Lusternik-Schnirelmann category of a sphere-bundle over a sphere

Dedicated to Professor J. R. Hubbuck on his 60th birthday

Norio Iwase

Faculty of Mathematics, Kyushu University, Ropponmatsu Fukuoka, JAPAN.

Abstract

We determine the L-S category of a total space of a sphere-bundle over a sphere in terms of primary homotopy invariants of its characteristic map, and thus providing a complete answer to Ganea's Problem 4. As a result, we obtain a necessary and sufficient condition for a total space N to have the same L-S category as its 'once punctured submanifold' $N \smallsetminus \{P\}, P \in N$. Also necessary and sufficient conditions for a total space M to satisfy Ganea's conjecture are described.

Key words: Lusternik-Schnirelmann category, higher Hopf invariant, sphere bundle, manifold, Ganea conjecture.
1991 MSC: Primary 55M30, Secondary 55P35, 55Q25, 55R35, 55S36.

1 Introduction

The (normalised) L-S category $\mathrm{cat}(X)$ of X is the least number m such that there is a covering of X by $m+1$ open subsets each of which is contractible in X, which equals to the least number m such that the diagonal map $\Delta_{m+1} : X \to \prod^{m+1} X$ can be compressed into the 'fat wedge' $\mathrm{T}^{m+1}(X)$ (see James [8] and Whitehead [21]). By definition, we have $\mathrm{cat}(\{*\}) = 0$.

A simple definition, however, does not always suggest a simple way of calculation. In fact, to *determine the L-S category of a sphere-bundle over a sphere in terms of homotopy invariants of its characteristic map* is listed as Problem 4 of

Email address: `iwase@math.kyushu-u.ac.jp` (Norio Iwase).
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Ganea [2] in 1971. Ganea's Problem 2 is also a basic problem on $\mathrm{cat}(X\times S^n)$, where we easily see that $\mathrm{cat}(X\times S^n) = \mathrm{cat}(X)$ or $\mathrm{cat}(X) + 1$: *Can the latter case only occur on any X and $n \geq 1$?* The affirmative answer had become known as "the Ganea conjecture" (see James [9]), particularly for manifolds.

Although a tight connection between L-S category and the Bar resolution (A_∞-structure) has been pointed out by Ginsburg [3] in 1963, a homological approach could not succeed to solve Ganea's problems on L-S category. By Singhof [18] followed by Montejano [11], Gómez-Larrañaga and González-Acuña [4], Rudyak [16,17] and Oprea and Rudyak [15], the conjecture is validated for a large class of manifolds. The first closed manifold counter-example to the conjecture was given by the author [7] as a total space of a sphere-bundle over a sphere, using the A_∞-method with concrete computations of Toda brackets depending on results by Toda [20] and Oka [14]. Also, Lambrechts, Stanley and Vandembroucq [10] and the author [7] provided manifolds each of which has the same L-S category as its once punctured submanifold.

The purpose of this paper is to determine the L-S category of a sphere-bundle over a sphere in terms of a primary homotopy invariant of the characteristic map of a bundle, providing simpler proofs of manifold examples in [7]. Using it, we could obtain many closed manifolds each of which has the same L-S category as its once punctured submanifold and many closed manifold counter-examples to Ganea's conjecture on L-S category.

Throughout this paper, we follow the notations in [6,7]: In particular for a map $f : S^k \to X$, a homotopy set of higher Hopf invariants $H^S_m(f) = \{[H^\sigma_m(f)] \mid \sigma \text{ is a structure map of } \mathrm{cat}\, X{\leq}m\}$ (or its stabilisation $\mathcal{H}^S_m(f) = \Sigma^\infty_* H^S_m(f)$) is referred simply as *a (stabilised) higher Hopf invariant* of f, which plays a crucial role in this paper. For a sphere map $f : S^k \to S^\ell$ with $k, \ell > 1$, we identify $H^S_1(f)$ and $\mathcal{H}^S_1(f)$ with their unique elements, $H_1(f)$ and $\mathcal{H}_1(f) = \Sigma^\infty H_1(f)$, since a sphere S^n has the unique structure $\sigma(S^n) : S^n \to \Sigma\Omega S^n$ for $\mathrm{cat}(S^n) = 1$, $n > 1$.

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2 L-S category of a sphere-bundle over a sphere

Let $r \geq 1, t \geq 0$ and E be a fibre bundle over S^{t+1} with fibre S^r. Then E can be described as $S^r \cup_\Psi S^r \times D^{t+1}$, with $\Psi : S^r \times S^t \to S^r$ (see Whitehead [21]). Hence E has a CW decomposition $S^r \cup_\alpha e^{t+1} \cup_\psi e^{r+t+1}$ with $\alpha : S^t \to S^r$ and

$\psi : S^{r+t} \to Q = S^r \cup_\alpha e^{t+1}$ given by the following formulae:

$$\alpha = \Psi|_{\{*\}\times S^t}, \quad \psi|_{S^{r-1}\times D^{t+1}} = \chi_\alpha \circ \mathrm{pr}_2,\ \psi|_{D^r\times S^t} = \Psi\circ(\omega_r\times 1_{S^t}),$$

where we denote by $\chi_f : (C(A), A) \to (C_f, B)$ the characteristic map for $f : A \to B$ and let $\omega_r = \chi_{(*:S^{r-1}\to\{*\})}$. When $r = 1$, the L-S categories of E and Q are studied by several authors; especially by Singhof [18] and Oprea-Rudyak [15] in the case when $r = t = 1$. We summarise known results.

Fact 2.1 *Let $r = 1$. Then we have the following.*

($t = 0$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.
($t = 1$, $\alpha = \pm 1$) $\mathrm{cat}(Q\times S^n) = 1$, $\mathrm{cat}(Q) = 0$, $\mathrm{cat}(E) = 1$, $\mathrm{cat}(E\times S^n) = 2$.
($t = 1$, $\alpha = 0$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.
($t = 1$, $\alpha \neq 0, \pm 1$) $\mathrm{cat}(Q\times S^n) = 3$, $\mathrm{cat}(Q) = 2$, $\mathrm{cat}(E) = 3$, $\mathrm{cat}(E\times S^n) = 4$.
($t > 1$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.

When $r > 1$, we identify $H^S_1(\alpha)$ with its unique element $H_1(\alpha)$. We summarise the known results (due to Berstein-Hilton [1]) from [7, Facts 7.1, 7.2].

Fact 2.2 *Let $r > 1$. Then we have the following.*

($t < r$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.
($t = r$, $\alpha = \pm 1_{S^r}$) $\mathrm{cat}(Q\times S^n) = 1$, $\mathrm{cat}(Q) = 0$, $\mathrm{cat}(E) = 1$, $\mathrm{cat}(E\times S^n) = 2$.
($t = r$, $\alpha \neq \pm 1_{S^r}$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.
($t > r$, $H_1(\alpha) = 0$) $\mathrm{cat}(Q\times S^n) = 2$, $\mathrm{cat}(Q) = 1$, $\mathrm{cat}(E) = 2$, $\mathrm{cat}(E\times S^n) = 3$.
($t > r$, $H_1(\alpha) \neq 0$) $\mathrm{cat}(Q\times S^n) = 3$ *or* 2, $\mathrm{cat}(Q) = 2$, $\mathrm{cat}(E) = 2$ *or* 3, $\mathrm{cat}(E\times S^n) = 3$ *or* 4.

By [6] and [7, Theorem 5.2, 5.3, 7.3], the following is also known.

Fact 2.3 *When $r > 1$, $t \geq r$ and $\alpha \neq \pm 1$, we also have the following.*

(1) $\Sigma^n H_1(\alpha) = 0$ *implies* $\mathrm{cat}(Q\times S^n) = 2$, *and* $\Sigma^{n+1} H_1(\alpha) \neq 0$ *implies* $\mathrm{cat}(Q\times S^n) = 3$.
(2) $\mathrm{cat}(E) = 2$ *if and only if* $H^S_2(\psi) \ni 0$, *and* $\mathrm{cat}(E) = 2$ *implies* $\mathrm{cat}(E\times S^n) = 3$ *for all* n.
(3) $\Sigma^n_* H^S_2(\psi) \ni 0$ *implies* $\mathrm{cat}(E\times S^n) = 3$, *and* $\Sigma^{n+r+1} h_2(\alpha) \neq 0$ *implies* $\mathrm{cat}(E\times S^n) = 4$.

Remark 2.4 *When α is in meta-stable range, $H_1(\alpha) : S^t \to \Omega S^r * \Omega S^r$ is given by the second James-Hopf invariant $h_2(\alpha) : S^t \to \Sigma S^{r-1} \wedge S^{r-1}$ composed with an appropriate inclusion to a wedge-summand. Thus we may regard*

$h_2(\alpha) = H_1(\alpha)$ *when* α *is in meta-stable range.*

Our main result is as follows:

Theorem 2.5 *Let* $\mathrm{cat}(Q) = 2$ *with* $t > r > 1$*, Then* $H_2^S(\psi)$ *contains* 0 *if and only if* $\Sigma^r H_1(\alpha) = 0$*. More generally for a co-H-map* $\beta : S^v \to S^{r+t}$ *with* $v < t + 2r - 1$*,* $H_2^S(\psi \circ \beta) = \beta^* H_2^S(\psi)$ *contains* 0 *if and only if* $\Sigma^r H_1(\alpha) \circ \beta = 0$*.*

The main result is obtained by the following lemma for Q of $\mathrm{cat}(Q) = 2$ with $t > r > 1$.

Lemma 2.6 $H_2^S(\psi) \ni \pm[(\hat{i} * 1_{\Omega Q * \Omega Q}) \circ \Sigma^r H_1(\alpha)]$*, where the bottom-cell inclusion* $\hat{i} : S^{r-1} \hookrightarrow \Omega Q$ *denotes the adjoint of the inclusion* $i : S^r \hookrightarrow Q$*.*

By combining above facts with Theorem 2.5, we obtain an answer to Ganea's Problem 4:

Theorem 2.7 (Table of L-S categories) *For an* S^r*-bundle* E *over* S^{t+1} *and its once-punctured submanifold* $E \smallsetminus \{P\} \simeq Q$*, we have the following table:*

<table>
<tr><th colspan="3">Conditions</th><th colspan="4">L-S categories</th></tr>
<tr><th>r</th><th>t</th><th>α</th><th>$Q \times S^n$</th><th>Q</th><th>E</th><th>$E \times S^n$</th></tr>
<tr><td rowspan="5">$r = 1$</td><td>$t = 0$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t = 1$</td><td>$\alpha = \pm 1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha = 0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$\alpha \neq 0, \pm 1$</td><td>3</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$t > 1$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="6">$r > 1$</td><td>$t < r$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="2">$t = r$</td><td>$\alpha = \pm 1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha \neq \pm 1$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t > r$</td><td>$H_1(\alpha) = 0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$H_1(\alpha) \neq 0$ & $\Sigma^r H_1(\alpha) = 0$</td><td rowspan="2">3 or 2
(1)</td><td rowspan="2">2</td><td>2</td><td>3</td></tr>
<tr><td>$\Sigma^r H_1(\alpha) \neq 0$</td><td>3</td><td>3 or 4
(2)</td></tr>
</table>

(1): $\begin{cases} \Sigma^n H_1(\alpha) = 0 \textit{ implies } \mathrm{cat}(Q \times S^n) = 2 \textit{ and} \\ \Sigma^{n+1} H_1(\alpha) \neq 0 \textit{ implies } \mathrm{cat}(Q \times S^n) = 3. \end{cases}$

(2): $\begin{cases} \Sigma^{r+n} H_1(\alpha) = 0 \textit{ implies } \mathrm{cat}(E \times S^n) = 3 \textit{ and} \\ \Sigma^{r+n+1} h_2(\alpha) \neq 0 \textit{ implies } \mathrm{cat}(E \times S^n) = 4. \end{cases}$

3 Applications and examples

Firstly, Theorem 2.7 yields the following result.

Theorem 3.1 *Let a manifold N be the total space of a S^r-bundle over S^{t+1} with a characteristic map $\Psi : S^r \times S^t \to S^r$, $t > r > 1$, and let $\alpha = \Psi|_{S^t}$. Then* $\mathrm{cat}(N \smallsetminus \{P\}) = \mathrm{cat}(N)$ *if and only if $H_1(\alpha) \neq 0$ and $\Sigma^r H_1(\alpha) = 0$.*

This theorem provides the following examples.

Example 3.2 *Let p be an odd prime and $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_1(2p)$. Then we have that $H_1(\alpha) = \alpha_1(3) \circ \alpha_1(2p) \neq 0$ and $\Sigma^2 H_1(\alpha) = 0$ by [20]. Let $N_p \to S^{4p-2}$ be the bundle with fibre S^2 induced by $\Sigma(\alpha_1(3) \circ \alpha_1(2p)) : S^{4p-2} \to S^4$ from the bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ with fibre $Sp(1)/U(1) = S^2$. By the arguments given in [7], we obtains that N_p has a CW-decomposition as $N_p \approx S^2 \cup_\alpha e^{4p-2} \cup_\psi e^{4p}$. Then Theorem 3.1 implies that* $\mathrm{cat}(N_p) = \mathrm{cat}(N_p \smallsetminus \{P\}) = 2$.

Example 3.3 ([7]) *Let p be a prime ≥ 5 and $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(2p)$ as in [7]. Then we have that $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(2p) \neq 0$ and $\Sigma^2 H_1(\alpha) = 0$ by [20]. Let $L_p \to S^{6p-4}$ be the bundle with fibre S^2 induced by $\Sigma(\alpha_1(3) \circ \alpha_2(2p)) : S^{6p-4} \to S^4$ from the bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ with fibre $Sp(1)/U(1) = S^2$. By the arguments given in [7], we obtains that L_p has a CW-decomposition as $L_p \approx S^2 \cup_\alpha e^{6p-4} \cup_\psi e^{6p-2}$. Then Theorem 3.1 implies that* $\mathrm{cat}(L_p) = \mathrm{cat}(L_p \smallsetminus \{P\}) = 2$.

Secondly, Theorem 2.7 also yields the following result.

Theorem 3.4 *Let a manifold M be the total space of a S^r-bundle over S^{t+1} with a characteristic map $\Psi : S^r \times S^t \to S^r$, $t > r > 1$, and let $\alpha = \Psi|_{S^t}$. If $\Sigma^r H_1(\alpha) \neq 0$ and $\mathcal{H}_1(\alpha) = 0$, then M is a counter-example to the Ganea's conjecture on L-S category; more precisely,* $\mathrm{cat}(M) = \mathrm{cat}(M \times S^n) = 3$ *if $\Sigma^r H_1(\alpha) \neq 0$ and $\Sigma^{n+r} H_1(\alpha) = 0$.*

This theorem provides the following manifold counter examples to Ganea's conjecture on L-S category.

Example 3.5 *Let $p = 2$ and $\alpha = \eta_2 \circ \eta_3^2 \circ \epsilon_5$. Then we have that $H_1(\alpha) = \eta_3^2 \circ \epsilon_5 \neq 0$, $\Sigma^2 H_1(\alpha) \neq 0$ and $\Sigma^6 H_1(\alpha) = 0$ by [20]. Let $M_2 \to S^{14}$ be the bundle with fibre S^2 induced by $\Sigma(\eta_3^2 \circ \epsilon_5) : S^{14} \to S^4$ from the bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ with fibre $Sp(1)/U(1) = S^2$. By the arguments given in [7] we obtains that M_2 has a CW-decomposition as $M_2 \approx S^2 \cup_\alpha e^{14} \cup_\psi e^{16}$. Then Theorem 3.4 implies that* $\mathrm{cat}(M_2 \times S^n) = \mathrm{cat}(M_2) = 3$ *for $n \geq 4$.*

Example 3.6 ([7]) *Let $p = 3$ and $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(6)$ as in [7]. Then we have that $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(6) \neq 0$, $\Sigma^2 H_1(\alpha) \neq 0$ and $\Sigma^4 H_1(\alpha) = 0$ by [20].*

Let $M_3 \to S^{14}$ *be the bundle with fibre* S^2 *induced by* $\Sigma(\alpha_1(3)\circ\alpha_2(6)) : S^{14} \to S^4$ *from the bundle* $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ *with fibre* S^2*. By the arguments given in [7] we obtains that* M_3 *has a CW-decomposition as* $M_3 \approx S^2 \cup_\alpha e^{14} \cup_\psi e^{16}$*. Then Theorem 3.4 implies that* $\mathrm{cat}(M_3{\times}S^n) = \mathrm{cat}(M_3) = 3$ *for* $n \geq 2$*.*

Finally, Theorem 2.5 and [7, Theorem 5.2] imply the following result.

Theorem 3.7 *Let a manifold* X *be the total space of a* S^r*-bundle over* S^{t+1} *with a characteristic map* $\Psi : S^r{\times}S^t \to S^r$*,* $t > r > 1$*, and let* $\alpha = \Psi|_{S^t}$*. When* $H_1(\alpha) \neq 0$ *and* β *is a co-H-map, we obtain that* $X(\beta) = S^r \cup_\alpha e^{t+1} \cup_{\psi\circ\beta} e^{v+1}$ *is of* $\mathrm{cat}(X(\beta)) = 3$ *if and only if* $\Sigma^r H_1(\alpha)\circ\beta \neq 0$*.*

Remark 3.8 *All examples obtained here still support the conjecture in [6].*

4 Proof of Lemma 2.6

Let $\mathrm{cat}(Q) = 2$ with $t > r > 1$. In the remainder of this paper, we distinguish a map from its homotopy class to make the arguments clear.

Here, let us recall the definition of a *relative Whitehead product*: For maps $f : \Sigma X \to M$ and $g : (C(Y), Y) \to (K, L)$, we denote by $[f, g]^{\mathrm{rel}} : X{*}Y = C(X){\times}Y \cup X{\times}C(Y) \to M{\times}L \cup \{*\}{\times}K$ the relative Whitehead product, which is given by

$$[f, g]^{\mathrm{rel}}|_{C(X)\times Y}(t{\wedge}x, y) = (f(t{\wedge}x), g(y)) \quad \text{and}$$
$$[f, g]^{\mathrm{rel}}|_{X\times C(Y)}(x, t{\wedge}y) = (*, g(t{\wedge}y)).$$

Also a pairing $F : M{\times}L \to M$ with axes 1_M and $h : L \to M$ (see Oda [13]) determines a map

$$(F \cup \chi_h) : (M{\times}L \cup \{*\}{\times}K) \to (M \cup_h K, M)$$

by $(F \cup \chi_h)|_{M\times L} = F$ and $(F \cup \chi_h)|_{\{*\}\times K} = \chi_h$, where $\chi_h : (K, L) \to (M \cup_h K, M)$ is a relative homeomorphism given by the restriction of the identification map $M \cup K \to M \cup_h K$. Then we can easily see that $\psi : S^{r+t} \to Q$ is given as

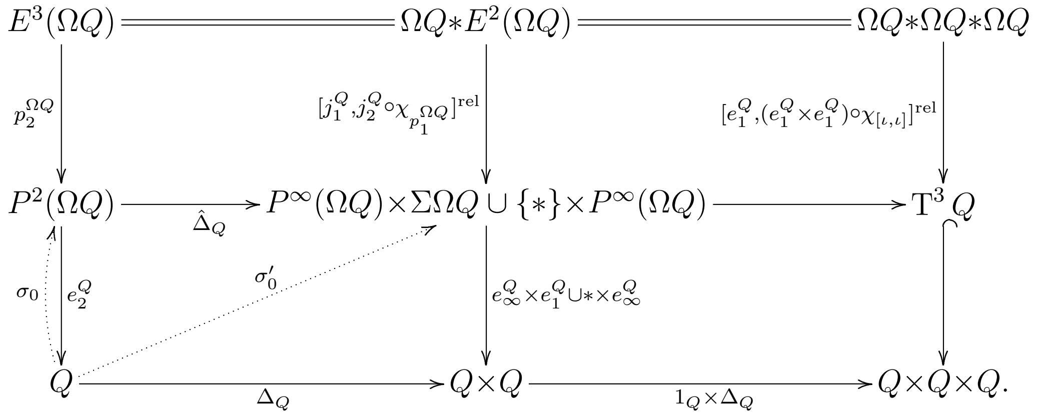
$$\psi = (\Psi \cup \chi_\alpha)\circ[\iota_r, C(\iota_t)], \tag{4.1}$$

where $\iota_k : S^k \to S^k$ and $C(\iota_k) : C(S^k) \to C(S^k)$ denote the identity maps.

We denote by $j_i^Q : P^i(\Omega Q) \hookrightarrow P^\infty(\Omega Q)$ the classifying map of the fibration $p_i^{\Omega Q} : E^{i+1}(\Omega Q) \to P^i(\Omega Q)$ and $e_i^Q = e_\infty^Q\circ j_i^Q$, where $e_\infty^Q : P^\infty(\Omega Q) \to Q$ is a homotopy equivalence extending the evaluation map $e_1^Q = ev : \Sigma\Omega Q \to Q$.

Let σ_∞ be the homotopy inverse of e^Q_∞. Then we may assume that $\sigma_\infty|_{S^r} = j^Q_1{\circ}\sigma(S^r)$ for dimensional reasons.

Proposition 4.1 *The following without the dotted arrows is a commutative diagram where the lower squares are pull-back diagrams.*



(4.2)

Therefore, there is a lifting σ'_0 of Δ_Q and hence a lifting σ_0 of the identity 1_Q.

Remark 4.2 *The homotopy fibre $\Omega Q{*}\Omega Q{*}\Omega Q \to \mathrm{T}^3\,Q$ of the inclusion*

$$\mathrm{T}^3\,Q = Q{\times}(Q{\vee}Q) \cup \{*\}{\times}(Q{\times}Q) \hookrightarrow Q{\times}(Q{\times}Q)$$

is given by a relative Whitehead product $[e^Q_1, (e^Q_1{\times}e^Q_1){\circ}\chi_{[\iota,\iota]}]^{\mathrm{rel}}$, where ι denotes the identity $1_{\Sigma\Omega Q}$ and

$$\chi_{[\iota,\iota]} : (C(\Omega Q{*}\Omega Q), \Omega Q{*}\Omega Q) \to (\Sigma\Omega Q{\times}\Sigma\Omega Q, \Sigma\Omega Q{\vee}\Sigma\Omega Q)$$

denotes a relative homeomorphism.

A lifting σ'_0 of Δ_Q in diagram (4.2) is given by the following data:

$$\sigma'_0|_{S^r}(y) = ((j^Q_1{\circ}\sigma(S^r))(y), \sigma(S^r)(y)) \quad \text{for } y \in S^r,$$

and for $u{\wedge}x \in (0,1]{\times}S^t/\{1\}{\times}S^t = Q \smallsetminus S^r$ with $\mu_t(x) = (x_1, x_2)$,

$$\sigma'_0|_{Q\smallsetminus S^r}(u{\wedge}x) = \begin{cases} ((j^Q_1{\circ}\sigma(S^r)){\circ}\alpha \times \sigma(S^r){\circ}\alpha){\circ}H_t(2u{\wedge}x), & \text{if } u \le \frac{1}{2} \\ (\hat{\chi}_\alpha(2u-1, x_1), \hat{\chi}_\alpha(2u-1, x_2)), & \text{if } u \ge \frac{1}{2}, \end{cases}$$

where H_t is a homotopy $\Delta_{S^t} \sim \mu_t$ in $S^t{\times}S^t$, $\mu_k = \Sigma^{k-1}\mu_1 : S^k \to S^k \vee S^k$ denotes the unique co-H-structure of S^k and $\hat{\chi}_\alpha$ is a null-homotopy $\sigma_\infty{\circ}\chi_\alpha : (C(S^t), S^t) \to (Q, S^r) \to (P^\infty(\Omega Q), \mathrm{im}(j^Q_1{\circ}\sigma(S^r)))$ of $j^Q_1{\circ}\sigma(S^r){\circ}\alpha \sim *$.

Since the lower left square of diagram (4.2) is a homotopy pullback diagram, σ'_0 and the identity 1_Q defines a lifting $\sigma_0 : Q \to P^2(\Omega Q)$ of 1_Q.

Proof of Proposition 4.1. By [6, Lemma 2.1] with $(X, A) = (P^\infty(\Omega Q), \{*\})$, $(Y, B) = (P^\infty(\Omega Q), \Sigma\Omega Q)$, $Z = P^\infty(\Omega Q)$ and $f = g = 1_{P^\infty(\Omega Q)}$, we have the following homotopy pushout-pullback diagram:

$$\begin{array}{ccccc}
E^2(\Omega Q) & \longrightarrow & \{*\} & & \\
\Big\downarrow{\scriptstyle p_1^{\Omega Q}} & HPO & \Big\downarrow & & \\
\Sigma\Omega Q & \longrightarrow & P^2(\Omega Q) & \xrightarrow{\hat{\Delta}_Q} & P^\infty(\Omega Q)\times\Sigma\Omega Q \cup \{*\}\times P^\infty(\Omega Q) \\
 & & \Big\downarrow{\scriptstyle e_2^Q} & HPB & \Big\downarrow{\scriptstyle e_\infty^Q\times e_1^Q\cup *\times e_\infty^Q} \\
 & & Q & \xrightarrow[\Delta_Q]{} & Q\times Q,
\end{array} \tag{4.3}$$

where we replaced $P^\infty(\Omega Q)$ by Q in the bottom, since $P^\infty(\Omega Q)$ is the homotopy equivalent with Q by $e_\infty^Q : P^\infty(\Omega Q) \to Q$ and $\sigma_\infty : Q \to P^\infty(\Omega Q)$.

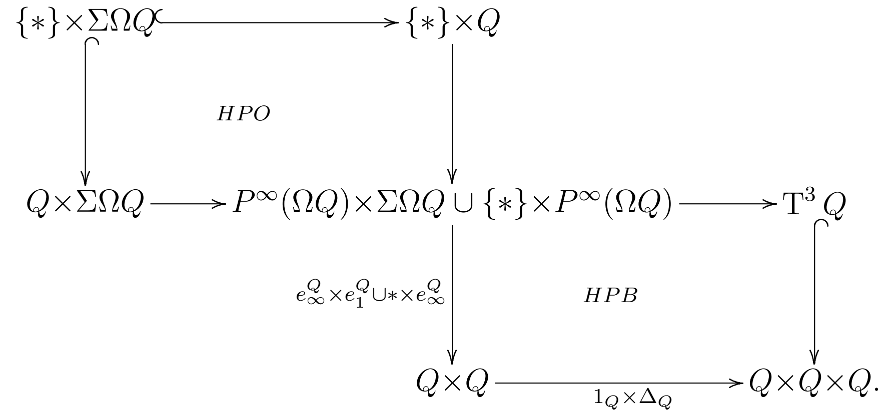
By [6, Lemma 2.1] with $(X, A) = (P^\infty(\Omega Q), \{*\})$, $(Y, B) = (P^\infty(\Omega Q), \Sigma\Omega Q)$, $Z = \{*\}$ and $f = g = *$, we have the following homotopy pushout-pullback diagram:

$$\begin{array}{ccccc}
\Omega Q\times E^2(\Omega Q) & \xrightarrow[\mathrm{pr}_1]{} & \Omega Q & & \\
\Big\downarrow{\scriptstyle \mathrm{pr}_2} & HPO & \Big\downarrow & & \\
E^2(\Omega Q) & \longrightarrow & \Omega Q * E^2(\Omega Q) & \xrightarrow{[j_1^Q, j_2^Q\circ\chi_{p_1^{\Omega Q}}]^{\mathrm{rel}}} & P^\infty(\Omega Q)\times\Sigma\Omega Q \cup \{*\}\times P^\infty(\Omega Q) \\
 & & \Big\downarrow & HPB & \Big\downarrow{\scriptstyle e_\infty^Q\times e_1^Q\cup *\times e_\infty^Q} \\
 & & \{*\} & \longrightarrow & Q\times Q,
\end{array} \tag{4.4}$$

where $\chi_{p_1^{\Omega Q}} : (C(E^2(\Omega Q)), E^2(\Omega Q)) \to (P^2(\Omega Q), \Sigma\Omega Q)$ is a relative homeomorphism.

The above constructions give a standard ΩQ-projective plane $P^2(\Omega Q)$ and a standard projection $p_2^{\Omega Q} : E^3(\Omega Q) \to P^2(\Omega Q)$. In fact, the diagonal map $\Delta_Q^3 : Q \to Q\times Q\times Q$ is the composition $(1_Q\times\Delta_Q)\circ\Delta_Q$ and there is the following homotopy pushout-pullback diagram by [6, Lemma 2.1] with $(X, A) = (Q, \{*\})$,

$(Y, B) = (Q{\times}Q, Q{\vee}Q)$, $Z = Q{\times}Q$, $f = \mathrm{pr}_1$ and $g = \Delta_Q{\circ}\,\mathrm{pr}_2$:



By combining this diagram with diagrams (4.3) and (4.4), we obtain the desired diagram. *QED.*

Since there is a right action of $S^t{\times}S^t$ on $S^r{\times}S^r$ by $\Psi^2 = (\Psi{\times}\Psi){\circ}(1{\times}T{\times}1) : S^r{\times}S^r{\times}S^t{\times}S^t \to S^r{\times}S^r$, we obtain the following.

Proposition 4.3 *The map* $\sigma_0'{\circ}\psi : S^t \to P^\infty(\Omega Q){\times}\Sigma\Omega Q \cup \{*\}{\times}P^\infty(\Omega Q)$ *satisfies*

$$\sigma_0'{\circ}\psi \sim (((j_1^Q{\circ}\sigma(S^r)) \times \sigma(S^r)){\circ}\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha)){\circ}[\mu_r, C(\mu_t)]^{\mathrm{rel}},$$

where $\Psi_0^2 = \Psi^2|_{(S^r\vee S^r)\times(S^t\vee S^t)} : (S^r{\vee}S^r){\times}(S^t{\vee}S^t) \to S^r{\vee}S^r$.

Proof. By (4.1), we know $\sigma_0'{\circ}\psi = \sigma_0'{\circ}(\Psi \cup \chi_\alpha){\circ}[\iota_r, C(\iota_t)]^{\mathrm{rel}} = \sigma_0'{\circ}(\Psi \cup \chi_\alpha) = (\sigma_0'|_{\mathrm{im}\,\sigma(S^r)}{\circ}\Psi \cup \sigma_0'{\circ}\chi_\alpha){\circ}[\iota_r, C(\iota_t)]^{\mathrm{rel}}$, where we have

$$\sigma_0'|_{\mathrm{im}\,\sigma(S^r)}{\circ}\Psi = j_1^Q{\circ}\sigma(S^r){\circ}\Delta_{S^r}{\circ}\Psi = j_1^Q{\circ}\sigma(S^r){\circ}\Psi^2{\circ}(\Delta_{S^r}{\times}\Delta_{S^t}) \quad \text{and}$$
$$\sigma_0'{\circ}\chi_\alpha = ((j_1^Q{\circ}\sigma(S^r)){\circ}\alpha{\times}(j_1^Q{\circ}\sigma(S^r)){\circ}\alpha){\circ}H_t + (\hat{\chi}_\alpha{\vee}\hat{\chi}_\alpha){\circ}C(\mu_t),$$

where the addition denotes the composition of homotopies. Using the same homotopy $H_t : \Delta_{S^t} \sim \mu_t$, we obtain homotopies

$$\sigma_0'|_{\mathrm{im}\,\sigma(S^r)}{\circ}\Psi \sim j_1^Q{\circ}\sigma(S^r){\circ}\Psi^2{\circ}(\Delta_{S^r}{\times}\mu_t) \quad \text{and} \quad \sigma_0'{\circ}\chi_\alpha \sim (\hat{\chi}_\alpha{\vee}\hat{\chi}_\alpha){\circ}C(\mu_t)$$

which fit together into a homotopy

$$\sigma_0'{\circ}(\Psi \cup \chi_\alpha) \sim (((j_1^Q{\circ}\sigma(S^r)) \times \sigma(S^r)){\circ}\Psi^2{\circ}(\Delta_{S^r}{\times}\mu_t) \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha){\circ}C(\mu_t)).$$

Then the homotopy $H_r : \Delta_{S^r} \sim \mu_r$ gives the homotopy relation

$$\sigma_0'{\circ}\psi \sim (((j_1^Q{\circ}\sigma(S^r)) \times \sigma(S^r)){\circ}\Psi^2{\circ}(\mu_r{\times}\mu_t) \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha){\circ}C(\mu_t)){\circ}[\iota_r, C(\iota_t)]^{\mathrm{rel}},$$

which yields $\sigma_0'\circ\psi \sim (((j_1^Q\circ\sigma(S^r))\times\sigma(S^r))\circ\Psi_0^2\cup(\hat{\chi}_\alpha\vee\hat{\chi}_\alpha))\circ[\mu_r, C(\mu_t)]^{\mathrm{rel}}$. *QED.*

Hence by the definition of σ_0 and ψ, we obtain the following proposition.

Proposition 4.4 *We have* $\hat{\Delta}_Q\circ p_2^{\Omega Q}\circ H_2^{\sigma_0}(\psi) \sim [j_1^Q\circ\sigma(S^r), \hat{\chi}_\alpha]^{\mathrm{rel}}$.

Proof. By the definition of σ_0, we obtain

$$\hat{\Delta}_Q\circ\sigma_0\circ\psi \sim \sigma_0'\circ\psi \sim (((j_1^Q\circ\sigma(S^r)) \times \sigma(S^r))\circ\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha))\circ[\mu_r, C(\mu_t)]^{\mathrm{rel}}.$$

Let $\mathrm{in}_i : Z \to Z\vee Z$ be the inclusion to the i-th factor. Then $[\mu_r, C(\mu_t)]^{\mathrm{rel}}$: $S^{r+t} \to (S^r\vee S^r)\times(S^t\vee S^t)$ can be deformed as

$$\begin{aligned}
[\mu_r, C(\mu_t)]^{\mathrm{rel}} &\sim [\mathrm{in}_1\circ\iota_r + \mathrm{in}_2\circ\iota_r, \mathrm{in}_1\circ C(\iota_t) + \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
&\sim [\mathrm{in}_1\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} + [\mathrm{in}_2\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
&\qquad + [\mathrm{in}_2\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} + [\mathrm{in}_1\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
&\sim [\mathrm{in}_1\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} + [\mathrm{in}_2\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
&\qquad + [\mathrm{in}_2\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} + [\mathrm{in}_1\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}}
\end{aligned}$$

in $(S^r\vee S^r)\times(S^t\vee S^t)$. Thus we have

$$\begin{aligned}
\hat{\Delta}_Q\circ\sigma_0\circ\psi \sim{}& (((j_1^Q\circ\sigma(S^r)) \times \sigma(S^r))\circ\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha))\circ[\mathrm{in}_1\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} \\
&+ (((j_1^Q\circ\sigma(S^r)) \times \sigma(S^r))\circ\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha))\circ[\mathrm{in}_2\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
&+ (((j_1^Q\circ\sigma(S^r)) \times \sigma(S^r))\circ\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha))\circ[\mathrm{in}_2\circ\iota_r, \mathrm{in}_1\circ C(\iota_t)]^{\mathrm{rel}} \\
&+ (((j_1^Q\circ\sigma(S^r)) \times \sigma(S^r))\circ\Psi_0^2 \cup (\hat{\chi}_\alpha \vee \hat{\chi}_\alpha))\circ[\mathrm{in}_1\circ\iota_r, \mathrm{in}_2\circ C(\iota_t)]^{\mathrm{rel}} \\
\sim{}& \mathrm{in}_1\circ(j_1^Q\circ\sigma(S^r)\circ\Psi \cup \hat{\chi}_\alpha)\circ[\iota_r, C(\iota_t)]^{\mathrm{rel}} \\
&+ \mathrm{in}_2\circ(j_1^Q\circ\sigma(S^r)\circ\Psi \cup \hat{\chi}_\alpha)\circ[\iota_r, C(\iota_t)]^{\mathrm{rel}} \\
&+ [\hat{\chi}_\alpha, j_1^Q\circ\sigma(S^r)]^{\mathrm{rel}}\circ\hat{T} + [j_1^Q\circ\sigma(S^r), \hat{\chi}_\alpha]^{\mathrm{rel}},
\end{aligned}$$

where $\hat{T} : S^{r+t} = S^{r-1}*S^t \to S^t*S^{r-1} = S^{r+t}$ is a switching map. Since $[\hat{\chi}_\alpha, j_1^Q\circ\sigma(S^r)]^{\mathrm{rel}} \sim *$ in $P^\infty(\Omega Q)\times\Sigma\Omega Q \cup \{*\}\times P^\infty(\Omega Q)$, we obtain

$$\hat{\Delta}_Q\circ\sigma_0\circ\psi \sim \mathrm{in}_1\circ\sigma_\infty\circ\psi + \mathrm{in}_2\circ\sigma_\infty\circ\psi + [j_1^Q\circ\sigma(S^r), \hat{\chi}_\alpha]^{\mathrm{rel}}.$$

On the other hand, we have

$$\begin{aligned}
\hat{\Delta}_Q\circ\Sigma\Omega\psi\circ\sigma(S^{r+t}) &= (j_1^Q\times j_1^Q)\circ\Delta_{\Sigma\Omega Q}\circ\Sigma\Omega\psi\circ\sigma(S^{r+t}) \\
&= (j_1^Q\times j_1^Q)\circ(\Sigma\Omega\psi\circ\sigma(S^{r+t}) \times \Sigma\Omega\psi\circ\sigma(S^{r+t}))\circ\Delta_{S^{r+t}} \\
&\sim (j_1^Q\circ\Sigma\Omega\psi\circ\sigma(S^{r+t})\vee j_2^Q\circ\Sigma\Omega\psi\circ\sigma(S^{r+t}))\circ\mu_{r+t} \\
&= \mathrm{in}_1\circ\sigma_\infty\circ\psi + \mathrm{in}_2\circ\sigma_\infty\circ\psi.
\end{aligned}$$

Since $p_2^{\Omega Q}\circ H_2^{\sigma_0}(\psi)$ is the difference between $\sigma_0\circ\psi$ and $j_1^Q\circ\Sigma\Omega\psi\circ\sigma(S^{r+t})$, we have the desired homotopy relation. *QED.*

Next we show the following description of $\hat{\chi}_\alpha$ up to homotopy.

Proposition 4.5 *For some $\delta_0 : S^{t+1} \to \Sigma\Omega Q$, there is a homotopy relation*

$$\hat{\chi}_\alpha \sim j_2^Q \circ \chi_{p_1^{\Omega Q}} \circ C(H_1(\alpha)) + j_1^Q \circ \delta_0 : (C(S^t), S^t) \to (P^\infty(\Omega Q), \mathrm{im}(j_1^Q \circ \sigma(S^r))),$$

where the addition is given by the coaction $(C(S^t), S^t) \to (C(S^t) \vee S^{t+1}, S^t)$.

Proof. Let $\chi'_\alpha : (C(S^t), S^t) \to (P^2(\Omega Q), \Sigma\Omega Q)$ be the map given by the deformation of α to $p_1^{\Omega Q} \circ H_1(\alpha)$ in $\Sigma\Omega Q$ and by $\chi_{p_1^{\Omega Q}} \circ C(H_1(\alpha)) : (C(S^t), S^t) \to (P^2(\Omega Q), \Sigma\Omega Q)$ as in [6, Lemma 5.4, Remark 5.5], where we denote by C the functor taking cones. Then by definition, we have $\chi'_\alpha \sim \chi_{p_1^{\Omega Q}} \circ C(H_1(\alpha))$ in $(P^2(\Omega Q), \Sigma\Omega Q)$ and $j_1^Q \circ \chi'_\alpha|_{S^t} = j_1^Q \circ \sigma(S^r) \circ \alpha = \hat{\chi}_\alpha|_{S^t}$. Hence the difference between $\hat{\chi}_\alpha$ and $j_2^Q \circ \chi'_\alpha$ is given by a map $\delta : S^{t+1} \to P^\infty(\Omega Q) \simeq Q$, which can be pulled back to $\delta_0 : S^{t+1} \to \Sigma\Omega Q$ ($\subset P^2(\Omega Q)$) (see the proof of [6, Theorem 5.6]). Thus we have $\hat{\chi}_\alpha \sim j_2^Q \circ \chi'_\alpha + j_1^Q \circ \delta_0 \sim j_2^Q \circ \chi_{p_1^{\Omega Q}} \circ C(H_1(\alpha)) + j_1^Q \circ \delta_0$. *QED.*

Now we prove Lemma 2.6 using Propositions 4.1, 4.4 and 4.5:

$$\begin{aligned}
[j_1^Q, j_2^Q \circ \chi_{p_1^{\Omega Q}}]^{\mathrm{rel}} \circ H_2^{\sigma_0}(\psi) &\sim \hat{\Delta}_Q \circ p_2^{\Omega Q} \circ H_2^{\sigma_0}(\psi) \sim [j_1^Q \circ \sigma(S^r), \hat{\chi}_\alpha]^{\mathrm{rel}} \\
&\sim [j_1^Q \circ \sigma(S^r), j_2^Q \circ \chi_{p_1^{\Omega Q}} \circ C(H_1(\alpha))]^{\mathrm{rel}} + [j_1^Q \circ \sigma(S^r), j_1^Q \circ \delta_0] \\
&= \pm [j_1^Q, j_2^Q \circ \chi_{p_1^{\Omega Q}}]^{\mathrm{rel}} \circ (\hat{i} * 1_{\Omega Q * \Omega Q}) \circ (1_{S^{r-1}} * H_1(\alpha)) + (j_1^Q \vee j_1^Q) \circ [\sigma(S^r), \delta_0].
\end{aligned}$$

Since $[\sigma(S^r), \delta_0] \sim 0$ in $\Sigma\Omega Q \times \Sigma\Omega Q$, we proceed as

$$[j_1^Q, j_2^Q \circ \chi_{p_1^{\Omega Q}}]^{\mathrm{rel}} \circ H_2^{\sigma_0}(\psi) \sim \pm [j_1^Q, j_2^Q \circ \chi_{p_1^{\Omega Q}}]^{\mathrm{rel}} \circ (\hat{i} * 1_{\Omega Q * \Omega Q}) \circ \Sigma^r H_1(\alpha).$$

Since the relative Whitehead product $[j_1^Q, j_2^Q \circ \chi_{p_1^{\Omega Q}}]^{\mathrm{rel}}$ induces a split monomorphism in homotopy groups, we have $H_2^{\sigma_0}(\psi) \sim \pm (\hat{i} * 1_{\Omega Q * \Omega Q}) \circ \Sigma^r H_1(\alpha)$. Thus we obtain $H_2^S(\psi) \ni [H_2^{\sigma_0}(\psi)] = \pm [(\hat{i} * 1_{\Omega Q * \Omega Q}) \circ \Sigma^r H_1(\alpha)]$. This completes the proof of Lemma 2.6.

5 Proof of Theorem 2.5

In this section, we always assume that $\beta : S^v \to S^{r+t}$ is a co-H-map and $v < t + 2r - 1$. If $[\Sigma^r H_1(\alpha) \circ \beta] = 0$, then we have $H_2^S(\psi \circ \beta) \ni [H_2^{\sigma_0}(\psi) \circ \beta] = \pm [(\hat{i} * 1_{\Omega Q * \Omega Q}) \circ \Sigma^r H_1(\alpha) \circ \beta] = 0$ by Lemma 2.6. Hence we show the converse. There are cofibre sequences as follows:

$$S^t \xrightarrow{\alpha} S^r \xhookrightarrow{i} Q \xrightarrow{q} S^{t+1}, \qquad S^{r+t} \xrightarrow{\psi} Q \xhookrightarrow{j} E \xrightarrow{\hat{q}} S^{r+t+1}.$$

By the arguments given in Section 4, we know there are 'standard' structures $\sigma(S^r) : S^r \to P^1(\Omega S^r)$ and $\sigma_0 : Q \to P^2(\Omega Q)$ for $\mathrm{cat}(S^r) = 1$ and $\mathrm{cat}(Q) = 2$, respectively, where $\sigma_0|_{S^r} = \sigma(S^r)$ in $P^2(\Omega Q)$.

Let σ be a structure for $\mathrm{cat}(Q) = 2$ with $H_2^\sigma(\psi)\circ\beta \sim 0$ in $E^3(\Omega Q)$. For dimensional reasons, $\sigma|_{S^r}$ is homotopic to $\sigma(S^r)$ which is given by the bottom-cell inclusion. We regard $e_2^Q : P^2(\Omega Q) \to Q$ as a fibration with fibre $E^3(\Omega Q) \overset{p_2^{\Omega Q}}{\to} P^2(\Omega Q)$ and σ_0 as a cross-section of e_2^Q. Then by the definition of a structure, we have $e_2^Q\circ\sigma \sim 1_Q$. Thus we obtain the following homotopy relations:

$$\sigma|_{S^r} \sim \sigma(S^r) = \sigma_0|_{S^r} \quad \text{in } P^2(\Omega Q), \quad e_2^Q\circ\sigma \sim e_2^Q\circ\sigma_0 = 1_Q.$$

Thus the difference between σ and σ_0 is given by a map $\gamma_0 : S^{t+1} \to P^2(\Omega Q)$ which can be lift to $E^3(\Omega Q)$:

$$\sigma \sim \sigma_0 + \gamma_0 \quad \text{in } P^2(\Omega Q),$$

where the addition is taken by the coaction $\mu : Q \to Q \vee S^{t+1}$ along the collapsing $q : Q \to S^{t+1}$. Thus we obtain that $\sigma\circ\psi \sim \{\sigma_0, \gamma_0\}\circ\mu\circ\psi$ in $P^2(\Omega Q)$, where $\{\sigma_0, \gamma_0\} : Q \vee S^{t+1} \to P^2(\Omega Q)$ is a map given by $\{\sigma_0, \gamma_0\}|_Q = \sigma_0$ and $\{\sigma_0, \gamma_0\}|_{S^{t+1}} = \gamma_0$.

By the definition of ψ, we have $\mathrm{pr}_1\circ\mu\circ\psi \sim \psi$ and $\mathrm{pr}_2\circ\mu\circ\psi \sim q\circ\psi \sim *$, and hence we obtain

$$\mu\circ\psi \sim (\psi \vee *)\circ\mu + a[\iota'_r, \iota''_{t+1}] \quad \text{in } Q \vee S^{t+1} \text{ for some } a \in \mathbb{Z},$$

where $\iota'_r : S^r \hookrightarrow Q \hookrightarrow Q \vee S^{t+1}$ and $\iota''_{t+1} : S^{t+1} \hookrightarrow Q \vee S^{t+1}$ are inclusions. Hence by putting $\gamma = a\gamma_0$, we obtain

$$\sigma\circ\psi \sim \sigma_0\circ\psi + [\sigma(S^r), \gamma] \quad \text{in } P^2(\Omega Q),$$

which yields the following homotopy relation in $P^2(\Omega Q)$ for a co-H-map β:

$$\begin{aligned} p_2^{\Omega Q}\circ H_2^\sigma(\psi)\circ\beta &\sim P^2(\Omega\psi)\circ\sigma(S^{r+t})\circ\beta - \sigma\circ\psi\circ\beta \\ &\sim P^2(\Omega\psi)\circ\sigma(S^{r+t})\circ\beta - (\sigma_0\circ\psi\circ\beta + [\sigma(S^r), \gamma]\circ\beta) \\ &\sim (P^2(\Omega\psi)\circ\sigma(S^{r+t}) - \sigma_0\circ\psi)\circ\beta - [\sigma(S^r), \gamma]\circ\beta \\ &\sim p_2^{\Omega Q}\circ H_2^{\sigma_0}(\psi)\circ\beta - [\sigma(S^r), \gamma]\circ\beta \\ &\sim \pm p_2^{\Omega S^r}\circ\Sigma^r H_1(\alpha)\circ\beta - [\sigma(S^r), \gamma]\circ\beta \end{aligned} \tag{5.1}$$

To proceed, we consider the following commutative ladder of fibre sequences.

$$\begin{array}{ccccccc} \Omega S^r & \hookrightarrow & E^3(\Omega S^r) & \xrightarrow{p_2^{\Omega S^r}} & P^2(\Omega S^r) & \xrightarrow{e_2^{S^r}} & S^r \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega Q & \hookrightarrow & E^3(\Omega Q) & \xrightarrow{p_2^{\Omega Q}} & P^2(\Omega Q) & \xrightarrow{e_2^{Q}} & Q. \end{array}$$

Since the pair $(E^3(\Omega Q), E^3(\Omega S^r))$ is $(t+2r-1)$-connected and $t+1 < r+t < t+2r-1$, $r > 1$, we have $\pi_{t+1}(E^3(\Omega Q)) \cong \pi_{t+1}(E^3(\Omega S^r))$ and $\pi_{r+t}(E^3(\Omega Q)) \cong \pi_{r+t}(E^3(\Omega S^r))$. Since γ can be lift to $E^3(\Omega Q)$ and we know $\pi_{t+1}(E^3(\Omega Q)) \cong \pi_{t+1}(E^3(\Omega S^r))$, we may regard that the image of γ is contained in $P^2(\Omega S^r)$. Hence γ vanishes in $P^\infty(\Omega S^r)$, and so is $[\sigma(S^r), \gamma]$. Thus $[\sigma(S^r), \gamma]$ can be lift to $\hat{\gamma} : S^{t+1} \to E^3(\Omega S^r)$ as $[\sigma(S^r), \gamma] \sim p_2^{\Omega S^r} \circ \hat{\gamma}$ in $P^2(\Omega S^r)$.

Therefore, the hypothesis $H_2^\sigma(\psi) \circ \beta \sim *$ together with the homotopy equation (5.1) implies the homotopy relation

$$p_2^{\Omega S^r}|_{S^{r-1} * E^2(\Omega S^r)} \circ \Sigma^r H_1(\alpha) \circ \beta \sim \pm p_2^{\Omega S^r} \circ \hat{\gamma} \circ \beta \quad \text{in } P^2(\Omega Q). \tag{5.2}$$

Since $p_2^{\Omega Q}$ induces a split monomorphism in homotopy groups and $\pi_v(E^3(\Omega Q)) \cong \pi_v(E^3(\Omega S^r))$ for $v < t + 2r - 1$, (5.2) implies a homotopy relation

$$p_2^{\Omega S^r}|_{S^{r-1} * E^2(\Omega S^r)} \circ \Sigma^r H_1(\alpha) \circ \beta \sim \pm [\sigma(S^r), \gamma] \circ \beta \quad \text{in } P^2(\Omega S^r).$$

To show $\Sigma^r H_1(\alpha) \circ \beta$ is trivial, we use the following proposition obtained by a straight-forward calculation (see Mac Lane [12], Stasheff [19] or [5], for example) of Bar resolution:

Proposition 5.1 *The composition map $\partial : E^{m+1}(\Omega S^r) \xrightarrow{p_m^{\Omega S^r}} P^m(\Omega S^r) \to P^m(\Omega S^r)/\Sigma\Omega S^r \simeq \Sigma E^m(\Omega S^r)$ induces a homomorphism*

$$\partial_* : \tilde{H}_*(\wedge^{m+1} \Omega S^r; \mathbb{Z}) \to \tilde{H}_*(\wedge^m \Omega S^r; \mathbb{Z}),$$

which is given by

$$\partial_*(x^{a_0} \otimes x^{a_1} \otimes \cdots \otimes x^{a_m}) = \sum_{i=1}^{m} (-1)^i x^{a_0} \otimes \cdots \otimes x^{a_{i-1}+a_i} \otimes \cdots \otimes x^{a_m},$$

where $a_0, \cdots, a_m \geq 1$ and $x \in H_{r-1}(\Omega S^r; \mathbb{Z})$ is the generator of the Pontryagin ring $H_*(\Omega S^r; \mathbb{Z})$.

Corollary 5.1.1 *The composition map $\partial' : S^{r-1} * E^2(\Omega S^r) \subset E^3(\Omega S^r) \xrightarrow{\partial} \Sigma E^2(\Omega S^r) \to \Sigma E^2(\Omega S^r)/\Sigma(S^{r-1} * \Omega S^r)$ induces an isomorphism*

$$\partial_* : \tilde{H}_*(S^{r-1} \wedge \Omega S^r \wedge \Omega S^r; \mathbb{Z}) \to \tilde{H}_*((\Omega S^r / S^{r-1}) \wedge \Omega S^r; \mathbb{Z}),$$

which is given by $\partial'_*(x \otimes x^j \otimes x^k) = -x^{j+1} \otimes x^k$ for $j, k \geq 1$.

Thus we obtain a left homotopy inverse of $p_2^{\Omega S^r}|_{S^{r-1} * E^2(\Omega S^r)} : S^{r-1} * E^2(\Omega S^r) \to P^2(\Omega S^r)$ as a composition map $P^2(\Omega S^r) \to P^2(\Omega S^r)/\Sigma\Omega S^r \approx \Sigma E^2(\Omega S^r) \to \Sigma E^2(\Omega S^r)/\Sigma(S^{r-1} * \Omega S^r) \simeq S^{r-1} * E^2(\Omega S^r)$, where the image of $\Sigma^r H_1(\alpha)$ lies in $S^{r-1} * E^2(\Omega S^r)$. On the other hand by the fact that $\operatorname{im} \sigma(S^r) \subset \Sigma\Omega S^r$, we also know that the Whitehead product $[\sigma(S^r), \gamma]$ vanishes in the quotient space

$P^2(\Omega S^r)/\Sigma\Omega S^r$, and hence never appears non-trivially in $S^{r-1}{*}E^2(\Omega S^r)$. Thus we conclude that $\Sigma^r H_1(\alpha)\circ\beta$ is trivial.

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