Ker}\, s^{s-1}, \end{aligned}$$

$$D_1^{s,t} = \pi_{t-s}(\mathrm{Map}_*(X, tot_s(\bar{R}^{\cdot} Y))).$$

Since $\bar{R}^s Y$ has the homotopy type of a generalized Eilenberg-Maclane space and X is a CW-complex, $\pi_t \,\mathrm{Map}^*(X, \bar{R}^{s+1}Y)$ is in one to one correspondence with $\mathrm{Hom}_{CA}(H_*(\Sigma^t X), H_*(\bar{R}^s Y))$ as sets and the proposition is obtained easily from the definitions.

Combining this proposition with Theorem 3.3, we obtain the following

THEOREM 3.6. *Let X be a CW-complex and Y a space with nondegenerate base point. Then the following three statements hold for the evaluation mapping by mod p homology*

$$H: \pi_i(\mathrm{Map}_*(X, Y\hat{p}), *) \longrightarrow \mathrm{Hom}_{\mathbf{CA}}(H_*(\Sigma^i X), H_*(Y)), \quad i \geqq 0.$$

a) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H_*\Sigma^t X), H_*(Y))=0$ *for* $t-i=s\geqq 1$, *then* H *is injective.*
b) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H_*(\Sigma^t X), H_*(Y))=0$ *for* $t-i+1=s\geqq 1$, *then* H *is surjective.*
c) *If* $\mathrm{Ext}^s_{\mathbf{CA}}(H_*(\Sigma^t X), H_*(Y))=0$ *for* $t-i=s\geqq 1$ *and for* $t-i+1=s\geqq 1$, *then* H *is bijective.*

PROOF. Clearly, a) and b) imply c). Firstly we prove a). By Theorem 3.3, $D_2^{s,s+i} \to D_2^{s-1,s-1+i}$ is injective and $D_2^{s,s+1+i} \to D_2^{s-1,s+i}$ is surjective for $s\geqq 1$. This implies $\text{proj-lim}^1_s D_2^{s,s+1+i}=0$ and $\pi_i(\mathrm{Map}_*(X, Y\hat{p}), *) \cong \text{proj-lim}_s D_2^{s,s+i}=D_2^{0,i}$. This implies a). Next, we prove b). Again by Theorem 3.3, $D_2^{s,s+i} \to D_2^{s-1,s-1+i}$ is surjective. This implies b) and so completes the proof of the theorem.

§4. Proof of Theorem 0.1

Before we prove Theorem 0.1, we introduce the notions of a category **MA** and an algebra functor **P** while recalling the notions of some other categories and functors in Miller [9, 10].

Let **U** be the category of unstable modules over **A**(p), **C** the category of graded cocommutative coassociative coalgebras over R and n**R** the category of graded modules over R as in Miller [9]. Also we introduce here the other category **MA** as the full subcategory of **U** whose objects have no elements in dimension $2pn$ for p odd or $2n$ for $p=2$, such that the image under the power operation $\mathscr{P}^n$ for p odd or the squaring operation S^n_q for $p=2$ is non-trivial. Then **MA** is not merely a category of desuspendable objects in **U** for $p>2$.

Let I, J and J' be the forgetful functors from **CA** to **C**, **U** to n**R** and **MA** to n**R** respectively; P and P' the functors from **CA** to **MA** and **C** to n**R** respectively, taking primitive modules; S' the functor from n**R** to **C**, taking maximal cocommutative tensor coalgebras; and $(\ )^+$ the functor from **MA** to **CA**, adding units and trivial comultiplications to modules.

Let **G**, **Q** (denoted by **F** in Miller [9]), **P** and **S**$'$ be the algebra functors of **CA**, of U, of **MA**, and of **C** respectively defined as

(4.1) $$\mathbf{G}=\mathbf{G}'I, \quad \mathbf{G}'(V)=H_*(\textstyle\prod K(V_n, n)),$$
$$\mathbf{Q}=\mathbf{Q}'J', \quad \mathbf{Q}'(V)_m=\mathrm{Hom}_{n\mathbf{R}}(P(m), V)=QH_m(\textstyle\prod K(V_n, n_\nu)),$$
$$\mathbf{P}=\mathbf{P}'J, \quad \mathbf{P}'(V)_m=\mathrm{Hom}_{n\mathbf{R}}(Q(m), V)=PH_m(\textstyle\prod K(V_n, n)),$$

and as $\mathbf{S}'=S'I'_0$, where $Q(m)_j=QH^m(K(R, j))$ and $P(m)_j=PH^m(K(R, j))$. Then,

$Q(m+1)=P(m)$ in **U** (not in **MA**), since $Q(m+1)_{j+1}=QH^{m+1}(K(R, j+1))\cong PH^{m}(K(R, j))=P(m)_j$ for $j\geqq 1$. By the definition we have $\mathbf{P}=P\mathbf{G}$ and the following formulae

$$\text{(4.2)} \qquad \mathrm{Hom}_{\mathbf{MA}}(Q(m), N)\cong N_m, \quad \mathrm{Hom}_{\mathbf{U}}(P(m), N)\cong N_m.$$

As in [10], we can easily get the relation $\mathrm{Hom}_{\mathbf{CA}}(M^+, C)=\mathrm{Hom}_{\mathbf{MA}}(M, PC)$ and the factorization

$$\mathrm{Hom}_{\mathbf{CA}}(M^+, \ )=\mathrm{Hom}_{\mathbf{MA}}(M, \ )P$$

for any object $M\in\mathbf{MA}$. In the above context, the following lemma is obtained essentially in [9, 10].

LEMMA 4.3. (1) *There is a convergent cohomological spectral sequence* $E_r^{s,t}$ *whose* E_2-*and* E_∞-*terms are described as*

$$E_2^{s,t}=\mathrm{Ext}^s_{\mathbf{MA}}(M, R^t_{\mathbf{P}}P(C)),$$

$$E^{t+t}_{\mathbf{CA}}=\mathrm{Ext}^{s+t}_{\mathbf{CA}}(M^+, C)$$

natural with respect to $M\in\mathbf{MA}$ *and* $C\in\mathbf{CA}$.
(2) *For each t, there is an isomorphism of graded vector spances*

$$J(R^*_{\mathbf{G}}P)(C)=(R^*_{\mathbf{S}'}P')(IC).$$

natural with respect to $C\in\mathbf{CA}$.

By replacing **U** and **F** with **MA** and **P** and by the fact that $P(H^*(K(R, 1)))=PH^*(K(R, 1))$ is injective in **MA**, we obtain this lemma similarly to the proof of Theorem 2.5 in [9, 10], which uses a general Grothendiek spectral sequence for the setting (4.4) and (4.5),

$$\text{(4.4)} \qquad \begin{array}{ccccc} \mathbf{CA} & \xrightarrow{P} & \mathbf{MA} & \xrightarrow{\mathrm{Hom}_{MA}(M,\ )} & n\mathbf{R} \\ \circlearrowleft & & \circlearrowleft & & \\ \mathbf{G} & & \mathbf{P} & & \end{array}$$

for $M\in\mathbf{MA}$ and

$$\text{(4.5)} \qquad \begin{array}{ccccc} \mathbf{CA} & \xrightarrow{I} & \mathbf{C} & \xrightarrow{P'} & n\mathbf{R} \\ \circlearrowleft & & \circlearrowleft & & \\ \mathbf{G} & & \mathbf{S}'. & & \end{array}$$

As Miller proved in [8, 9 and 10] for any prime p, $\bar{H}_*(\Sigma B(\mathbf{Z}/p\mathbf{Z})^n)=\Sigma \bar{H}^*(B(\mathbf{Z}/p\mathbf{Z})^n)$ is a direct summand of the direct limit of $\Sigma P(n)=Q(n+1)$; *if* $n=1$, then $\bar{H}^*(\Sigma B(\mathbf{Z}/p\mathbf{Z})^n)$ is a direct summand of

$$Q_{2n+1}=\lim\{Q(2n+1)\xrightarrow{\mathscr{P}^n}Q(2pn+1)\xrightarrow{\mathscr{P}^{pn}}\cdots\}$$
$$=\lim\{\Sigma P(2n)\xrightarrow{\mathscr{P}^n}\Sigma P(2pn)\xrightarrow{\mathscr{P}^{pn}}\cdots\},$$

Since $\mathrm{Hom}_{\mathbf{MA}}(Q(m), N)=N_m$, $Q(m)$ is projective in **MA**, and

$$\mathrm{Ext}^s_{\mathbf{MA}}(\bar{H}_*(\Sigma(B\mathbf{Z}/p\mathbf{Z})^n), N)=0$$

for $s\geqq 1$ and $N\in\mathbf{MA}$ of finite type.

As $\mathrm{Hom}_{\mathbf{MA}}(Q(m), N)=N_m$, the right adjoint Ω of Σ is described as $\mathrm{Hom}_{\mathbf{MA}}(\Sigma Q(m),\ )$. And the right derived functors $R^n\Omega$ are trivial for $n\geqq 1$. If N is of finite type, so are ΩN and $R^n\Omega N$ (see Miller [8, 9 and 10]). And then we can easily obtain

PROPOSITION 4.6. *Let p be any prime. If $N\in\mathbf{MA}$ is of finite type, then* $\mathrm{Ext}^s_{\mathbf{MA}}(\bar{H}_*(\Sigma^t(B\mathbf{Z}/p\mathbf{Z})^n), N)=0$ *for $s\geqq t\geqq 1$.*

Assume Y is a space whose mod p cohomology is a free commutative algebra and of finite type. Then $I(H_*(Y))$ is injective in **C** and $R^tP(H^*(Y))=0$ for $t\geqq 1$. This implies

$$\mathrm{Ext}^s_{\mathbf{CA}}(\bar{H}_*(\Sigma^t(B\mathbf{Z}/p\mathbf{Z})^n), H_*(Y))$$
$$=\mathrm{Ext}^s_{\mathbf{MA}}(\bar{H}_*(\Sigma^t(B\mathbf{Z}/p\mathbf{Z})^n), PH_*(Y))=0$$

for $t=s\geqq 1$ or $t+1=s\geqq 1$. Hence, the following evaluation by mod p cohomology is a natural bijection of sets with base points by Theorem 3.6:

$$\pi_0(\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, Y_p^{\hat{}}), *)\longrightarrow \mathrm{Hom}_{\mathbf{CA}}(H_*(B(\mathbf{Z}/p\mathbf{Z})^n), H_*(Y))$$
$$\longrightarrow \mathrm{Hom}_{\mathbf{A}}(H^*(Y), H^*(B(\mathbf{Z}/p\mathbf{Z})^n)).$$

This completes the proof of Theorem 0.1.

The corollaries are obtained directly.

Finally, we give some examples for Corollary 0.5.

EXAMPLE 4.7. *Ley p be any prime. Then there exist following bijections*:

1) $\pi_0\,\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z})^n, BSp(1))\cong\pi_0\,\mathrm{Hom}((\mathbf{Z}/p\mathbf{Z})^n, Sp(1))$,
2) $\pi_0\,\mathrm{Map}_*(B(\mathbf{Z}/p\mathbf{Z}), BU(n))\cong\pi_0\,\mathrm{Hom}((\mathbf{Z}/p\mathbf{Z}), U(n))$.

PROOF. We show the Bar construction functor B induces the bijection. Composing B with the cohomology functor, we have the mapping is monic. So we show it is epic. By simple computations, π_0 Hom $((\mathbf{Z}/p\mathbf{Z})^n, Sp(1))=\{b\in\mathbf{Z}/p\mathbf{Z};\ b=a^2 \text{ for some } a\in\mathbf{Z}/p\mathbf{Z}\}^n$ and π_0 Hom $((\mathbf{Z}/p\mathbf{Z})^n, U(m))=(SP^m(\mathbf{Z}/p\mathbf{Z}))^n$. Then 1) follows from a computation of the commutativity of a homomorphism and the cohomology operations $\mathscr{P}=\prod_i \mathscr{P}^i$ and β by using the well-knwon fromulae

$$\mathscr{P}v_1=v_1,\quad \beta v_1=v_2,\quad \mathscr{P}v_2=v_2+v_2^p,\quad \beta v_2=0,$$

$$\mathscr{P}w_4=w_4+2w_4^{(p-1)/2}+w_4^p,\quad \beta w_4=0$$

on the modulo p cohomology rings $H^*(B(\mathbf{Z}/p\mathbf{Z}))=E(v_1)\otimes\mathbf{Z}_p[v_2]$ and $H^*(BSp(1))=\mathbf{F}_p[w_4]$. So, we omit the proof of 1). We show 2). Using the well-known formulae

$$\mathscr{P}^j u_m=u_m\cdot Q_j(u_1,\dots, u_m),$$

$$Q_j(S_1(x_1,\dots, x_m),\dots, S_m(x_1,\dots, x_m))=S_j(x_1^{p-1},\dots, x_m^{p-1}),$$

where $S_i(t_1,\dots, t_m)$ is the elementary symmetric polynomial of $t_1,\dots, t_m$ for the cohomology ring $H^*(BU(n))=\mathbf{F}_p[u_1,\dots, u_n]$, and the elementary algebraic fact; in the algebraic closure $\bar{\mathbf{F}}_p$ of $\mathbf{F}_p$, $a^{p-1}=1$ implies a is in $\mathbf{F}_p$; we may take elements a_j in $\mathbf{F}_p$ and the image of u_j is described as $S_j(a_1,\dots, a_m)v_2^p$. So we obtain the example.

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Department of Mathematics,
Kyushu University 33
Hakozaki, Fukuoka 812,
Japan