Ganea's conjecture on Lusternik-Schnirelmann category

Norio Iwase*

Permanent Address: Graduate School of Mathematics, Kyushu University, Japan.
Current: Department of Mathematical Sciences, University of Aberdeen, United Kingdom.
e-mail: n.iwase@maths.abdn.ac.uk

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Abstract

A series of complexes Q_p indexed by all primes p is constructed with $\operatorname{cat} Q_p = 2$ and $\operatorname{cat} Q_p \times S^n = 2$ for either $n \geq 2$ or $n = 1$ and $p = 2$. This disproves Ganea's conjecture on LS category, or Lusternik-Schnirelmann category.

1 Introduction

Problem 2 paused by Ganea [4], Ganea's conjecture on LS category states the following: The LS category of a space is increased by one by taking the product with a sphere. A major advance in this subject has been made by Hess [6] and Jessup [7] working in the rational category: The rational version of the conjecture is true. Also by Singhof [10] and Rudyak [8], the conjecture has been verified for a large class of manifolds.

In this paper, we work in the category of CW complexes with base points and the LS category is considered as *normalized*, i.e., $\operatorname{cat} X$ is the least number n such that the diagonal map $\Delta : X \to X^{n+1}$ can be compressed into the 'fat wedge' $X^{[n+1]}$. Hence $\operatorname{cat}\{*\} = 0$. We introduce the p-local version of category $\operatorname{cat}_p X$ for a nilpotent space X as the least number n such that the diagonal map $\Delta : X \to X^{n+1}$ can be compressed into $X^{[n+1]}$, at the prime p. This immediately implies that $\operatorname{cat}_p X \leq \operatorname{cat} X$ for a nilpotent space X.

Let us recall that an A_∞-space, in the sense of Stasheff [11], is a space with an A_∞-form. Stasheff has shown that any given A_∞-space is homotopy equivalent to the loop space of some space, which is often called the A_∞-structure of the given A_∞-space. Our point of view is the

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other way around: For a given space, its loop space is an A_∞-space with the given space as its A_∞-structure. More precisely, every space X has a filtration given by the projective spaces $P^m(\Omega X)$ of its loop space ΩX. From this point of view, we recover the following fundamental result due to Ganea (see [2]).

Theorem 1.1 *(Ganea) Let X be a connected CW complex. Then* $\operatorname{cat} X \leq m$ *if and only if the canonical inclusion* $e^X_m : P^m(\Omega X) \subset P^\infty(\Omega X) \simeq X$ *has a right homotopy inverse.*

From the product structure of the projective spaces, we have the following well-known facts:

Theorem 1.2 *Let X and Y be connected CW complexes. Then* $\operatorname{cat} X{\times}Y \leq m$ *if and only if the canonical inclusion* $\bigcup_{a+b=m} P^a(\Omega X){\times}P^b(\Omega Y) \subset P^\infty(\Omega X){\times}P^\infty(\Omega Y) \simeq X{\times}Y$ *has a right homotopy inverse.*

Corollary 1.2.1 $\operatorname{cat} X{\times}Y \leq \operatorname{cat} X + \operatorname{cat} Y$*. Hence* $\operatorname{cat} X{\times}S^n$ *is either* $\operatorname{cat} X$ *or* $\operatorname{cat} X + 1$.

Corollary 1.2.2 *Let* $\operatorname{cat} X = m$*. Then* $\operatorname{cat} X{\times}S^n = m$ *if and only if $X{\times}S^n$ is dominated by* $P^m(\Omega X) \cup P^{m-1}(\Omega X){\times}S^n$.

The proofs of the above results suggest that the obstruction to the existence of a compression of X into $P^{m-1}\Omega X$ is given by a map to the m-fold join of ΩX; its n-fold suspension gives the essential obstruction to the existence of a compression of $X{\times}S^n$ into $P^m\Omega X \cup P^{m-1}\Omega X{\times}S^n$. This suggests how one might obtain counter examples to Ganea's conjecture and, using Toda's results on the homotopy groups of spheres, we establish the existence of such examples. Although some results below are well-known to the experts, we reprove them in a manner which illuminates the computations needed for the above counter examples. The main result of this paper is as follows.

Theorem 1.3 *There exists a series of 1-connected 2 cell complexes Q_p indexed by all primes p. For an odd prime p, Q_p satisfies* $\operatorname{cat} Q_p = \operatorname{cat} Q_p{\times}S^n = 2$ *and* $\operatorname{cat}_p Q_p = \operatorname{cat}_p Q_p{\times}S^n = 2$ *for* $n \geq 2$*. For* $p = 2$*, Q_2 satisfies* $\operatorname{cat} Q_2 = \operatorname{cat} Q_2{\times}S^n = 2$ *and* $\operatorname{cat}_2 Q_2 = \operatorname{cat}_2 Q_2{\times}S^n = 2$ *for* $n \geq 1$*. In addition, when $n = 1$ and p odd, we have* $\operatorname{cat} Q_p{\times}S^1 = \operatorname{cat}_p Q_p{\times}S^1 = 3$.

These examples are in a sharp contrast to the Hess-Jessup theorem for rational case (see [7] and [6]), or the Singhof-Rudyak theorem for manifolds (see [10] and [8]). We remark that this construction in the case $p = 2$ is strongly related to the fact that S^7 is a Hopf space but S^{15} is not (Toda [13]), and that we could not give examples of Q at odd primes p with $\operatorname{cat}_p Q{\times}S^1 = \operatorname{cat}_p Q$. They also suggest the following conjecture.

Conjecture 1.4 *If* $\mathrm{cat}\, X{\times}S^k = \mathrm{cat}\, X$ *for some* k*, then* $\mathrm{cat}\, X{\times}S^n = \mathrm{cat}\, X$ *for all* $n \geq k$*.*

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2 Push-out pull-back lemma

Let (X, A) and (Y, B) be CW pairs with $i : A \subset X$ and $j : B \subset Y$ the inclusions. We denote by Ω_i and Ω_j the mapping fibres of i and j. For given $f : Z \to X$ and $g : Z \to Y$, we also define some pull-backs:

$$\begin{aligned}
\Omega_i &= \{(a, \ell_X) \in A \times L(X) | {*} = \ell_X(0), i(a) = \ell_X(1)\} \cong \{\ell_X \in L(X) | {*} = \ell_X(0), \ell_X(1) \in A\},\\
\Omega_j &= \{(b, \ell_Y) \in B \times L(Y) | {*} = \ell_Y(0), j(b) = \ell_Y(1)\} \cong \{\ell_Y \in L(Y) | {*} = \ell_Y(0), \ell_Y(1) \in B\},\\
\Omega_{i,f} &= \{(z, \ell_X) \in Z \times L(X) | f(z) = \ell_X(0), \ell_X(1) \in A\},\\
\Omega_{j,g} &= \{(z, \ell_Y) \in Z \times L(Y) | g(z) = \ell_Y(0), \ell_Y(1) \in B\},
\end{aligned}$$

where $L(-)$ denotes the space of free paths on the space $-$. Similarly, for maps $i{\times}j : A{\times}B \subset X{\times}Y$, $k : X{\times}B{\cup}A{\times}Y \subset X{\times}Y$ and $(f, g) = (f{\times}g)\Delta_Z : Z \to X{\times}Y$, we define

$$\begin{aligned}
\Omega_{i\times j} &= \{(\ell_X, \ell_Y) \in L(X) \times L(Y) | {*} = \ell_X(0), {*} = \ell_Y(0), \ell_X(1) \in A, \ell_Y(1) \in B\} = \Omega_i{\times}\Omega_j,\\
\Omega_k &= \{(\ell_X, \ell_Y) \in L(X) \times L(Y) | {*} = \ell_X(0), {*} = \ell_Y(0) \,\text{and}\, (\ell_X(1), \ell_Y(1)) \in A{\times}Y{\cup}X{\times}B\},\\
\Omega_{i\times j,(f,g)} &= \{(z, \ell_X, \ell_Y) \in Z \times L(X) \times L(Y) | f(z) = \ell_X(0), g(z) = \ell_Y(0), (\ell_X, \ell_Y) \in \Omega_{i\times j}\},\\
\Omega_{k,(f,g)} &= \{(z, \ell_X, \ell_Y) \in Z \times L(X) \times L(Y) | f(z) = \ell_X(0), g(z) = \ell_Y(0), (\ell_X, \ell_Y) \in \Omega_k\}.
\end{aligned}$$

Then there are natural projections $\phi : \Omega_{i\times j,(f,g)} \to \Omega_{i,f}$ and $\psi : \Omega_{i\times j,(f,g)} \to \Omega_{j,g}$ given by

$$\phi(z, \ell_X, \ell_Y) = (z, \ell_X), \quad \psi(z, \ell_X, \ell_Y) = (z, \ell_Y).$$

We establish the following lemma.

Lemma 2.1 *Let* (X, A) *and* (Y, B) *be connected CW pairs and* Z *a connected CW complex with maps* $f : Z \to X$ *and* $g : Z \to Y$*. Then the homotopy pull-back* $\Omega_{k,(f,g)}$ *of* $(f, g) : Z \to X{\times}Y$ *and* $k : X{\times}B{\cup}A{\times}Y \subset X{\times}Y$ *has naturally the homotopy type of the homotopy push-out of* $\phi : \Omega_{i\times j,(f,g)} \to \Omega_{i,f}$ *and* $\psi : \Omega_{i\times j,(f,g)} \to \Omega_{j,g}$*.*

$$\begin{tikzcd}
\Omega_{i\times j,(f,g)} \arrow[r, "\phi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "HPO"] & \Omega_{i,f} \arrow[d, hook] & \\
\Omega_{j,g} \arrow[r, hook] & \Omega_{k,(f,g)} \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & X\times B \cup A\times Y \arrow[d, hook, "k"] \\
& Z \arrow[r, "(f,g)"] & X\times Y
\end{tikzcd}$$

Proof. We can determine subspaces E_1, E_2 and E_0 in $E = \Omega_{k,(f,g)}$ as follows:

$$\begin{aligned}
E_1 =& \{(z,\ell_X,\ell_Y) \in E | \ell_Y(1) \in B\} \supset \{(z, c(f(z)), \ell_Y) \in E | \ell_Y(0) = g(z), \ell_Y(1) \in B\} \cong \Omega_{j,g},\\
E_2 =& \{(z,\ell_X,\ell_Y) \in E | \ell_X(1) \in A\} \supset \{(z, \ell_X, c(g(z))) \in E | \ell_X(0) = f(z), \ell_X(1) \in A\} \cong \Omega_{i,f},\\
E_0 =& \{(z,\ell_X,\ell_Y) \in E | \ell_X(0) = f(z), \ell_Y(0) = g(z), \ell_X(1) \in A, \ell_Y(1) \in B\} = \Omega_{i\times j,(f,g)},
\end{aligned}$$

where $c(w)$ denotes the constant path at w. Then we have $E = E_1 \cup E_2$ and $E_1 \cap E_2 = E_0$. We can easily show that $\Omega_{j,g}$ and $\Omega_{i,f}$ are deformation retracts of E_1 and E_2, respectively. Also, the inclusions of E_0 in E_1 and E_2 are, up to homotopy, given by ψ and ϕ. Hence E has the homotopy type of the (unreduced) homotopy push-out $\Omega_{j,g} \cup \{[0,1] \times \Omega_{i\times j,(f,g)}\} \cup \Omega_{i,f}$. *QED.*

3 Proof of Theorem 1.1

Let E^{m+1} be the homotopy fibre of the inclusion $X^{[m+1]} \to X^{m+1}$ and P^m, which is so-called the *Ganea space*, be the homotopy pull-back of

$$\begin{tikzcd}
& X^{[m+1]} \arrow[d, hook] \\
X \arrow[r, "\Delta_{m+1}"] & X^{m+1},
\end{tikzcd}$$

where $X^{[m+1]} = \{(x_0, ...x_m) \in X^{m+1} | x_t = * \text{ for some } t\}$ and Δ_{m+1} denotes the diagonal.

Let us recall that $\operatorname{cat} X \leq m$ if and only if the diagonal map Δ_{m+1} is compressible into $X^{[m+1]}$. The latter condition is clearly equivalent to the existence of a homotopy cross-section of the projection $P^m \to X$.

Now we take $Z = X$, $Y = X^m$, $f = 1_X$, $g = \Delta_m$, $A = \{*\}$, and $B = X^{[m]}$ and we then have

$\Omega_{k,(f,g)} = P^m$, $\Omega_{i,f} \simeq *$, $\Omega_{j,g} = P^{m-1}$ and the following pull-back diagram:

$$\begin{tikzcd}
\Omega_j \arrow[r] \arrow[d, equal] & \Omega_{i\times j,(f,g)} \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & \Omega_{i,f} \arrow[d] \\
\Omega_j \arrow[r] & \Omega_{j,g} \arrow[r] & Z.
\end{tikzcd}$$

Since $f = 1_X$ and $A = \{*\}$, $\Omega_{i,f}$ is contractible, and hence $\Omega_{i\times j,(f,g)}$ is homotopy equivalent to Ω_j the fibre of $\Omega_{j,g} \to Z$, in this case. Here j is the inclusion map $X^{[m]} \subset X^m$, and hence Ω_j is E^m by definition. Thus we have the following push-out and pull-back diagram:

$$\begin{tikzcd}
E^m \arrow[r] \arrow[d] \arrow[dr, phantom, "HPO"] & P^{m-1} \arrow[d, hook] & \\
\{*\} \arrow[r, hook] & P^m \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & X\times X^{[m]} \cup \{*\}\times X^m \arrow[d, hook, "k"] \\
 & X \arrow[r, "\Delta_{m+1}"] & X\times X^m
\end{tikzcd}$$

Hence P^m has the homotopy type of a (unreduced) mapping cone of the canonical inclusion $E^m \subset P^{m-1}$, $m \geq 1$.

Similarly using Lemma 2.1, we have the following push-out and pull-back diagram:

$$\begin{tikzcd}
\Omega X\times E^m \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "HPO"] & E^m \arrow[d, hook] & \\
\Omega X \arrow[r, hook] & E^{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & X\times X^{[m]} \cup \{*\}\times X^m \arrow[d, hook, "k"] \\
 & \{*\} \arrow[r, "*"] & X\times X^m
\end{tikzcd}$$

Hence E^{m+1} has the homotopy type of the (unreduced) join of ΩX and E^m. This implies that $\{(E^{m+1}, P^m); m \geq 0\}$ gives the A_∞-structure for ΩX in the sense of Stasheff. Thus P^m has the homotopy type of $P^m(\Omega X)$ the ΩX-projective m-space. This implies Theorem 1.1.

4 Product formulas

Firstly we prove Theorem 1.2. We define a modified A_∞-structure for $\Omega X \times \Omega Y$ as follows:

$$\begin{aligned}\hat{P}^m &= \bigcup_{a+b=m} P^a(\Omega X)\times P^b(\Omega Y) \subset P^\infty(\Omega X)\times P^\infty(\Omega Y),\\ \hat{E}^{m+1} &= \bigcup_{a+b=m} E^{a+1}(\Omega X)\times E^{b+1}(\Omega Y) \subset E^\infty(\Omega X)\times E^\infty(\Omega Y).\end{aligned}$$

Then we immediately obtain that $\hat{E}^m$ is contractible in $\hat{E}^{m+1}$ and $\hat{P}^{m+1}$ has the homotopy type of the mapping cone of the projection $\hat{E}^{m+1} \to \hat{P}^m$. By Stasheff [11], this gives an A_∞-structure for $\Omega X \times \Omega Y$ and the inclusion $P^m(\Omega X \times \Omega Y) \subset P^\infty(\Omega X \times \Omega Y) = P^\infty(\Omega X)\times P^\infty(\Omega Y)$ can be deformed into the subspace $\hat{P}^m \subset P^\infty(\Omega X)\times P^\infty(\Omega Y)$. Also we know that $\operatorname{cat} \hat{P}^m \leq m$. Then by Theorem 1.1, Theorem 1.2 follows.

Remark 4.1 *Since $\hat{P}^m$ has the homotopy type of the mapping cone of $\hat{E}^m \to \hat{P}^{m-1}$, $\operatorname{cat} \hat{P}^m \leq m$ for all $m \leq \infty$.*

This immediately implies Corollary 1.2.1.

Next we show Corollary 1.2.2: Let X satisfy $\operatorname{cat} X \times S^n = m = \operatorname{cat} X$. Then by Theorem 1.2, $X \times S^n$ is dominated by $\bigcup_{a+b=m} P^a(\Omega X)\times P^b(\Omega S^n)$ and hence by $P^m(\Omega X)\times\{*\} \cup P^{m-1}(\Omega X)\times P^\infty(\Omega S^n) \simeq P^m(\Omega X) \cup P^{m-1}(\Omega X)\times S^n$. This implies the Corollary 1.2.2.

5 Counter Examples to Ganea's conjecture

To show Theorem 1.3, it is sufficient to construct the following

Example 5.1 *1) For an odd prime p, let α be the generator of the p-primary summand of $\pi_{4p-3}(S^2)$ which is isomorphic with $\mathbb{Z}/p\mathbb{Z}$ and $Q_p = S^2 \cup_\alpha e^{4p-2}$. Then $\operatorname{cat} Q_p = \operatorname{cat}_p Q_p = 2$ and $\operatorname{cat} Q_p \times S^1 = \operatorname{cat}_p Q_p \times S^1 = 3$, but $\operatorname{cat} Q_p \times S^n = \operatorname{cat}_p Q_p \times S^n = 2$ for $n \geq 2$.*

2) For the prime 2, let α be the generator of the direct summand $\mathbb{Z}/4\mathbb{Z}$ of $\pi_{29}(S^8) \cong \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ and $Q_2 = S^8 \cup_{2\alpha} e^{30}$. Then $\operatorname{cat} Q_2 = \operatorname{cat}_2 Q_2 = 2$, while $\operatorname{cat} Q_2 \times S^n = \operatorname{cat}_2 Q_2 \times S^n = 2$ for $n \geq 1$.

In each case, we know that $\operatorname{cat}_q Q_p = 1$ and $\operatorname{cat}_q Q_p \times S^n = 2$ for $0 \leq q \neq p$ and $n \geq 1$.

All the examples in Example 5.1 are obtained by similar methods. We will concentrate on part 1) of Example 5.1. First of all, let us recall that the Hopf map $\eta : S^3 \to S^2$ induces an isomorphism $\eta_* : \pi_*(S^3) \to \pi_*(S^2)$ for $* \geq 3$. In particular, $\eta_* : \pi_{4p-3}(S^3) \to \pi_{4p-3}(S^2) \cong \mathbb{Z}/3\mathbb{Z}$ is an isomorphism. So let α and $\beta = \alpha_1^2(3)$ be the corresponding generators in $\pi_{4p-3}(S^2)$ and $\pi_{4p-3}(S^3)$. Let Q_p be the mapping cone of α. To avoid too much calculation of homotopy groups, we consider β rather than α. We show the following lemma which is well-known for experts.

Lemma 5.2 *The map $\beta = \alpha_1^2(3)$ is not a suspension map but a co-Hopf map of order p, whose iterated suspensions $\Sigma^t\beta$ are trivial for $t \geq 2$ but $\Sigma\beta \neq 0$.*

Proof. We can easily obtain the latter part of the lemma by examing Theorem 13.4 in [14]. In fact, $\pi_{4p-1}(S^5)$ has no elements of order p. Thus $\Sigma^2\beta$ is trivial. However we know that the suspension homomorphism $\pi_*(X) \to \pi_{*+1}(\Sigma X)$ is a split monomorphism for any Hopf space X (due to James). Thus $\pi_{4p-3}(S^3) \to \pi_{4p-2}(S^4)$ is a split monomorphism, and hence, $\Sigma\beta$ gives a non-trivial generator of a direct summand of order p.

Thus it remains to show the first part of the lemma: Since the finite group $\pi_{4p-4}(S^2)$ has no p-torsion, β cannot be a suspension. In [9], Saito has extended the results of Berstein-Hilton [1] which describes the obstruction for a general map to be a co-Hopf map using Ganea's criterion for a co-Hopf space: Let $f : X \to Y$ be a map of simply connected co-Hopf spaces. Then the obstruction to f being a co-Hopf map is an element $H(f) \in [X, \Omega Y{*}\Omega Y]$, where H is the generalised Hopf invariant homomorphism $H : [X,Y] \to [X, \Omega Y{*}\Omega Y]$. In our case, $H(\beta)$ lies in

$$\begin{aligned}
\pi_{4p-3}(\Omega S^3{*}\Omega S^3) &\cong \pi_{4p-3}(\Omega S^3 \wedge \Sigma\Omega S^3)\\
&\cong \pi_{4p-3}(\Omega S^3 \wedge \Sigma(S^2 \cup e^4 \cup ... \cup e^{4p-4} \cup (\text{higher cells} \geq 4p-2)))\\
&\cong \pi_{4p-3}(\Omega S^3 \wedge \Sigma(S^2 \vee S^4 \vee ... \vee S^{4p-4}))\\
&\cong \pi_{4p-3}(\Sigma(S^2 \vee S^4 \vee ... \vee S^{4p-4}) \wedge (S^2 \vee S^4 \vee ... \vee S^{4p-4}))\\
&\cong \pi_{4p-3}(\Sigma\{S^{2+2} \vee S^{4+2} \vee S^{2+4} \vee ... \vee S^{4p-6+2} \vee ... \vee S^{2+4p-6}\})\\
&\cong \pi_{4p-3}(\Sigma\{S^4 \vee S^6 \vee S^6 \vee ... \vee S^{4p-4} \vee ... \vee S^{4p-4}\})\\
&\cong \pi_{4p-4}(J(S^4 \vee S^6 \vee S^6 \vee ... \vee S^{4p-4} \vee ... \vee S^{4p-4}))\\
&\cong \pi_{4p-4}(J(S^4) \times J(S^6 \vee S^6 \vee ... \vee S^{4p-4} \vee ... \vee S^{4p-4}))\\
&\cong \pi_{4p-4}(J(S^4) \times J(S^6) \times J(S^6) \times ... \times J(S^{4p-4}) \times ... \times J(S^{4p-4}))\\
&\cong \pi_{4p-3}(S^5) \oplus \pi_{4p-3}(S^7) \oplus \pi_{4p-3}(S^7) \oplus ... \oplus \pi_{4p-3}(S^{4p-3}) \oplus ... \oplus \pi_{4p-3}(S^{4p-3}),
\end{aligned}$$

which has no element of order p by [14], where $J(X)$ denotes the James' reduced product space of X (see Whitehead [15]). Since the order of β is p, $H(\beta)$ is trivial and we obtained the lemma. *QED.*

We show the following proposition which was shown by Gilbert [5] working with the notion of *wcat*.

Proposition 5.3 *The map* $\alpha = \eta\beta = \eta\alpha_1^2(3)$ *is not a co-H-map and the obstruction is described by the 2nd James-Hopf invariant* $h_2(\alpha) = \beta$*, which is a generator of the p-primary summand of* $\pi_{4p-3}(S^3)$ *which is isomorphic with* $\mathbb{Z}/p\mathbb{Z}$*:*

$$\mu_2\alpha \simeq (\alpha \vee \alpha)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta$$

where we denote by $\mu_k : S^k \to S^k \vee S^k$ *the (unique) co-Hopf structure of the sphere* S^k *and by* $+_k$ *the multiplication induced by the co-Hopf structure of sphere* S^k*.*

Proof. There is a well-known formula for the Hopf map η:

$$\mu_2\eta \simeq (\eta \vee \eta)\mu_3 +_3 [i_1, i_2]$$

in $\pi_3(S^2 \vee S^2)$ where $i_t : X \to X \vee X$ is the inclusion to the t-th factor. Since $\alpha \simeq \eta\beta$, we have the homotopy relation $\mu_2\alpha \simeq \mu_2\eta\beta \simeq \{(\eta \vee \eta)\mu_3 +_3 [i_1, i_2]\}\beta$ in $\pi_{4p-3}(S^2 \vee S^2)$. Since β is a co-Hopf map by Lemma 5.2, this is homotopy equivalent to

$$(\eta \vee \eta)\mu_3\beta +_{4p-3} [i_1, i_2]\beta \simeq (\eta\beta \vee \eta\beta)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta \simeq (\alpha \vee \alpha)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta.$$

This implies that $h_2(\alpha) \simeq \beta$ which gives the obstruction to α being a co-Hopf map and $h_k(\alpha) = 0$ for $k \geq 3$. *QED.*

To determine the LS category of Q_p and $Q_p \times S^n$, we need to show the following lemma.

Lemma 5.4 *The following diagram, without the dotted arrow, commutes up to homotopy.*

$$\begin{tikzcd}
S^{4p-3} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & S^2 \arrow[r, hook, "i"] \arrow[dd, hook, "\Sigma\Omega i\Sigma j_1"] & Q_p \arrow[dd, "1_{Q_p}"] \arrow[ddd, dashed, bend right, "\lambda"'] \\
S^1 * S^1 \arrow[d, "(\Omega i * \Omega i)(j_1 * j_1)"'] & & \\
\Omega Q_p * \Omega Q_p \arrow[r, "p_1^{Q_p}"] & \Sigma\Omega Q_p \arrow[r, "ev_{Q_p}"] \arrow[dr, "\iota_1^{Q_p}"'] & Q_p \\
& & P^2\Omega Q_p \arrow[u, "e_2^{Q_p}"']
\end{tikzcd}$$

where $i : S^2 \to Q_p$ *and* $j_t : S^t \to \Omega S^{t+1}$ *give the bottom cell inclusions and* $p_1^{Q_p}$ *denotes the Hopf construction of the loop addition of* ΩQ_p*,* $\iota_1^{Q_p} : \Sigma\Omega Q_p \to P^2\Omega Q_p$ *denotes the inclusion to the mapping cone of* $p_1^{Q_p}$ *and* $e_t^{Q_p} : P^t\Omega Q_p \subset P^\infty\Omega Q_p \simeq Q_p$ *denotes the canonical inclusion.*

Proof. The commutativity of the right half square of the diagram and the triangle below are clear. So we concentrate on showing the commutativity of the left half square of the diagram. There is the following homotopy commutative diagram due to Ganea:

$$\begin{tikzcd}
S^1 * S^1 \arrow[r, "{[i_1,i_2]}"] \arrow[d, hook, "(\Omega i * \Omega i)(j_1 * j_1)"'] & S^2 \vee S^2 \arrow[r, hook] \arrow[d, hook, "i \vee i"'] & S^2 \times S^2 \arrow[d, hook, "i \times i"'] \\
\Omega Q_p * \Omega Q_p \arrow[r, "q_1^{Q_p}"] & Q_p \vee Q_p \arrow[r, hook] & Q_p \times Q_p.
\end{tikzcd}$$

By Proposition 5.3, we have

$$(i \vee i)\mu_2 \alpha \simeq (i\alpha \vee i\alpha)\mu_{4p-3} +_{4p-3} (i \vee i)[i_1, i_2]\beta \simeq (i \vee i)[i_1, i_2]\beta \simeq q_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta.$$

Also Ganea showed, for any co-Hopf space X, that there exists a map (shown as a dotted arrow) corresponding uniquely to the co-Hopf structure so that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
 & \Omega X * \Omega X \arrow[r, equal] \arrow[dd, "p_1^X"] & \Omega X * \Omega X \arrow[dd, "q_1^X"] \\
X \arrow[rrd, "\mu_X"] \arrow[rd, dashed] \arrow[rdd, "1_X"'] & & \\
 & \Sigma\Omega X \arrow[r, "\pi^X"] \arrow[d, "ev_X"] \arrow[rd, phantom, "HPB"] & X \vee X \arrow[d, hook] \\
 & X \arrow[r, "\Delta_X"] & X \times X.
\end{tikzcd} \tag{5.1}$$

Since a sphere has a unique co-Hopf structure, we have $\mu_2 \simeq \pi^{S^2}\Sigma j_1$ and hence

$$(i \vee i)\mu_2 \simeq (i \vee i)\pi^{S^2}\Sigma j_1 \simeq \pi^{Q_p}\Sigma\Omega i \Sigma j_1.$$

Thus we get the following relation:

$$\pi^{Q_p}\Sigma\Omega i\Sigma j_1\alpha \simeq (i \vee i)\mu_2\alpha \simeq q_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta \simeq \pi^{Q_p}p_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta.$$

Here the diagram 5.1 is a pull-back diagram. Since q_1^X induces a split monomorphism on homotopy groups, $\Sigma\Omega i\Sigma j_1\alpha$ is determined, up to homotopy, by the equations

$$\begin{aligned}
ev_{Q_p}\Sigma\Omega i\Sigma j_1\alpha &\simeq iev_{S^2}\Sigma j_1\alpha \simeq i1_{S^2}\alpha \simeq * \simeq ev_{Q_p}p_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta, \\
\pi^{Q_p}\Sigma\Omega i\Sigma j_1\alpha &\simeq \pi^{Q_p}p_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta.
\end{aligned}$$

Therefore we have that $\Sigma\Omega i\Sigma j_1\alpha \simeq p_1^{Q_p}(\Omega i * \Omega i)(j_1 * j_1)\beta$. *QED.*

Remark 5.5 *There exists a map* $\lambda : Q_p \to P^2\Omega Q_p$ *given by the homotopy deforming* $\Sigma\Omega i\Sigma j_1\alpha$ *in* $\Sigma\Omega Q_p$ *to* $\alpha' = p_1^{Q_p}|_{S^1*S^1}\beta$ *and by* $\hat{\chi}|_{C(S^1*S^1)}C(\beta)$*, where we denote by* C *the functor taking cones and* $\hat{\chi} : (C(\Omega Q_p*\Omega Q_p), \Omega Q_p*\Omega Q_p) \to (P^2\Omega Q_p, \Sigma\Omega Q_p)$ *the characteristic map of the attached cone of the mapping cone space* $P^2\Omega Q_p$ *of* $p_1^{Q_p}$.

The following theorem is a special case of a result of Berstein-Hilton [1], or Gilbert [5]. However we include a proof as it contains the idea used to determine the LS category of $Q_p{\times}S^n$.

Theorem 5.6 $\operatorname{cat}_p Q_p = \operatorname{cat} Q_p = 2$ *but* $\operatorname{cat}_q Q_p = 1$ *for* $q \neq p$.

Proof. For the prime p, we compute the homotopy group $\pi_{4p-3}(\Omega Q_p*\Omega Q_p)$, where the element $(\Omega i*\Omega i)(j_1*j_1)\beta$ lies:

$$\begin{aligned}\pi_{4p-3}(\Omega Q_p*\Omega Q_p) &\cong \pi_{4p-3}((\Omega S^2 \cup (\text{higher cells} \geq 4p-3))\wedge\Sigma(\Omega S^2 \cup (\text{higher cells} \geq 4p-3)))\\ &\cong \pi_{4p-3}(\Omega S^2\wedge\Sigma\Omega S^2)\\ &\cong \pi_{4p-3}(\Sigma\{S^{1+1} \vee S^{2+1} \vee S^{1+2} \vee (\text{higher spheres} \geq 4)\}).\end{aligned}$$

Hence $\pi_{4p-3}(S^1*S^1)$ is a direct summand of $\pi_{4p-3}(\Omega Q_p*\Omega Q_p)$. As $(\Omega i*\Omega i)(j_1*j_1)$ is the bottom cell inclusion, $(\Omega i*\Omega i)(j_1*j_1)\beta$ gives a generator of p-torsion subgroup of $\pi_{4p-3}(\Omega Q_p*\Omega Q_p)$.

By Sugawara [12], the projection $p_1^{Q_p}$ is a quasi-fibration with the fibre ΩQ_p which is contractible in the total space $\Omega Q_p*\Omega Q_p$. Thus we have the following (split) short exact sequence:

$$0 \to \pi_t(\Omega Q_p*\Omega Q_p) \to \pi_t(\Sigma\Omega Q_p) \to \pi_t(Q_p) \to 0 \tag{5.2}$$

Since $\Sigma\Omega i\Sigma j_1$ is the bottom cell inclusion, it gives a generator of $\pi_2(\Sigma\Omega Q_p) = \mathbb{Z}$. Hence, if $\operatorname{cat} Q_p = 1$, in other words, if Q_p is dominated by $\Sigma\Omega Q_p$, then there is an embedding of Q_p in $\Sigma\Omega Q_p$ whose restriction to S^2 is given by $\Sigma\Omega i\Sigma j_1$ and hence $\Sigma\Omega i\Sigma j_1\alpha$ should be trivial. This contradicts the exactness of (5.2) at $t = 4p-3$, and hence we obtain $\operatorname{cat}_p Q_p = 2$.

On the other hand, if $q \neq p$, then $\beta = 0$ and, by Lemma 5.4, the bottom cell inclusion $\Sigma\Omega i\Sigma j_1$ can be extended to a map $\lambda'_1 : Q_p \to P^2\Omega Q_p$. The difference of 1_{Q_p} and λ'_1 in Q_p is described by $\gamma'_1 \in \pi_{4p-2}(Q_p)$. By the exactness of (5.2) at $t = 4p-2$, γ'_1 can be pulled back on $\Sigma\Omega Q_p$ to $\gamma_1 \in \pi_{4p-2}(\Sigma\Omega Q_p)$. Thus we can obtain the genuine compression λ_1 of 1_{Q_p} to $\Sigma\Omega Q_p$ by adding γ_1 to λ'_1. This implies that $\operatorname{cat}_q Q_p = 1$ for $q \neq p$ and it completes the proof of the theorem. *QED.*

Remark 5.7 *The difference between the identity 1_{Q_p} and the map $e_2^{Q_p}\lambda$ is given by an element $ev_{Q_p}\gamma \in \pi_{4p-2}(Q_p)$, where $\gamma \in \pi_{4p-2}(\Sigma\Omega Q_p)$, since $\pi_{4p-2}(\Sigma\Omega Q_p) \to \pi_{4p-2}(Q_p)$ is a split surjection.*

Finally we calculate the LS category of $Q_p{\times}S^n$. The attaching map of the top cell of $Q_p{\times}S^n$ is the map

$$\hat{\alpha} : S^{4p-2}{*}S^{n-1} = D^{4p-2}{\times}S^{n-1} \cup S^{4p-3}{\times}D^n \to Q_p{\times}\{*\} \cup S^2{\times}S^n$$

which is given by

$$\begin{aligned} \hat{\alpha}|_{D^{4p-2}\times S^{n-1}} &= \chi{\times}{*} \\ \hat{\alpha}|_{S^{4p-3}\times D^n} &= \alpha{\times}\chi_n \end{aligned}$$

where $\chi : (D^{4p-2}, S^{4p-3}) \to (Q_p, S^2)$ denotes the characteristic map of the top cell of Q_p and $\chi_n : (D^n, S^{n-1}) \to (S^n, \{*\})$ denotes the relative homeomorphism. Thus we have the following equations for $(\lambda{\times}\{*\} \cup (\Sigma\Omega i\Sigma j_1){\times}1_{S^n})\hat{\alpha}$:

$$\begin{aligned} (\lambda{\times}\{*\} \cup (\Sigma\Omega i\Sigma j_1){\times}1_{S^n})\hat{\alpha}|_{D^{4p-2}\times S^{n-1}} &= \lambda\chi{\times}{*} \\ (\lambda{\times}\{*\} \cup (\Sigma\Omega i\Sigma j_1){\times}1_{S^n})\hat{\alpha}|_{S^{4p-3}\times D^n} &= \Sigma\Omega i\Sigma j_1\alpha{\times}\chi_n \end{aligned}$$

As for the space $Q_p{\times}S^n$, the space $P^2\Omega Q_p{\times}S^n$ is also the mapping cone of

$$\hat{p}_1^{Q_p} : (\Omega Q_p{*}\Omega Q_p){*}S^{n-1} = C(\Omega Q_p{*}\Omega Q_p){\times}S^{n-1} \cup (\Omega Q_p{*}\Omega Q_p){\times}D^n \to P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n$$

which is given by

$$\begin{aligned} \hat{p}_1^{Q_p}|_{C(\Omega Q_p*\Omega Q_p)\times S^{n-1}} &= \hat{\chi}{\times}{*} \\ \hat{p}_1^{Q_p}|_{(\Omega Q_p*\Omega Q_p)\times D^n} &= p_1^{Q_p}{\times}\chi_n \end{aligned}$$

where $\hat{\chi} : (C(\Omega Q_p{*}\Omega Q_p), \Omega Q_p{*}\Omega Q_p) \to (P^2\Omega Q_p, \Sigma\Omega Q_p)$ denotes the characteristic map of the attached cone of the mapping cone $P^2\Omega Q_p$. By Remark 5.5, the bottom cell inclusion $\Sigma\Omega i\Sigma j_1$ can be extended to $\lambda : Q_p \to P^2\Omega Q_p$ which is a compression of the identity. More precisely, λ is the homotopy given by the composition of the homotopy of $\Sigma\Omega i\Sigma j_1\alpha$ in $\Sigma\Omega Q_p$ to $\alpha' = p_1^{Q_p}|_{S^3}\beta$ and the null-homotopy $C(\beta)$ in $C(\Omega Q_p{*}\Omega Q_p)$. The former part of the homotopy λ also gives the homotopy of $\Sigma\Omega i\Sigma j_1\alpha{\times}\chi_n$ to $\alpha'{\times}\chi_n$. Thus we have that $(\lambda{\times}\{*\} \cup (\Sigma\Omega i\Sigma j_1){\times}1_{S^n})\hat{\alpha}$ is homotopic to $\hat{\alpha}'$ which is given by

$$\begin{aligned} \hat{\alpha}'|_{D^{4p-2}\times S^{n-1}} &= \hat{\chi}|_{C(S^1*S^1)}C(\beta){\times}{*} = \hat{p}_1^{Q_p}|_{C(S^1*S^1)\times S^{n-1}}(C(\beta){\times}1_{S^{n-1}}), \\ \hat{\alpha}'|_{S^{4p-3}\times D^n} &= (p_1^{Q_p}|_{S^1*S^1}\beta){\times}\chi_n = \hat{p}_1^{Q_p}|_{(S^1*S^1)\times D^n}(\beta{\times}1_{D^n}). \end{aligned}$$

Thus $\hat{\alpha}$ is homotopic in $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n$ to $\hat{p}_1^{Q_p}|_{(S^1*S^1)*S^{n-1}}(\beta{*}1_{S^{n-1}})$. This yields the following proposition.

Proposition 5.8 *The following diagram, without the dotted arrow, commutes up to homotopy.*

$$\begin{tikzcd}
S^{4p-3}{*}S^{n-1} \arrow[r, "\hat{\alpha}"] \arrow[d, "\beta{*}1_{S^{n-1}}"] & Q_p{\times}\{*\} \cup S^2{\times}S^n \arrow[r, hook] \arrow[dd, hook, "\lambda{\times}\{*\}\cup(\Sigma\Omega i\Sigma j_1){\times}1_{S^n}"] & Q_p{\times}S^n \arrow[dd, "\lambda{\times}1_{S^n}"] \arrow[ddr, dashed, "1_{Q_p}{\times}1_{S^n}"] & \\
(S^1{*}S^1){*}S^{n-1} \arrow[d, hook, "((\Omega i{*}\Omega i)(j_1{*}j_1)){\times}1_{S^{n-1}}"] & & & \\
(\Omega Q_p{*}\Omega Q_p){*}S^{n-1} \arrow[r, "\hat{p}_1^{Q_p}"] & P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n \arrow[r, hook] & P^2\Omega Q_p{\times}S^n \arrow[r, "e_2^{Q_p}{\times}1_{S^n}"] & Q_p{\times}S^n
\end{tikzcd}$$

Since $\beta{*}1_{S^{n-1}} \simeq \pm\Sigma(\beta{\wedge}1_{S^{n-1}}) \simeq \pm\Sigma^n\beta$, we have established the following result.

Proposition 5.9 $1_{Q_p}{\times}1_{S^n}$ *can be compressed into* $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n$*, for* $n \geq 2$*.*

Proof. In the case when $n \geq 2$, $\beta{*}1_{S^{n-1}}$ is trivial. Since the inclusion $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n \to P^2\Omega Q_p{\times}S^n$ induces a split epimorphism in the homotopy groups, a similar argument to that used in the proof of Theorem 5.6 leads us the conclusion that there is a compression δ of $\lambda{\times}1_{S^n}$ to $P^2\Omega Q_p{\times}\{*\} \cup \Sigma\Omega Q_p{\times}S^n$. Moreover, we may assume the compression homotopy leaves the subspace $Q_p{\times}\{*\} \cup S^2{\times}S^n$ fixed. By Remark 5.7, the identity 1_{Q_p} is given from $e_2^{Q_p}\lambda$ by adding an element $ev_{Q_p}\gamma$, $\gamma \in \pi_{4p-2}(\Sigma\Omega Q_p)$. We define a map δ_2 by

$$\delta_2 : Q_p{\times}S^n \overset{\mu\times 1_{S^n}}{\longrightarrow} (Q_p \vee S^{10}){\times}S^n = \ Q_p{\times}S^n \cup S^{10}{\times}S^n \xrightarrow{\delta\cup(\gamma\times 1_{S^n})} P^2\Omega Q_p{\times}\{*\} \cup \Sigma\Omega Q_p{\times}S^n,$$

where μ denotes the co-action of S^{4p-2}. Since δ is homotopic to λ in $P^2\Omega Q_p{\times}S^n$ with the subspace $\{*\}{\times}S^n$ left fixed, δ_2 is homotopic to

$$(\lambda+\gamma){\times}1_{S^n} : Q_p{\times}S^n \overset{\mu\times 1_{S^n}}{\longrightarrow} (Q_p \vee S^{10}){\times}S^n = \ Q_p{\times}S^n \cup S^{10}{\times}S^n \xrightarrow{(\lambda\times 1_{S^n})\cup(\gamma\times 1_{S^n})} P^2\Omega Q_p{\times}S^n,$$

in $P^2\Omega Q_p{\times}S^n$ which is a compression of $1_{Q_p}{\times}1_{S^n}$. Thus $\delta_2 : Q_p{\times}S^n \to P^2\Omega Q_p{\times}\{*\} \cup \Sigma\Omega Q_p{\times}S^n$ gives the compression of $1_{Q_p}{\times}1_{S^n}$. *QED.*

Thus we have $2 = \mathrm{cat}_p\, Q_p \leq \mathrm{cat}_p\, Q_p{\times}S^n \leq \mathrm{cat}\, Q_p{\times}S^n \leq \mathrm{cat}\,(P^2\Omega Q_p{\times}\{*\} \cup \Sigma\Omega Q_p{\times}S^n) \leq 2$, for $n \geq 2$, and hence we have established our main theorem.

Theorem 5.10 $\mathrm{cat}\, Q_p{\times}S^n = \mathrm{cat}_p\, Q_p{\times}S^n = 2$*, for* $n \geq 2$*, while* $\mathrm{cat}\, Q_p{\times}S^1 = \mathrm{cat}_p\, Q_p{\times}S^1 = 3$*.*

In the case when $n = 1$, we have $\Sigma\beta \neq 0$. Then a similar argument to that used in the proof of Theorem 5.6 leads us the conclusion that $\mathrm{cat}\, Q_p{\times}S^1 = \mathrm{cat}_p\, Q_p{\times}S^1 = 3$ while $\mathrm{cat}\, Q_p = \mathrm{cat}_p\, Q_p = 2$. The details are left to the reader.

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